Thermal Hall Effect and Geometry with Torsion

Andrey Gromov[1] and Alexander G. Abanov[2]
[1]Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794, USA
[2]Department of Physics and Astronomy and Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, New York 11794, USA
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We formulate a geometric framework that allows us to study momentum and energy transport in nonrelativistic systems. It amounts to a coupling of the nonrelativistic system to the Newton-Cartan (NC) geometry with torsion. The approach generalizes the classic Luttinger's formulation of thermal transport. In particular, we clarify the geometric meaning of the fields conjugated to energy and energy current. These fields describe the geometric background with nonvanishing temporal torsion. We use the developed formalism to construct the equilibrium partition function of a nonrelativistic system coupled to the NC geometry in $2+1$ dimensions and to derive various thermodynamic relations.

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Introduction.—In the seminal work of 1964, Luttinger developed a linear response theory for thermoelectric transport [1]. An essential part of his approach is the coupling of the many body system to an auxiliary external "gravitational potential" conjugated to the energy density. The evolution of the energy density is defined by the divergence of energy current; the latter is a fundamental object in the theory of thermal transport. In this Letter we identify the appropriate sources of the momentum, energy, and energy current in nonrelativistic systems. We use the developed general formalism to derive thermodynamic relations involving thermal Hall current in the presence of external fields.

In relativistic systems the energy density and the corresponding current are naturally combined into a stress-energy tensor $T^{\mu\nu}$ coupled to an external gravitational field described by the spacetime metric. The energy-momentum and charge conservation laws can be written as

$$\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \qquad \partial_\mu J^\mu = 0, \tag{1}$$

where $T^{\mu\nu}$ is a stress-energy tensor defined as a response to the external metric $g_{\mu\nu}$. Here, we introduced an electric current J^μ and an external electromagnetic field $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$. Given a matter action S we can compute the energy-momentum tensor and the electric current as

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}, \qquad J^\mu = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_\mu}. \tag{2}$$

In the absence of the external sources the first equation in Eq. (1) encodes two conservation laws: conservation of momentum and conservation of energy,

$$\dot{P}^j + \partial_i T^{ij} = 0, \qquad \dot{\varepsilon} + \partial_i J^i_E = 0, \tag{3}$$

where we introduced momentum, energy, and energy current as $P^j \equiv T^{0j}$, $\varepsilon = T^{00}$ and $J^i_E = T^{i0}$. These notations will be very natural later on. In relativistic systems the stress-energy tensor $T^{\mu\nu}$ (being defined as the response to the external spacetime metric) is symmetric. This implies equality of momentum and energy current $P^i = J^i_E$.

In nonrelativistic systems this equality no longer holds. For example, for a single massive nonrelativistic particle with mass m moving with velocity v^i, we have $P^i = mv^i$ and $J^i_E = (mv^2/2)v^i$.

The first result of this Letter is the identification of the appropriate sources for the momentum, energy, and energy current. We introduce a nonrelativistic analogue of Eq. (2). This is achieved by replacing the spacetime metric $g_{\mu\nu}$ by different geometric data known as Newton-Cartan (NC) geometry with torsion. We explain how to couple a given nonrelativistic system to the NC geometry. Our analysis does not assume Galilean symmetry and is valid in systems without boost symmetry. The NC geometry has appeared in the context of the quantum Hall effect [2], nonrelativistic (Lifshitz) holography [3], and fluid dynamics [4]. The relation between the thermal transport and geometry with (and without) torsion was also discussed in Refs. [5,6]. The torsional responses in relativistic systems were discussed in Refs. [7–9].

While the coupling to NC geometry can be used in any nonrelativistic field theory, we are mainly motivated by applications to nonrelativistic fluid dynamics. In fluid dynamics, in addition to the standard symmetry constraints of field theory, there is an additional set of conditions that ensure that solutions of Eq. (3) are compatible with the (local) second law of thermodynamics [10]. Recently these constraints became a subject of active research in relativistic hydrodynamics [11,12]. It turns out that some of these constraints can be obtained systematically, demanding that solutions of Eq. (1) consistently describe thermal equilibrium in the presence of static external sources [11,13]. Here we are interested in nonrelativistic applications of these ideas.

The second result of this Letter is a construction of the generating functional of Euclidean static correlation functions consistent with local spacetime and gauge symmetries. Consistency of these static correlation functions with stationary solutions of nonrelativistic hydrodynamics provides constraints on the latter. We note here that equilibrium analysis should be valid for rather general, not necessarily Galilean invariant, systems. Throughout the Letter we assume that we are in $2+1$ dimensions, but most of the analysis is valid in any dimension with obvious modifications.

Coupling to Newton-Cartan geometry.—Conservation laws (3) follow from the space and time translation symmetries. In what follows we will introduce external fields that naturally couple to momentum, energy, and energy current by making these symmetries local.

Before going to general formulations, we consider an example of free fermions. The action is given by

$$S = \int dtd^2x\Big(i\Psi^\dagger\partial_0\Psi - \frac{1}{2m}(\partial_A\Psi)^\dagger(\partial_A\Psi)\Big). \qquad (4)$$

In order to make this action coordinate independent—i.e., gauge the time and space translations—we introduce frame fields (or vielbeins) E^μ_a and their inverse e^a_μ [14] and replace the derivatives in Eq. (4) as follows:

$$\partial_A \to E^\mu_A\partial_\mu, \qquad \partial_0 \to E^\mu_0\partial_\mu. \qquad (5)$$

The second replacement can be understood as a material derivative so that the vielbein E^μ_0 is the velocity field. Then the action (4) takes the form

$$S = \int dV\mathcal{L},$$
$$\mathcal{L} = \Big(\frac{i}{2}v^\mu(\Psi^\dagger\partial_\mu\Psi - \partial_\mu\Psi^\dagger\Psi) - \frac{h^{\mu\nu}}{2m}\partial_\mu\Psi^\dagger\partial_\nu\Psi\Big). \qquad (6)$$

Our conventions are $a,b,\ldots=0,1,2$ and $\mu,\nu,\ldots=0,1,2$; also $A,B,\ldots=1,2$ and $i,j,\ldots=1,2$. General coordinate transformations act on the Greek indices and local frame transformations act on the Latin $a,b,\ldots$ indices.

We have defined a degenerate "metric" $h^{\mu\nu}=\delta^{AB}E^\mu_AE^\nu_B$, a 1-form $n_\mu=e^0_\mu$, and a vector $v^\mu=E^\mu_0$. Notice that the spatial part of the metric h^{ij} is an (inverse) metric on a fixed time slice: it is symmetric and invertible. The introduced objects are not independent, but obey the relations

$$v^\mu n_\mu = 1, \qquad h^{\mu\nu}n_\nu = 0. \qquad (7)$$

These are precisely the conditions satisfied by the NC geometry data [2,15,16]. Some detailed discussion of the first order (i.e., using the vielbeins) formulation of the NC geometry can be found in Refs. [17,18].

The action (6) can be viewed as an action (4) written in an arbitrary coordinate system. The invariant volume element is $dV=edtd^2x$, with $e=\sqrt{\det(e^a_\mu e^a_\nu)}$. Because of the spatial isotropy of Eq. (4), the vielbeins naturally combine into the degenerate metric $h^{\mu\nu}$. Similarly, the temporal components of the vielbeins (denoted by v^μ and n_μ) stand aside in Eq. (6), explicitly breaking the (local) Lorentz symmetry down to $SO(2)$. If the physical system were anisotropic the replacement (5) would still make sense, but one would have to treat each vielbein as an independent object, i.e., not constrained by any local symmetries of the tangent space.

To couple a generic matter action to the NC geometry, one has to proceed in the same way as in the example considered above. Namely, one should modify the derivatives according to Eq. (5). Then the objects v^μ, n_μ, and $h^{\mu\nu}$ (NC data) will naturally arise (we assume spatial isotropy from now on). When the 1-form n_μ is not closed, we define the Newton-Cartan temporal torsion 2-form [19] as

$$\mathcal{T}_{\mu\nu} = \partial_\mu n_\nu - \partial_\nu n_\mu. \qquad (8)$$

In practice, it is convenient to use a particular parametrization of the NC background fields. Let us specify the spatial part h^{ij} of the degenerate metric and assume that $n_\mu=(n_0,n_i)$ and $v^\mu=(v^0,v^i)$ are also specified and that they satisfy the first relation in Eq. (7). Then we find, from other relations in Eq. (7),

$$h^{\mu\nu} = \begin{pmatrix} \frac{n^2}{n_0^2} & -\frac{n^i}{n_0} \\ -\frac{n^i}{n_0} & h^{ij} \end{pmatrix},$$

where we define $n^i=h^{ij}n_j$, $n^2=n_in_jh^{ij}$. In this parametrization the invariant volume element is given by $dV=\sqrt{h}n_0dtd^2x$, where we have denoted $\det(h^{ij})=h^{-1}$.

The momentum, stress, energy, and energy current are identified as responses to the NC geometry as follows [19]:

$$P_i = -v^0\frac{\delta S}{\delta v^i}, \qquad T_{ij} = -2\frac{\delta S}{\delta h^{ij}}, \qquad (9)$$

$$\varepsilon = -\Big(n_0\frac{\delta S}{\delta n_0} - v^0\frac{\delta S}{\delta v^0}\Big), \qquad J^i_E = -n_0\frac{\delta S}{\delta n_i}, \qquad (10)$$

where we turn off the fields n_i and v_i after the variation is taken.

The introduced NC geometry is general and reduces to some cases considered in the literature. For example, the choice $n_\mu=(1,0,0)$, $v=(1,v^i)$ corresponds to the torsionless NC background, which turned out to be convenient in studying Galilean invariant actions [2,20–23].

Another particular limit is given by $n_\mu = (e^\psi, 0, 0)$, $v^\nu = (e^{-\psi}, 0, 0)$. This is an example of the NC geometry with temporal torsion. The torsion is given by

$$\mathcal{T} = e^\psi(\partial_i \psi) dx^i \wedge dt. \tag{11}$$

In this case the only nonvanishing component of the torsion tensor is $\mathcal{T}_{0i}$. This NC geometry essentially appeared in the procedure introduced by Luttinger [1,24]. The field ψ is precisely the gravitational potential introduced in Ref. [1]. The disadvantage of this choice of geometry is the absence of the field n_i that couples to the energy current.

In the following we consider a general case keeping all of the components of NC geometry turned on.

Before proceeding let us illustrate how one can derive expressions for conserved currents using the coupling to NC geometry.

Consider a system of free fermions. We have already introduced the NC fields into the action of free fermions in Eq. (6). Then the direct application of Eq. (10), using equations of motion and turning off NC fields after the variations, we obtain the familiar expressions for energy and energy current in flat space

$$\varepsilon = -\frac{1}{2m}(\partial_i \Psi)^\dagger (\partial_i \Psi), \tag{12}$$

$$J_i^E = \frac{i}{4m^2}(\partial^2 \Psi^\dagger \partial_i \Psi - \partial_i \Psi^\dagger \partial^2 \Psi). \tag{13}$$

Equilibrium.—We construct the most general partition function, consistent with time independent, local space and time translations, and gauge symmetries. The partition function can be written as a Euclidian functional integral,

$$W = -\ln \operatorname{tr} \exp\left\{-\frac{H - \bar{\mu} N}{\bar{T}}\right\} = -\ln \int D\Psi D\Psi^\dagger e^{-S_E}. \tag{14}$$

Here we introduced a Euclidean action [25]

$$S_E[\{\Psi, \Psi^\dagger\}; A_\mu, n_\mu, v^\mu, h^{ij}] = \int d^2x \sqrt{h} \oint_0^{1/\bar{T}} d\tau n_0 \mathcal{L}_E, \tag{15}$$

where $\{\Psi, \Psi^\dagger\}$ refers to a collection of matter fields. This action is coupled to the NC geometry, as explained in the previous section. We have also coupled the theory to the external e/m field described by the vector potential A_μ.

The time independent field n_0 can be viewed as an inhomogeneous temperature $T(x)$ defined according to

$$\oint_0^{1/\bar{T}} d\tau n_0 \to \oint_0^{1/T(x)} d\tau', \qquad \frac{1}{T(x)} = \frac{n_0}{\bar{T}}. \tag{16}$$

The NC geometry allows us to introduce spatial variations in the size of the compact imaginary time direction.

Rescaling the Euclidean time $\tau \to \tau'/\bar{T}$ in Eq. (15) and correspondingly transforming the fields n_0, A_0, v^0 we find that the action depends on $\bar{T}$ as follows:

$$S_E = S_E\left[\Psi, \Psi^\dagger; \frac{A_0}{\bar{T}}, \frac{n_0}{\bar{T}}, v^0 \bar{T}, A_i, \frac{n_i}{n_0}\bar{T}, v^i, h^{ij}\right]. \tag{17}$$

In (local) equilibrium, external fields do not depend on Euclidean time. The generating functional W depends on the temperature T and external sources. We also assume that W can be written as an integral of a local density so that

$$W = \int d^2x \sqrt{h} \frac{n_0}{\bar{T}} \mathcal{P}\left(\frac{A_0}{\bar{T}}, \frac{n_0}{\bar{T}}, v^0 \bar{T}, A_i, \frac{n_i}{n_0}\bar{T}, v^i, h^{ij}\right), \tag{18}$$

where we have already replaced the integral over Euclidean time by the overall factor $1/\bar{T}$. It is worth noting that results derived from the Euclidean generating functional can be used to obtain the zero frequency correlation functions in real time upon a Wick rotation.

Local time shifts.—We are mainly interested in the thermal transport, so from now on we set the external field at $v^i = 0$ and parametrize $v^0 = (1/n_0) \equiv e^{-\psi}$ in order to satisfy Eq. (7). This field configuration is preserved by the time independent space and time translations.

The transformation law of the external field n_i under a local time shift $t \to t + \zeta(x)$ takes the form

$$\delta(e^{-\psi} n_i) = -\partial_i \zeta; \tag{19}$$

i.e., the field $e^{-\psi} n_i$ transforms like a $U(1)$ gauge field under a local time shift. This field can be regarded as a connection on an S^1 bundle over the base manifold, where S^1 is the thermal circle. The field strength is related to the NC temporal torsion.

It is convenient to introduce $\mathcal{A}_i = A_i - A_0 e^{-\psi} n_i$. This field transforms like a gauge field under electromagnetic gauge transformations and it is invariant under local time shifts.

Invariance of the generating functional with respect to the transformation (19) implies a local conservation law of the thermal current

$$J_Q^i = -\frac{\bar{T}}{\sqrt{h}}\left(\frac{\delta W}{\delta e^{-\psi} n_i} + A_0 \frac{\delta W}{\delta A_i}\right) = J_E^i - A_0 J^i. \tag{20}$$

This current is conserved,

$$\nabla_i J_Q^i = 0, \tag{21}$$

where $\nabla_i X^i = \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} X^i)$ is the covariant divergence.

Generating functional in derivative expansion.—We present the partition function as an expansion in derivatives of the external NC and electromagnetic fields. We consider the following generating functional:

$$W = \int d^2x\sqrt{h}\frac{1}{T}\mathcal{P}(\mu, T, \mathcal{B}, B_E), \tag{22}$$

where we have defined the local chemical potential and temperature in terms of external fields,

$$\frac{1}{T(x)} = \frac{e^{\psi}}{\bar{T}}, \qquad \mu(x) = e^{-\psi}A_0(x), \tag{23}$$

and defined gauge invariant (pseudo)scalars

$$\mathcal{B} = \epsilon^{ij}\partial_i\mathcal{A}_j, \qquad B_E = \epsilon^{ij}\partial_i(e^{-\psi}n_j). \tag{24}$$

Writing Eq. (22), we assumed that both $\mathcal{B}$ and B might be large, while their derivatives are small and can be neglected. We also assumed that the gradients of both μ and T are small.

The generating functional (22) encodes various local thermodynamic quantities and relations. For example, the energy (in flat space) can be found with the help of Eq. (10), appropriately modified for the presence of the gauge field

$$\begin{aligned}\varepsilon &= \bar{T}\frac{\delta W}{\delta e^{\psi}} + TA_0\frac{\delta W}{\delta A_0} = \frac{\partial(\mathcal{P}/T)}{\partial(1/T)} - \mu\frac{\partial\mathcal{P}}{\partial\mu}\\ &= \mathcal{P} + sT + n\mu,\end{aligned} \tag{25}$$

where we made the identifications

$$n(x) = \bar{T}\frac{\delta W}{\delta A_0} = -\frac{\partial\mathcal{P}}{\partial\mu} \tag{26}$$

and

$$s(x) = -\frac{\partial\mathcal{P}}{\partial T}. \tag{27}$$

The relation (25) suggests that $\mathcal{P}(\mu, T, \mathcal{B}, B_E)$ is the density of the grand thermodynamic potential (in the presence of external fields) and that Eq. (25) is the local version of the known thermodynamic relation $\mathcal{P} = E - \bar{T}S - \bar{\mu}N$.

It is instructive to find the pressure in the presence of external fields, also known as internal pressure:

$$P_{\text{int}} = \bar{T}\frac{\delta W}{\delta h^i_i} = P_{(0)} - M\mathcal{B} - M_E B_E, \tag{28}$$

where we have introduced the magnetization $M = e^{\psi}(\partial\mathcal{P}/\partial\mathcal{B})$ and the energy magnetization $M_E = e^{\psi}(\partial\mathcal{P}/\partial B_E)$, and $P_{(0)}$ is the pressure at zero magnetic field.

The additional contribution to the pressure given by the second term in Eq. (28) comes from the Lorentz force acting on magnetization currents. The last term of Eq. (28) gives a contribution similar to the one present in the nonvanishing background field B_E.

Magnetization currents.—While all transport currents vanish in thermal equilibrium, there are still electric and energy magnetization currents circulating in a material—even at equilibrium. These currents cannot be measured in transport experiments [24]. However, e.g., the electric magnetization current can, in principle, be observed in spectroscopy experiments or by measuring the magnetic field created by moving charges. The energy current can (at least in principle) be observed by the frame drag [26] due to distortions in the gravitational field created by the flow of energy. In the presence of the inhomogeneous external fields, magnetization currents can flow in the bulk of the material; otherwise, they are concentrated on the boundary of the sample.

Knowing magnetization currents is important, as this knowledge can be used to separate transport currents from the magnetization ones for systems driven out of equilibrium [24]. Also, for a particular case of the chemical potential lying in the excitation gap, the magnetization currents are the only currents responsible for the Hall effect [27].

In the following we consider both electric and thermal magnetization currents. They are given, respectively, by

$$J^i = \bar{T}\frac{\delta W}{\delta A_i} = \epsilon^{ij}\partial_j M, \tag{29}$$

$$J^i_Q = \epsilon^{ij}\partial_j M_E. \tag{30}$$

The currents (29) and (30) are conserved in the presence of arbitrary temperature profile $T(x)$, set by Eq. (23), and coincide with the ones found in Refs. [24,28,29] at the level of linear response.

We note here that the energy magnetization M_E is usually defined by Eq. (30), while the NC "magnetic field" B_E (usually denoted as B_g and referred to as gravimagnetic field) is defined as a quantity thermodynamically conjugated to M_E. In this work we clarified how one can systematically introduce external fields n_i in a nonrelativistic system and couple the system to B_E (24). Previous approaches explicitly used the presence of Lorentz symmetry [26,29] and cannot be applied in a majority of condensed matter systems.

Streda formulas.—It is possible to express the Hall conductivity and other parity odd responses purely in terms of the derivatives of thermodynamic quantities. We define electric and thermal conductivities as

$$J^i = \epsilon^{ij}(\sigma_H\partial_i\mu + \sigma^T_H\partial_i T), \tag{31}$$

$$J^i_Q = \epsilon^{ij}(\kappa^{\mu}_H\partial_i\mu + \kappa_H\partial_i T). \tag{32}$$

Comparing with Eqs. (29) and (30) we obtain, using the Maxwell's relations [30],

$$\sigma_H = \left(\frac{\partial M}{\partial \mu}\right)_{T,\mathcal{B},B_E} = \left(\frac{\partial n}{\partial \mathcal{B}}\right)_{T,\mu,B_E}, \qquad (33)$$

$$\sigma_H^T = \left(\frac{\partial M}{\partial T}\right)_{\mu,\mathcal{B},B_E} = \left(\frac{\partial s}{\partial \mathcal{B}}\right)_{T,\mu,B_E}, \qquad (34)$$

$$\kappa_H^\mu = \left(\frac{\partial M_E}{\partial \mu}\right)_{T,\mathcal{B},B_E} = \left(\frac{\partial n}{\partial B_E}\right)_{T,\mu,\mathcal{B}}, \qquad (35)$$

$$\kappa_H = \left(\frac{\partial M_E}{\partial T}\right)_{\mu,\mathcal{B},B_E} = \left(\frac{\partial s}{\partial B_E}\right)_{T,\mu,\mathcal{B}}. \qquad (36)$$

These are thermodynamic Streda-type formulas [31,32] for the response coefficients.

Galilean and Lorentz symmetries.—So far we have assumed that the (unperturbed) system under consideration is gauge invariant, spatially isotropic and homogeneous, and time translation invariant. In this general case there are no additional relations between electric current, momentum, and energy current. Several new relations appear if additional symmetries are present. For simplicity, we assume below that the underlying microscopic system consists of charged particles of a single species or several species with the same e/m (electric charge to mass) ratio.

If the system is Galilean invariant the electric current is proportional to the momentum $J^i = (e/m)P^i$; therefore, the magnetization density is proportional to the density of the angular momentum $M = (e/m)L_z$. Then from Eq. (33) we have

$$\sigma_H = \frac{e}{m}\left(\frac{\partial L_z}{\partial \mu}\right)_{T,\mathcal{B},B_E}; \qquad (37)$$

that is, Hall conductivity can be expressed in terms of derivatives of the angular momentum.

If the system is Lorentz invariant, then there is an additional equality between momentum and the energy current, as we pointed out in the introduction, $J_E^i = P^i$, and, therefore, $M_E = L_z$. Therefore, we have another version of the Streda formula for thermal Hall conductivity [29],

$$\kappa_H = \left(\frac{\partial L_z}{\partial T}\right)_{\mu,\mathcal{B},B_E}. \qquad (38)$$

In a general case, when no additional symmetries are present the angular momentum is not related to either electric or thermal magnetization and the relations (37) and (38) do not hold.

Conclusions.—To conclude, it is shown that coupling the physical system to the Newton-Cartan geometry introduces the appropriate sources for energy, momentum, and energy current. Variations of the action with respect to different components of the NC geometry give familiar expressions for energy, momentum, and energy current densities. It turns out that, in order to introduce the temperature gradients, one has to couple a physical system to the NC geometry with temporal torsion. We stress that the formalism does not assume either Lorentz or Galilean symmetry. Those symmetries can be imposed afterwards to restrict the responses of the physical system.

The developed formalism was used to construct a general local equilibrium partition function of a nonrelativistic system. With the partition function at hand, known thermodynamic relations have been obtained in the presence of external gauge and Newton-Cartan fields. It was found that upon linearization the resulting general expressions for electric and thermal magnetization currents agree with the linear response expressions known in the literature.

The constructed formalism is expected to have many potential applications in condensed matter systems and hydrodynamics. For example, the general geometric effective action constructed in the presence of the torsional NC background will not be restricted by the Lorentz symmetry and, therefore, is more natural in condensed matter context. The Galilean symmetry can be implemented by adding additional constraints on the action coupled to NC geometry. The generalization to systems with internal degrees of freedom such as spin may prove to be of interest in the context of the spin Hall effect.

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Note added.—Recently, we were made aware of complementary results [33]. We have learned about Ref. [34], where the NC geometry with torsion was related to the energy transport in Galilean invariant systems.

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