Anyonic Liquids in Nearly Saturated Spin Chains

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Most Heisenberg-like spin chains flow to a universal free-fermion fixed point near the magnetic-field induced saturation point. Here, we show that an exotic fixed point, characterized by two species of low-energy excitations with mutual anyonic statistics, may also emerge in such spin chains if the dispersion relation has two minima. By using bosonization, two-magnon exact calculations, and numerical density-matrix-renormalization-group calculations, we demonstrate the existence of this anyonic-liquid fixed point in an XXZ spin chain with up to second-neighbor interactions. We also identify a range of microscopic parameters, which support this phase.

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Magnetic-field induced saturation of quantum magnets is one of the most widely studied quantum critical points (QCPs) of nature: magnets with axial symmetry along the field axis become fully polarized at a critical field value. In two and three spatial dimensions, the corresponding QCP that separates the fully and partially polarized states belongs to the "Bose-Einstein condensate" (BEC) universality class. [1–5] The magnets can be treated as a dilute gas of bosons in the vicinity of the QCP by mapping the spins that are antiparallel to the field into hard-core bosons. In contrast, in most one-dimensional ($d = 1$) models studied thus far, the weakly interacting quasiparticles near the field-induced QCP have fermionic statistics [6]. Here, we demonstrate that a much richer spectrum of QCPs, including novel anyonic liquids, may emerge in nearly saturated axially symmetric spin chains.

The essential ingredient is magnetic frustration, which can provide natural realizations of single-particle dispersions with degenerate minima at multiple wave vectors $\mathbf{Q}$ [7]. Such single-particle dispersions do not change the universality class of the BEC QCP in $d > 1$, but can give rise to multi-$\mathbf{Q}$ condensates [8–11] such as long-range ordered magnetic vortex crystals [12,13]. In contrast, long-range order is suppressed in $d = 1$ due to strong quantum fluctuations. In this case, a Jordan-Wigner (J-W) transformation [14,15] allows us to describe the magnet as a dilute gas of interacting fermions near the QCP. The Pauli exclusion principle renders all fermion-fermion interactions irrelevant (in a renormalization-group sense), resulting in a free-fermion fixed point with a single-minimum dispersion relation [6]. The central question addressed in this Letter is the fate of the $d = 1$ QCP when magnetic frustration generates a dispersion relation with two degenerate minima.

We show that frustration can stabilize a novel anyonic liquid near the field-induced QCP of spin chains. This result extends the classification of QCPs for saturated quantum magnets from simple theories of free bosons ($d > 1$), and free fermions ($d = 1$), to an exotic line of QCPs with emergent Abelian anyonic statistics that interpolate between these two fixed points. Our anyonic liquid consists of two species of quasiparticles originating from the two degenerate minima (with two species of anyons, inversion symmetry breaking is not necessary and we only consider models with inversion symmetry [16,17]). Quasiparticles of different species do not interact with each other yet their commutation relations imply that they are Abelian anyons as opposed to simple bosons or fermions. In fact, similar theories of $d = 1$ Abelian anyons [18] have been envisioned in the field-theory literature through abstract flux attachment to free bosonic theories [19–32]. However, no experimentally relevant microscopic models have been shown to support such anyonic liquids. By combining bosonization, renormalization-group arguments, and numerical density-matrix renormalization group (DMRG) computations [33,34], we provide an experimentally relevant realization for these elusive anyonic liquids in the context of frustrated magnetism. Moreover, we propose experimental signatures, which should facilitate their observation.

The corresponding XXZ Hamiltonian [35–41]

$$H = \sum_{j;a=1,2}\left[\frac{J_a}{2}(S_j^+S_{j+a}^- + S_{j+a}^+S_j^-) + \Delta_a J_a\left(S_j^zS_{j+a}^z - \frac{1}{4}\right)\right] \tag{1}$$

is illustrated in Fig. 1(a). It includes up to second-neighbor exchange interactions and a Zeeman term which allows us to tune $S_T^z = \sum_j S_j^z$ with an external magnetic field B_z (S_T^z is conserved because $[H, S_T^z] = 0$). For a possible physical realization in a bilayer zigzag ladder, see Refs. [42,43].

After a J-W transformation $S_j^- = c_j \exp(-i\pi\sum_{k<j} n_k)$ and $S_j^z = n_j - \frac{1}{2}$, with $n_j = c_j^\dagger c_j$, we can reinterpret $H = H_0 + H_I$ as a model for interacting spinless fermions:

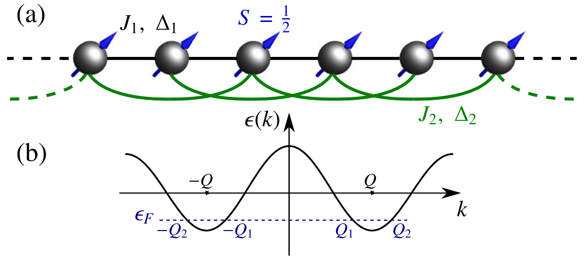

FIG. 1 (color online). The dispersion of Eq. (2) with two minima at $\pm Q$. The Fermi points are at $\pm Q_i$, $i = 1, 2$ with corresponding Fermi velocities v_i.

$$H_0 = \sum_{x;a=1,2}\left(\frac{J_a}{2} c_x^\dagger c_{x+a} + \text{H.c.}\right) = \sum_k \epsilon(k) c_k^\dagger c_k, \quad (2)$$

$$H_I = \sum_{x;a=1,2}(\Delta_a J_a n_x n_{x+a}) - J_2 \sum_x (c_x^\dagger n_{x+1} c_{x+2} + \text{H.c.}). \quad (3)$$

Here, we have dropped the chemical-potential terms (including B_z), which just tune the conserved $\sum_x n_x$. The single-particle dispersion relation is $\epsilon(k) = J_1 \cos(k) + J_2 \cos(2k)$. We assume $J_1 < 0$ and $|J_1| < 4|J_2|$ to guarantee that $\epsilon(k)$ has two minima at $k = \pm Q$ with $\cos(Q) = -(J_1/4J_2)$ [see Fig. 1(b)]. The condition of having a nearly saturated spin chain directly leads to a low density of fermions, i.e., the dilute limit, in which the Fermi momenta $Q_1, Q_2 \to Q$ [see Fig. 1(b)].

To bosonize H, we introduce creation and annihilation operators in the vicinity of the Fermi points: $\psi_a(p) \equiv c(Q_a + p)$ and $c(-Q_a + p) \equiv \bar{\psi}_a(p)$ for $a = 1, 2$. A Fourier transform of these fields leads to their real space version

$$c_x = e^{iQ_1 x}\psi_1(x) + e^{-iQ_1 x}\bar{\psi}_1(x) + e^{iQ_2 x}\psi_2(x) + e^{-iQ_2 x}\bar{\psi}_2(x). \quad (4)$$

The chiral fields $\psi_1(x)$ and $\bar{\psi}_1(x)$ vary slowly in space. This is similar to standard bosonization, but with twice the number of species. After linearizing the dispersion relation $\epsilon(\pm Q_1 + p) = \mp v_1 p$ and $\epsilon(\pm Q_2 + p) = \pm v_2 p$ [see Fig. 1(b)], ψ_2 and $\bar{\psi}_1$ (ψ_1 and $\bar{\psi}_2$) become right (left) movers, and the chiral fermions can be represented in terms of bosonic fields

$$\psi_{1,2}(x) = \frac{1}{\sqrt{2\pi}} e^{\pm i\phi_{1,2}(x)}, \qquad [\partial_x \phi_{1,2}(x), \phi_{1,2}(x')] = \pm 2\pi i \delta(x - x'),$$

$$\bar{\psi}_{1,2}(x) = \frac{1}{\sqrt{2\pi}} e^{\mp i\bar{\phi}_{1,2}(x)}, \qquad [\partial_x \bar{\phi}_{1,2}(x), \bar{\phi}_{1,2}(x')] = \mp 2\pi i \delta(x - x').$$

The chiral current operators [44] can be written as $j_a(x) \equiv \psi_a^\dagger(x)\psi_a(x) = (1/2\pi)\partial_x \phi_a(x)$ and $\bar{j}_a(x) \equiv \bar{\psi}_a^\dagger(x)\bar{\psi}_a(x) = (1/2\pi)\partial_x \bar{\phi}_a(x)$.

The noninteracting part of the Hamiltonian density can be written in terms of diagonal chiral current bilinears $j_a(x) j_a(x)$ as $H_0 = \pi \sum_{a=1,2} \int dx [v_a j_a(x) j_a(x) + v_a \bar{j}_a(x) \bar{j}_a(x)]$. The interacting part, which describes various scattering processes, has the general form

$$\begin{aligned} H_I = \int dx [& g_{1\bar{1}} j_1(x) \bar{j}_1(x) + g_{12} j_1(x) j_2(x) + g_{1\bar{2}} j_1(x) \bar{j}_2(x) \\ & + g_{\bar{1}2} \bar{j}_1(x) j_2(x) + g_{\bar{1}\bar{2}} \bar{j}_1(x) \bar{j}_2(x) + g_{2\bar{2}} j_2(x) \bar{j}_2(x) \\ & + g_c(\psi_1^\dagger(x)\bar{\psi}_1^\dagger(x)\psi_2(x)\bar{\psi}_2(x) + \text{H.c.})], \end{aligned} \quad (5)$$

where the coefficients g represent the effective interactions at the fixed point, where the renormalization-group flow stops. A derivation of the bare coupling constants in terms of the microscopic parameters of the XXZ chain is provided in the Supplemental Material [45].

We now introduce the fields

$$\varphi(x) = \frac{1}{2}[\phi_1(x) + \phi_2(x)], \qquad \bar{\varphi}(x) = \frac{1}{2}[\bar{\phi}_1(x) + \bar{\phi}_2(x)], \quad (6)$$

and their conjugate momenta $\Pi(x) = -\frac{1}{2\pi}[\partial_x \phi_1(x) - \partial_x \phi_2(x)]$ and $\bar{\Pi}(x) = \frac{1}{2\pi}[\partial_x \bar{\phi}_1(x) - \partial_x \bar{\phi}_2(x)]$. Physically, $\Pi(x)$ and $\bar{\Pi}(x)$ are proportional to current operators from fermions in the vicinity of the right and left minimum, respectively [see Fig. 1(b)]. Similarly, $\partial_x \varphi(x)$ and $\partial_x \bar{\varphi}(x)$ are proportional to densities near these minima.

We are interested in the dilute limit of a small (but finite) density of electrons, for which $v_1 \approx v_2 = v$. When approaching the saturation QCP (zero density), the velocity v vanishes as $Q_1 - Q_2$. The momentum cutoff around the Fermi points also decreases proportional to the density. As the renormalized coupling constants continuously approach their value at the QCP, we argue that by approaching saturation, g_{12} and $g_{\bar{1}\bar{2}}$ continuously approach zero as they are irrelevant at the QCP for precisely the same reason as for the single-minimum case: the Pauli exclusion principle forbids interactions like $\psi_x^\dagger \psi_x^\dagger \psi_x \psi_x$ so the most relevant interactions must have two derivatives $\psi_x^\dagger \partial_x \psi_x^\dagger \psi_x \partial_x \psi_x$, making them irrelevant perturbations to the free-fermion fixed point (see Ref. [6]). Moreover, the spatial derivative that appears in the fermionic currents $i(\psi_x^\dagger \partial_x \psi_x - \partial_x \psi_x^\dagger \psi_x)$ makes the coefficient of $\Pi(x)\bar{\Pi}(x)$ irrelevant (the terms proportional to Π^2 and $\bar{\Pi}^2$ are, however, relevant as the fermionic anticommutation relations yield relevant terms of the type $\partial_x \psi^\dagger \partial_x \psi$ for the same species). In addition, inversion symmetry requires $g_{1\bar{2}} = g_{\bar{1}2}$.

The general form of the Hamiltonian in the dilute limit is then given by

$$H=\left(\frac{1}{2\pi}\right)^2\int dx\{2\pi v[(\partial_x\varphi)^2+(\partial_x\bar{\varphi})^2]+2\pi^3 v(\Pi^2+\bar{\Pi}^2)$$
$$+g\pi(\partial_x\varphi\bar{\Pi}-\partial_x\bar{\varphi}\Pi)+g'\partial_x\varphi\partial_x\bar{\varphi}$$
$$+2g_c\cos[2(\bar{\varphi}-\varphi)]\}, \qquad (7)$$

where $g\equiv g_{1\bar{1}}-g_{2\bar{2}}$, $g'\equiv g_{1\bar{1}}+2g_{1\bar{2}}+g_{2\bar{2}}$, and the explicit dependence of the fields on x is suppressed. Since we have used the limiting values of the coupling constants in the limit of vanishing density, it is important to bear in mind that our results are valid only over large length scales in comparison with the interparticle spacing (inverse of the cutoff for linearized dispersion).

If the term proportional to g_c becomes relevant, it can open a gap and destroy criticality. However, we have a quantum liquid if this term is irrelevant (to be checked *a posteriori*). If g' also flows to zero for a certain range of microscopic parameters, we can rewrite the Hamiltonian as

$$H=\frac{u}{2\pi}\int dx\sum_{\sigma=\pm}\left[\frac{1}{K}(\partial_x\varphi_\sigma)^2+K(\pi\Pi_\sigma)^2\right], \qquad (8)$$

where the new fields are related to the old ones through the following anyonic gauge transformation:

$$\varphi_+\equiv\varphi,\quad \Pi_+\equiv\Pi-\frac{\alpha}{\pi^2}\partial_x\bar{\varphi},\quad \varphi_-\equiv\bar{\varphi},\quad \Pi_-\equiv\bar{\Pi}+\frac{\alpha}{\pi^2}\partial_x\varphi$$

with $\alpha\equiv(g/4v)$, $K=1/\sqrt{1-(\alpha/\pi)^2}$, and $u=v/K$ [46]. Note that the momentum of one species is shifted by a gauge field times the density of the other species. This is equivalent to attaching a flux to each particle in such a way that the new "composite" particles obey anyonic commutation relations [19]: α represents the mutual statistical phase for exchanging the two types of particles. In other words, the anyonic nature of the new quasiparticles corresponds to a generalized J-W transformation (discussed below) and can be inferred from the commutation relations given below Eq. (4) [19]. Because the scaling dimension of $\cos[2(\bar{\varphi}-\varphi)]$ is $2K$ for the anyonic liquid, g_c indeed flows to zero.

In fact, the Hamiltonian (8) is a direct generalization of the Shastry-Schulz model of noninteracting anyons [19]. Just like in the Shastry-Schulz model, the two anyonic species are completely decoupled (there is a unique statistics of quasiparticles for which the theory breaks into two decoupled sectors). The Shastry-Schulz model, however, corresponds to the special case of $K=1$, indicating no intraspecies interactions. The α-dependent K in our model results in a continuous interpolation from free bosons ($\alpha\to\pi$, $H=(\pi v/2)\int dx\sum_\sigma\Pi_\sigma^2$) to free fermions ($\alpha=0$, $K=1$).

The key to realizing the anyonic liquid (8), however, is a vanishing renormalized g' at the fixed point. Although it is difficult to express g' in terms of microscopic parameters, an exact two-magnon calculation allows us to determine the microscopic parameters for which $g'=0$. We use the analogy with free fermions [a Luttinger liquid (LL) with Luttinger parameter $K=1$]. For such a noninteracting LL, the two-particle state $c^\dagger_{k_1}c^\dagger_{k_2}|0\rangle$ is an exact eigenstate of the Hamiltonian. As soon as K moves away from unity, this state scatters into other two-particle states and will not remain an eigenstate. Thus, if the effective Hamiltonian has the general Luttinger-liquid form and $c^\dagger_{k_1}c^\dagger_{k_2}|0\rangle$ is an exact eigenstate of the microscopic Hamiltonian, the Luttinger parameter must be equal to unity (free-fermion fixed point). Similarly, we require that a two-anyon state be an exact eigenstate of the Hamiltonian (1).

Going back to Eq. (1), we perform a generalized J-W transformation to anyons with statistical phase ϕ and annihilation operator a_x on site x: $S^-_x=a_x e^{-i\phi\sum_{y<x}n_y}$ and $S^z_x=n_x-\frac{1}{2}$ with $n_x=a^\dagger_x a_x$. The anyonic statistics of these particles can be observed in the relationship $a^\dagger_x a^\dagger_y=e^{-i\phi}a^\dagger_y a^\dagger_x$ for $x<y$ (see Ref. [43] for the physical interpretation of anyons in terms of spins). In the dilute limit, the possible momenta are $\pm Q$. We need to find a relationship between the microscopic parameters so that the two-particle state $a^\dagger_Q a^\dagger_{\bar{Q}}|0\rangle$, with $\bar{Q}\equiv -Q$, where a_Q is the Fourier transform of a_x defined above at momentum Q, is an exact eigenstate of Eq. (1). The Hamiltonian has the same form as Eqs. (2) and (3) in terms of anyonic operators (with c replaced by a), except for the correlated hopping term (the term in H_I proportional J_2), which now reads $(J_2/2)\sum_x n_{x+1}[(e^{i\phi}-1)a^\dagger_x a_{j+2}+(e^{-i\phi}-1)a^\dagger_{x+2}a_x]$. Requiring $Ha^\dagger_Q a^\dagger_{\bar{Q}}|0\rangle=\epsilon a^\dagger_Q a^\dagger_{\bar{Q}}|0\rangle$ leads to

$$\Delta_1=\cos(Q)+\frac{\sin(Q)}{2}[\tan(Q)+\tan(Q+\phi/2)], \qquad (9)$$

$$\Delta_2=\cos(2Q)+\sin(2Q)\tan(2Q+\phi/2) \qquad (10)$$

with the energy given by $\epsilon=-2(\Delta_1 J_1+\Delta_2 J_2)+2J_1\cos(Q)+2J_2\cos(2Q)$. Note that eliminating ϕ between Eqs. (9) and (10) gives a relationship between the microscopic parameters Δ_1 and Δ_2 for a given J_1/J_2 ($\cos Q=-J_1/4J_2$). This relationship is achieved by tuning only one microscopic parameter and it allows the system to realize an anyonic liquid with an emergent statistical angle ϕ determined by the above equations. Because there is only one anyon of each species in $a^\dagger_Q a^\dagger_{\bar{Q}}|0\rangle$, the intraspecies interactions characterized by the parameter K play no role in the above argument.

If the effective theory of the system is given by Eq. (7), the above values of Δ_1 and Δ_2 guarantee the absence of scattering between the two anyonic species. The effective Hamiltonian must then reduce to Eq. (8) with $\alpha=\pi-\phi$. In other words, we have a family of Hamiltonians characterized by two parameters Q and ϕ, which can potentially

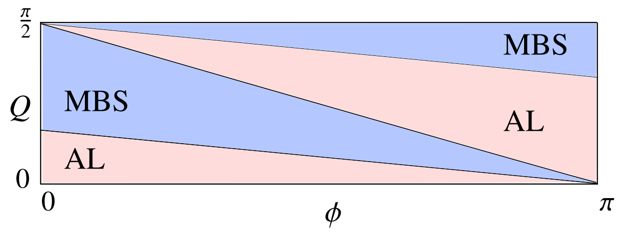

FIG. 2 (color online). The phase diagram of the Hamiltonian (1) with $(J_1/4J_2) = -\cos(Q)$ and other coupling constants given by Eqs. (9) and (10). The phases are, respectively, denoted by AL (anyonic liquid) and MBS (magnon bound state).

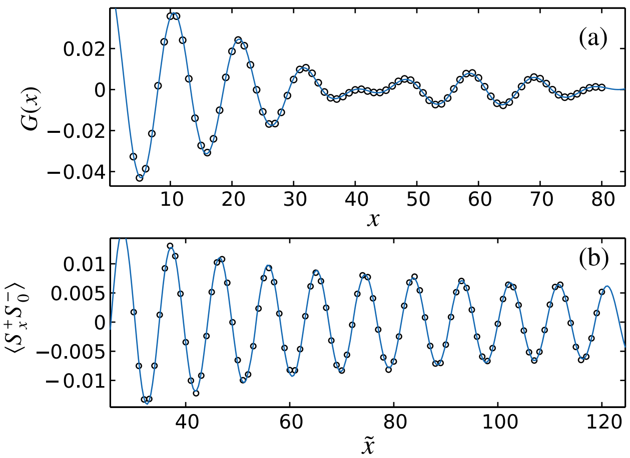

FIG. 3 (color online). (a) The fermionic Green's function for $\phi/\pi = 0.615$ and $Q/\pi = 0.2$ at density $\rho_0 = 0.05$. The black circles (blue line) represent(s) the numerical results (fit). Fitting to Eq. (11) gives $Q_1/\pi = 0.16$, $Q_2/\pi = 0.21$, and an exponent 0.108 in excellent agreement with analytical predictions $Q_1/\pi = 0.17$, $Q_2/\pi = 0.22$, and an exponent 0.108. (b) The spin-spin correlation function for the same parameter. Fitting to Eq. (12) gives an exponent 0.70 in good agreement with the analytical prediction 0.67.

flow to the anyonic-liquid fixed point (8). However, the formation of low-energy bound states may lead to either a first-order phase transition from the saturated state (the number of particles changes discontinuously at the saturation field) or a continuous transition into a state with dominant nematic (BEC of pairs) or higher-order multipolar fluctuations. As discussed in the Supplemental Material [45], by using exact two-magnon calculations [47], we found the range of parameters that give rise to low-energy bound states, destabilizing the anyonic liquid, and obtained the phase diagram of Fig. 2.

Returning to the anyonic liquid, we now present analytical predictions for different correlation functions, which are numerically verified with the DMRG method. For the fermionic Green's function $G(x) = \langle c_y^\dagger c_{x+y} \rangle$, we find

$$G(x) \propto [\sin(Q_1 x + \omega_1) - \sin(Q_2 x + \omega_2)] x^{-1/\sqrt{1-\lambda^2}}, \tag{11}$$

for $x\rho_0 \gg 1$, with $\lambda = \alpha/\pi$ in the dilute limit. In general, the ordering vectors change at finite densities (because of the string operator that relates fermions to anyons), but the change is negligible in the limit of small density considered here [19]. Moreover, the ordering vectors Q_i have an uncertainty of order $1/L$ in a finite system of length L. We therefore compare the above prediction with the numerical results by fitting the numerically computed correlation function to expression (11), with the ordering vectors, the overall coefficient, and the exponent as fitting parameters (using the fact that the exponents are relatively close to 1 we neglect the phase shifts in the oscillatory prefactor [45] in fitting the data). An exponent close to $-(1/\sqrt{1-\lambda^2})$ and ordering vectors close to the computed (for the given density of fermions) Q_1 and Q_2 would corroborate our analytical prediction for an anyonic liquid.

We performed the DMRG calculations for a chain of length $L = 400$ with periodic boundary conditions (implemented by constructing two parallel chains of length $L/2$ and connecting the endpoints [48]). We compared the results with a calculation for $L = 200$ and chose the range of x where the two data sets overlap. Excellent convergence was obtained by keeping 1000 states in the DMRG iterations. As seen in Fig. 3(a), the exponent of the correlation function differs from $\delta = 1$ (free fermion fixed point) and it is consistent with the exponents of an anyonic liquid. The ordering momenta are also very close to our analytical predictions (the agreement cannot be perfect because of the finite value of the density $\rho_0 = 0.05$).

The statistical angle also changes the asymptotic behavior of the two-point spin-spin correlators. This angle can then be obtained by measuring the k dependence of the transverse magnetic susceptibility $\chi_{xx} = \chi_{yy}$, which is determined by the Fourier transform of the correlator $\langle S_x^+ S_0^- \rangle$. At low densities, we can neglect the average density ρ_0 in $\sum_{y<x} n_y = \int_{-\infty}^x dy[\rho_0 + \sum_a (j_a(y) + \bar{j}_a(y))]$ and write $S_x^- \sim c_x e^{-i[\varphi(x)+\bar{\varphi}(x)]}$. By using Eqs. (4) and (8), we find that the four terms in $\langle S_x^+ S_0^- \rangle$ fall into two categories, respectively decaying to leading order as $x^{-(1/2)[\sqrt{1-\lambda^2}+(1\pm\lambda)^2/\sqrt{1-\lambda^2}]}$ (where $\lambda = (\alpha/\pi)$), with the leading dilute-limit behavior given by

$$\langle S_x^+ S_0^- \rangle \propto \sin(Q_1 x + \omega) x^{-(1/2)[\sqrt{1-\lambda^2}+(1-\lambda)^2/\sqrt{1-\lambda^2}]}, \tag{12}$$

for $x\rho_0 \gg 1$ (ω is a phase shift). We also checked the above expression with DMRG computations. The bosonic correlators have a stronger finite-size dependence so in fitting the data we replaced x in $x^{-(1/2)[\sqrt{1-\lambda^2}+(1-\lambda)^2/\sqrt{1-\lambda^2}]}$ with its finite-size counterpart $\tilde{x} = (L/\pi)\sin(\pi(x/L))$. The agreement is excellent as shown in Fig. 3(b). The anyonic fixed point can be detected by comparing the above exponent with the exponent of the correlator that determines the longitudinal susceptibility χ_{zz}: the oscillatory

$[k = \pm(Q_2 - Q_1)]$ components of $\langle S_x^z S_0^z \rangle$ decay as $x^{-(1/2K)} = x^{-(1/2)\sqrt{1-\lambda^2}}$ for $x\rho_0 \gg 1$ [49]. Finally, we note that disorder is a relevant perturbation for magnetic saturation QCP's [50]. However, the exponents that we are predicting for the two-spin correlators can still be measured if the characteristic length scale associated with the disorder is much longer than the average interparticle distance $1/\rho_0$.

In summary, by studying the effects of strong magnetic frustration in nearly saturated spin chains, we extended the classification of the saturation QCPs from the standard paradigm of simple free fermionic (bosonic) theories in $d = 1$ $(d > 1)$ [6] to an exotic continuous line of anyonic liquids,. These liquids are characterized by two species of anyonic quasiparticles with vanishing interspecies interactions. The emergent statistical phase of the quasiparicles interpolates continuously between bosons and fermions. While envisioned in the field-theory literature, anyonic liquids had thus far remained as an abstract theoretical construction. Our results provide natural realizations of one-dimensional anyonic liquids in a simple and experimentally relevant model, opening a promising direction in the search for anyons in frustrated magnets. As only one exchange parameter needs to be tuned in order to realize our anyonic liquids (apart from the magnetic field, which can be easily brought to the vicinity of the critical point), physical or chemical pressure could drive generic highly frustrated one-dimensional magnetic materials into the anyonic-liquid phase. Relationships between the transverse and longitudinal magnetic susceptibilities serve as experimental signatures of this exotic phase. The fate of higher-dimensional systems realized by coupling these anyonic wires [51–54] poses an interesting challenge for future investigations. For certain anyonic phases [55], novel two-dimensional topological phases might emerge (see Refs. [56,57] for such constructions).

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