Onset of a Limit Cycle and Universal Three-Body Parameter in Efimov Physics

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The Efimov effect is the only experimentally realized universal phenomenon that exhibits the renormalization-group limit cycle with the three-body parameter parametrizing a family of universality classes. Recent experiments in ultracold atoms have unexpectedly revealed that the three-body parameter itself is universal when measured in units of an effective range. By performing an exact functional renormalization-group analysis with various finite-range interaction potentials, we demonstrate that the onset of the renormalization-group flow into the limit cycle is universal, regardless of short-range details, which connects the missing link between the two universalities of the Efimov physics. A close connection between the topological property of the limit cycle and few-body physics is also suggested.

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In the early 1970s, Efimov predicted a counterintuitive quantum phenomenon, in which resonantly interacting three bosons form an infinite series of three-body bound states even if the interaction is too weak to support a two-body bound state [1]. This Efimov effect emerges in a wide range of systems including identical bosons [1], mass-imbalanced fermions [2,3], particles in mixed dimensions [4], nucleons [5], magnons [6], and macromolecules such as DNA [7]. Besides its universality, the Efimov spectrum exhibits a discrete scale invariance, which implies that the energy eigenvalues of the trimers are related to one another by a universal scaling factor of 22.7^2. This peculiar property provides a unique example of a renormalization-group (RG) limit cycle [8], which refers to a periodic behavior of a RG flow and had been elusive until the emergence of the Efimov effect. Because of the universality and uniqueness, the Efimov effect has been extensively studied in various fields of physics such as atomic, chemical, nuclear, and particle physics. In particular, experimental observations of the Efimov effect in ultracold atoms [9–14] have given an enormous impetus to the development of Efimov physics.

Among discoveries in recent ultracold atom experiments is the universality in the three-body parameter κ^* (or equivalently the scattering length a_- at the triatomic resonance) [15–17], which sets the energy scale of the lowest-lying Efimov state (see Fig. 1). While low-energy two-body observables are universally described by the s-wave scattering length a, the existence of Efimov states implies an additional dependence of the low-energy three-body observables on the three-body parameter, which encapsulates short-range details of the three-body physics and had therefore been considered to be nonuniversal. Recent experiments in ultracold atoms, however, have revealed that a_- takes on almost the same value when measured in units of the van der Waals length r_{vdW} for different atomic species, internal states, and Feshbach resonances, suggesting some underlying physics that makes such an agreement possible. Recently, it was suggested [18–21] that systems other than the atomic van der Waals systems such as nuclear systems fall into a similar universality class if the three-body parameter is measured in units of the effective range that characterizes the range of interactions. It has been shown [21] that for each of two different classes of deep two-body potentials, the three-body parameter measured in units of r_{eff} has a

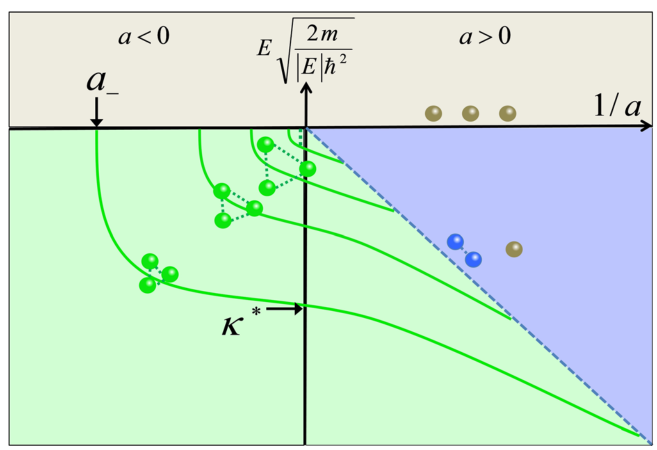

FIG. 1 (color online). Energy spectrum of three-identical bosons with resonant interaction (not to scale). The abscissa shows the inverse s-wave scattering length a^{-1}, and the ordinate shows the square root of the energy eigenvalue multipled by its sign. The Efimov states are related to one another by the universal scaling factor of $e^{2\pi/s_0} \simeq 22.7^2$ with $s_0 \simeq 1.00624$. The three-body parameters that set the energy scale of the lowest-lying Efimov state are labeled here as a_- and κ^*. Here, a_- denotes the s-wave scattering length at the triatomic resonance and κ^* is the binding wave number of the lowest-lying Efimov state at unitarity. They are uniquely related to each other.

universal value; especially, for potentials decaying faster than $1/r^6$, the three-body parameter falls within a universal window of $\kappa^* r_{\rm eff} = 0.44$–$0.52$.

In this Letter, we address an as yet unexplored fundamental question: Does this experimentally found and numerically vindicated universality bring about a new universality in the renormalization-group limit cycle? We answer this question in the affirmative by focusing on identical bosons at the unitarity limit $a = \pm\infty$. Furthermore, how topological properties of a renormalization-group flow can be related to universal aspects of few-body physics is an intriguing but so far untapped problem. We will touch upon this point at the end of this Letter, taking four-body physics as an illustrative example.

Most previous RG analyses [22–24] of the Efimov effect have been limited to the zero-range model $V_z(\mathbf{r}) = g\delta(\mathbf{r})$, in which κ^* is treated as an input parameter, precluding any statement about its universality. Since the finite-range effect plays an essential role in this universality, we should extend the RG analysis to systems with finite-range interactions. In previous works, the zero-range model was adopted to discuss the limit cycle; however, it predicts the periodic RG flow not only in the infrared limit but also in the ultraviolet limit, which indicates the unphysical Thomas collapse [25].

In contrast, since we are interested in the universality of the three-body parameter, in which a finite-range nature of the interaction plays a crucial role [18–21], we have performed a functional renormalization-group (FRG) analysis for different Hamiltonians with various finite-range interactions. We have obtained exact RG flows for the three-body coupling constant, which is defined as a dimensionless particle-dimer scattering amplitude (see Fig. 2). We have found that, in contrast with the zero-range model, each RG flow starts at a point away from the limit cycle and exhibits characteristic behavior that depends on the short-range details of each individual interaction potential; however, in the infrared regime, the flow begins to show the limit-cycle behavior, the onset of which is found to give a universal value of $k^* r_{\rm eff} = 0.49(4)$, which lies in the universal "window" of the three-body parameter $\kappa^* r_{\rm eff} = 0.44 - 0.52$ [21]. We here used the same definition of the effective range $r_{\rm eff}$ as in Ref. [21]. The onset k^* is evaluated as the RG scale at which the first divergence of the coupling constant occurs. By using the relation $\kappa^* a_- = 2.13$ [26], we obtain $a_- = -4.3(3) r_{\rm eff}$, corresponding to $a_- = -11.8(9) r_{\rm vdW}$ in units of the van der Waals length $r_{\rm vdW}$, in good agreement with the experimental results $a_- = -9.5(4) r_{\rm vdW}$ [15], $-9.7(7) r_{\rm vdW}$ [16], $-10.9(7) r_{\rm vdW}$ [17]. We, thus, identify the universality of the onset of the limit cycle with that of the three-body parameter. We note that the nonperturbative nature of the FRG has played a decisive role in revealing this relation, since we

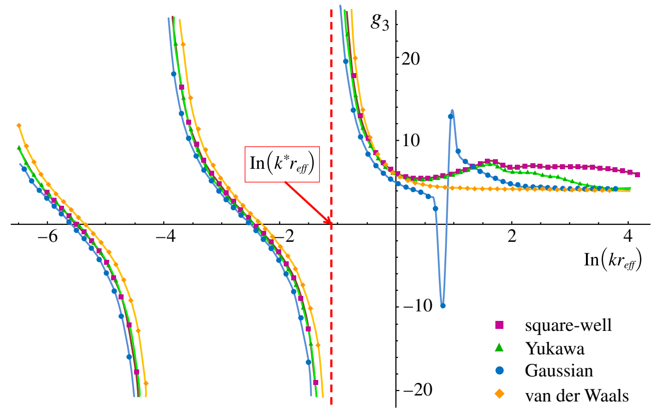

FIG. 2 (color online). Exact RG flows of the three-body coupling constant g_3. The abscissa shows the logarithm of the RG scale $\ln(k r_{\rm eff})$ in units of the effective range $r_{\rm eff}$, and the ordinate shows the three-body coupling constant g_3. Different curves correspond to different interaction potentials: Gaussian (circle), van der Waals (diamond), Yukawa (triangle), and square-well (square) potentials. The universal onset point of the limit cycle coincides with the universal value of the three-body parameter $\kappa^* r_{\rm eff} = 0.49$.

have to deal with a diverging coupling constant, which the perturbative Wilsonian RG cannot deal with. It is striking that the geometrical property (i.e., the onset point) of the limit cycle can be related to the universality of the three-body parameter in the Efimov physics. This observation can be further generalized to the topological constraint of the limit-cycle behavior on the relationship between an Efimov state and its four-body companions as we discuss later.

It has been pointed out that the three-body parameter deviates from the universal value for narrow Feshbach resonances [27–29] and that a deviation from the universal scaling occurs for deep Efimov states [27]. The former occurs because closed-channel molecules are not fully gapped and their dynamics cannot be disregarded; thus, a two-channel model description is essential. Concerning the latter, the scaling factor between the lowest and second-lowest lying Efimov states obtained from our calculation is 22.8(4), which is consistent with the value of 23.04 obtained in Ref. [27] within the accuracy of our calculation.

We now present our theoretical framework for obtaining these results. To perform an exact RG analysis on finite-range interactions, we use a simple microscopic model that accurately reproduces pair correlations of model interaction potentials and can be solved exactly for three particles. We use a separable-potential model whose interaction Hamiltonian is written in the form of a projection operator $\hat{V}_f = \xi|\chi\rangle\langle\chi|$, which retains the simplicity of the zero-range (δ function) interaction $\hat{V}_z = g|\mathbf{r} = 0\rangle\langle\mathbf{r} = 0|$. The microscopic action for identical bosons is then written as

$$
\begin{aligned}
S[\psi,\psi^*] = &\int_Q \psi^*(Q)(iq^0 + q^2 - \mu)\psi(Q) \\
&+ \frac{\xi}{4}\int_{Q_1 Q_2 Q_1{}' Q_2{}'} \delta(Q_1 + Q_2 - Q_1{}' - Q_2{}') \\
&\times \chi\left(\frac{\mathbf{q}_1{}' - \mathbf{q}_2{}'}{2}\right)\chi^*\left(\frac{\mathbf{q}_2 - \mathbf{q}_1}{2}\right) \\
&\times \psi^*(Q_1{}')\psi^*(Q_2{}')\psi(Q_2)\psi(Q_1),
\end{aligned}
\tag{1}
$$

where Q denotes the four momentum consisting of Matsubara frequency q_0 and momentum $\mathbf{q}$, μ is the chemical potential, $\chi(q) \coloneqq \langle \mathbf{q}|\chi\rangle$ in momentum representation, ψ denotes the bosonic field, and $\int_Q = \int (dq^0 d^3q/(2\pi)^4)$. Throughout this Letter, we employ the units $\hbar = k_B = 2m = 1$, where k_B is the Boltzmann constant and m is the mass of the particle.

We can choose an appropriate $|\chi\rangle$ of $\hat{V}_f$ so that $\hat{V}_f$ reproduces the low-energy pair correlation, including a nonzero range, of model potentials. This approximation can be systematically developed with arbitrarily high accuracy by adding another projection term to $\hat{V}_f$ [30]. The construction procedure of $|\chi\rangle$ is described in Refs. [21,30]. Despite its simplicity, $\chi(q)$ reproduces two-body observables, including phase shifts and bound-state energies, of exact model potentials with high accuracy. Here we use four different types of the separable models: van der Waals, Yukawa, infinite square-well, and Gaussian models, which are available in Refs. [20,21].

On the basis of this model, we perform an exact RG analysis based on FRG, which provides a nonperturbative RG scheme dealing with strongly correlated situations as Efimov physics. We start from the Wetterich equation [31],

$$
\partial_k \Gamma_k[\psi,\psi^*] = \frac{1}{2}\mathrm{Tr}\tilde{\partial}_k \ln\left(\frac{\delta^2\Gamma_k}{\delta\psi(q)\delta\psi^*(q)} + R_k(\mathbf{q})\right),
\tag{2}
$$

where Γ_k is the one-particle irreducible (1PI) effective action of the scale-dependent action $S_k = S + \int_Q R_k(\mathbf{q})\psi^*(Q)\psi(Q)$ and reduces in the ultraviolet limit $k = \Lambda$ to the microscopic action S and in the infrared limit $k = 0$ to the usual effective action Γ, defined as the Legendre transform of the Schwinger functional. The symbol Tr implies the sum over momenta, Matsubara frequencies, and internal indices. The symbol $\tilde{\partial}_k$ acts only on the Litim's optimized regulator [32] $R_k(\mathbf{q}) \coloneqq (k^2 - q^2)\Theta(k^2 - q^2)$, where Θ is the unit-step function. To deal with the RG flow of the three-body coupling constant, we perform a vertex expansion [33] of Eq. (2) with respect to the field variables to derive the RG equations for 1PI vertices,

$$
\begin{aligned}
\Gamma_k = &\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\int_{\substack{K_1,\ldots,K_n \\ K_1',\ldots,K_n'}} \Gamma_k^{(2n)}(K_1,\ldots,K_n;K_n',\ldots,K_1') \\
&\times \delta(K_1 + \cdots + K_n - K_n' - \cdots - K_1') \\
&\times \psi^*(K_1)\cdots\psi^*(K_n)\psi(K_n')\cdots\psi(K_1'),
\end{aligned}
\tag{3}
$$

where $\Gamma_k^{(2n)}$ is the $2n$th-order 1PI vertex, which represents the correlation of n particles. Since we are interested only in the three-body physics, we have only to consider terms up to $n = 3$. Indeed, the exact RG flow equations are closed up to $n = 3$ since in the vacuum limit (i.e., the limits of diverging inverse temperature $\beta \to \infty$ and the vanishing number density of particles $n \to 0$), the physics of four or more number of particles does not affect the three-body physics [34]. In other words, in the vacuum limit, the diagrams containing particle-hole loops vanish because of the infinitely large chemical potential, which leads to decoupling of higher-order vertices from lower-order ones, allowing an exact treatment of the RG equations.

We first consider one- and two-body sectors, which renormalize the three-body coupling constant. The exact RG equations in the vacuum limit for one- and two-body sectors are depicted in Figs. 3(a) and 3(b), respectively. Noting the ultraviolet boundary condition $\Gamma_k = S(k = \Lambda)$, we find that the one-body sector is given as

$$
\Gamma_k^{(2)}(P) = ip^0 + p^2 - \mu,
\tag{4}
$$

which is consistent with the fact that the self-energy correction is absent in the particle vacuum. Because of the separate dependence on the relative momentum of the separable model, the two-body sector can be decomposed into the total-momentum and the relative-momentum parts as depicted in Fig. 3(c), providing an analytical solution as follows:

$$
\Gamma_k^{(4)}(P;P_1,P_2) = \chi^*(\mathbf{p}_2)\Gamma_k^S(P)\chi(\mathbf{p}_1),
\tag{5}
$$

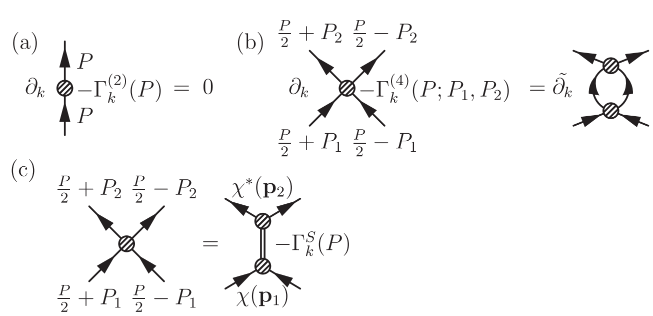

FIG. 3. Diagrammatic representations of the exact FRG equations for (a) the one-body sector and (b) the two-body sector. The derivative $\tilde{\partial}_k$ acts only on a regulator which is contained in the internal propagator. (c) Decomposition of the four-point 1PI vertex into the total-momentum and the relative-momentum parts.

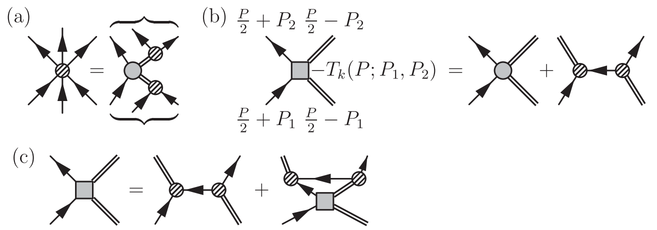

FIG. 4. (a) Decomposition of the six-point 1PI vertex, which describes the three-body scattering process. The double lines and the shaded vertices are the same as in Fig. 3(c). The curly brackets mean symmetrization with respect to external lines. (b) Definition of the particle-dimer scattering amplitude T_k represented as a square vertex. (c) Integral form of the exact FRG equation for the three-body sector. Integration of the RG equation with respect to the cutoff scale k has been analytically performed.

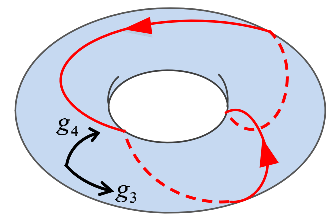

FIG. 5 (color online). Schematic illustration of the limit cycle on the space of the three- and four-body coupling constants g_3 and g_4. A torus of g_3-g_4 space can be constructed by enclosing the space periodically. The limit cycle winds twice in the g_4 direction as it winds once in the g_3 direction. This reflects the fact that each Efimov state is associated with two four-body bound states [37,38]. We suggest that such a nontrivial topology of the limit cycle may support the robustness of the number of bound states against a continuous change of the Hamiltonian.

$$\frac{1}{\Gamma_k^S(P)} = \frac{1}{16\pi a} - \frac{1}{2}\int \frac{d^3 l}{(2\pi)^3} \frac{ip^0 + \frac{\mathbf{p}^2}{2} - 2\mu + R_k(\frac{\mathbf{p}}{2}+\mathbf{l}) + R_k(\frac{\mathbf{p}}{2}-\mathbf{l})}{[ip^0 + \frac{\mathbf{p}^2}{2} + 2\mathbf{l}^2 - 2\mu + R_k(\frac{\mathbf{p}}{2}+\mathbf{l}) + R_k(\frac{\mathbf{p}}{2}-\mathbf{l})]2\mathbf{l}^2} |\chi(\mathbf{l})|^2, \tag{6}$$

where a is the s-wave scattering length.

The three-body sector can then be solved numerically based on these analytical expressions. Following Ref. [35], we decompose the six-point 1PI vertex as described in Fig. 4(a). The exact RG flow equation for the three-body sector can then be analytically integrated with respect to k, resulting in a simple form as depicted in Fig. 4(c). We note that in the infrared limit $k = 0$, this integrated RG equation reduces to the Skornyakov-Ter-Martirosyan equation [36] for the separable model. Since we are only interested in the spatially isotropic s-wave component, which is relevant for Efimov physics, we make a projection onto $T_k(p,q) \coloneqq \int d\theta_{\mathbf{pq}} T_k(P^0_{\mathrm{onshell}} = 3\mu; \mathbf{p}, \mathbf{q})$, and define the dimensionless three-body coupling constant g_3 as a rescaled particle-dimer scattering amplitude:

$$g_3 \coloneqq k^2 T_k(p=0, q=0). \tag{7}$$

By solving the exact RG equation of the three-body coupling constant for the four different types of interparticle interaction numerically, we obtain Fig. 2. We can see that the RG flows for four different potentials show the interaction-dependent behavior at high energy; however, in the infrared regime, flows converge to the limit cycle. The average onset point of the limit cycle is evaluated to be $kr_{\mathrm{eff}} = 0.49(4) =: k^* r_{\mathrm{eff}}$ in excellent agreement with the universal three-body parameter. Specifically, we have obtained $kr_{\mathrm{eff}} = 0.53(2)$ for van der Waals, $kr_{\mathrm{eff}} = 0.49(5)$ for square well, $kr_{\mathrm{eff}} = 0.48(5)$ for Yukawa, and $kr_{\mathrm{eff}} = 0.44(5)$ for Gaussian models, all of which agree with the values of the three-body parameter obtained in Ref. [21]. This observation suggests that the universality of the three-body parameter can be understood from the RG point of view as the universality of the onset of the limit cycle, which provides the first example relating the geometrical aspect of the limit cycle to the universal property of few-body physics. Our result also suggests that the three-body parameter can be regarded as the energy scale below which the universal discrete scale invariance of the system emerges. Below the energy scale set by κ^*, we find a convergence of the three-body coupling constant g_3, which is defined through the particle-dimer scattering amplitude [see Eq. (7)]. This convergence strongly suggests that, below the energy scale of κ^*, the three-body correlation function takes roughly the same (discrete-scale-invariant) form irrespective of short-range details of the Hamiltonian.

In this Letter, we have connected the missing link between the two universalities of Efimov physics, namely, the universal discrete scaling of the energy spectrum and the universal three-body parameter, by demonstrating that the renormalization-group limit cycle starts at a universal point, regardless of short-range details. An intriguing extension of the present work is to relate topological aspects of the limit cycle with universal properties of few-body physics. For example, when four identical bosons interact via a resonant interaction, two four-body bound states universally appear associated with one Efimov state exhibiting a universal scaling [37,38]. We may relate these universal four-body bound states with a topological aspect of the RG limit cycle. The previous RG analysis of four-body physics has shown that the four-body coupling constant g_4 forms a closed RG limit cycle, which is solely induced by the limit cycle of the three-body coupling constant g_3 [39]. From this result we suggest that if we constitute a torus of the g_3-g_4 space by enclosing the space periodically, the closed limit cycle winds twice on the torus as schematically illustrated in Fig. 5. Since the winding number of the limit cycle on the torus is topological, we may conclude that the number of four-body bound states is a topological winding number regardless of the details of interparticle interactions. This may afford a fundamental example that relates a topological property of a limit cycle to a universal property of few-body physics with discrete scale invariance.

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