Density-Curvature Response and Gravitational Anomaly

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We study constraints imposed by the Galilean invariance on linear electromagnetic and elastic responses of two-dimensional gapped systems in a background magnetic field. Exact relations between response functions following from the Ward identities are derived. In addition to the viscosity-conductivity relations known in the literature, we find new relations between the density-curvature response and the thermal Hall response.

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Introduction.—Strongly interacting two-dimensional electron gas in a magnetic field is notorious for defying analytical approaches. Recently some progress was achieved in understanding these systems with the use of local Galilean symmetry [1–3]. This symmetry is present in the simplest models of noninteracting electrons. It is also possible to add nontrivial interactions to this model that preserve the local Galilean invariance (LGI) [1]. Thus, locally Galilean invariant systems may serve, at the very least, as toy models for the fractional quantum Hall states.

In this Letter we find the constraints on the linear response functions imposed by LGI. In addition to the known electromagnetic responses, we include responses to an external gravitational field. The latter can be used to compute various viscoelastic responses. For systems in a background magnetic field some of these constraints were found in Ref. [1]. Those constraints relate wave vector dependent Hall conductivity with Hall viscosity. Later, more general relations of this type were obtained [4]. The relations between various linear response functions derived in this work include the generalization of a viscosity-conductivity relation to the arbitrary gyromagnetic ratio g_s, Kohn's theorem for electric susceptibility and its gravitational analogue. These Ward identities impose an infinite number of constraints on the coefficients in wave vector and frequency expansions of response functions.

It is well known that the Hamiltonian of a charged particle with the gyromagnetic ratio $g_s = 2$ is factorizable and has a macroscopic degeneracy of the first Landau level even in the presence of an inhomogeneous magnetic field and spatial curvature [5,6]. In Refs. [2,6] it was argued that at this special value of g_s the response functions are regular in the limit of the cyclotron frequency going to infinity. We use this argument together with LGI to relate the chiral central charge to a bulk density-curvature response. This relation allows us to predict the value of this bulk response for states described by a K matrix with $\nu \le 1$. A similar relation for the Laughlin's functions was recently found in Ref. [7].

Galilean invariance.—In Galilean invariant systems with one species of particles or with multiple species with equal e/m for each particle, there is a relation between the mass current T^{0i} and the electric current J^i (see, e.g., Ref. [8]):

$$J^i = \frac{e}{m} T^{0i}. \tag{1}$$

Here we consider the local version of the Galilean symmetry following Ref. [9]. The expectation values of the electric current J^i and the stress tensor T^{ij} in the background e/m and gravitational fields can be computed as

$$J^\mu = \frac{1}{\sqrt{g}} \frac{\delta S_{\mathrm{eff}}}{\delta A_\mu}, \qquad T^{ij} = \frac{2}{\sqrt{g}} \frac{\delta S_{\mathrm{eff}}}{\delta g_{ij}}. \tag{2}$$

Here S_{eff} is the effective action that encodes the response of the underlying microscopic theory to external perturbations and J^0 is the charge density (denoted as ρ later on). The local symmetry of the action that insures Eq. (1) (for $g_s = 0$) is [9]

$$\begin{aligned} \delta A_i &= -\xi^k F_{ki} - m g_{ik} \dot{\xi}^k - \partial_i(\alpha + A^k \xi_k), \\ \delta A_0 &= -\xi^k F_{k0} - \partial_0(\alpha + A^k \xi_k) + \frac{g_s}{4} \frac{\epsilon^{ij}}{\sqrt{g}} \partial_i(g_{jk} \dot{\xi}^k), \\ \delta g_{mn} &= -\xi^k \partial_k g_{mn} - g_{mk} \partial_n \xi^k - g_{nk} \partial_m \xi^k, \end{aligned} \tag{3}$$

where $F_{ik} = \partial_i A_k - \partial_k A_i$ is the field strength tensor. The last term in Eq. (3) accounts for the effective magnetic moment of electrons equal to $g_s/4$ [10].

These transformations combine a local version of Galilean transformations parametrized by $\xi^k(x,t)$ and gauge transformations $\alpha(x,t)$. In the following we use Galilean transformations accompanied by a particular gauge transformation $\alpha = -A_k \xi^k$, so that Eqs. (3) have an explicitly gauge invariant form. Conventional (global) Galilean transformations corresponding to a constant velocity v^k are given by $\xi^k(x,t) = v^k t$.

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Background field separation.—In the following we assume that the background e/m and the gravitational fields are small and smooth deviations from the constant background magnetic field $\bar{F}_{12} = B_0$ and the flat metric $\bar{g}_{ik} = \delta_{ik}$. The vector potential can be written as $\bar{A}_i + A_i$, where the first term corresponds to the constant magnetic field. The second term generates time-dependent, inhomogeneous electromagnetic fields.

The constant part of the external magnetic field B_0 is a parameter of the macroscopic theory and will enter the coefficients in the gradient expansion of the effective action. We do not transform it under Galilean transformations but instead absorb the corresponding part into the transformation laws of the vector potential A_i [compare to Eq. (3)] as

$$\delta A_i = -\xi^k \bar{F}_{ki} - \xi^k F_{ki} - m g_{ik} \dot{\xi}^k. \tag{4}$$

The external metric is a small perturbation over the flat background $g_{ik} = \delta_{ik} + \delta g_{ik}$.

Building blocks for quadratic effective action.—To restrict the form of the effective action we use the rotational invariance, locality, gauge invariance, and similarities between electromagnetism and gravity.

The gauge invariance requires that the effective action depends on the vector potential A_μ only through electric field E_i and magnetic field B. The only exception is the Chern-Simons term, which is gauge invariant only up to boundary terms. We also assume that the system under consideration is gapped. Therefore, linear response functions are local, i.e., can be written as Taylor series in frequency and momentum, so that the quadratic effective action is constructed as an expansion in derivatives. As transformations (3) mix *different* orders in the gradient expansion, we expect nontrivial relations between the universal response coefficients and higher order gradient corrections thereof.

We analyze the gravitational terms in a similar way by introducing an Abelian gauge field that encodes coupling to the background curvature. This field is a nonrelativistic spin connection [1] $\omega_0 = -\frac{1}{2}\epsilon^{ab} e^{aj} \dot{e}^b_j$, $\omega_i = -\frac{1}{2}\epsilon^{ab} e^{aj} \partial_i e^b_j - (1/2\sqrt{g})\epsilon^{jk}\partial_j g_{ik}$, where e^a_j are the time-dependent zweibeins [11]. The spin connection depends only on the metric and transforms as an Abelian gauge field under local $SO(2)$ spatial rotations $\omega_\mu \to \omega_\mu + \partial_\mu \alpha$.

With the spin connection at hand, we construct the gravielectric $\mathcal{E}_i = \dot{\omega}_i - \partial_i \omega_0$ and gravimagnetic $\frac{1}{2}\sqrt{g}R = \partial_1\omega_2 - \partial_2\omega_1$ fields which are explicitly invariant under the local $SO(2)$ rotations. Notice that the parity properties of e/m fields and their elastic cousins are different: R is a scalar, while B is a pseudoscalar and $\mathcal{E}_i$ is an axial vector.

In the linear order in deviations from the flat background, we explicitly have

$$R \approx \partial_i\partial_j g_{ij} - \Delta g^i_i, \qquad \mathcal{E}_i \approx -\frac{1}{2}\epsilon^{jk}\partial_j \dot{g}_{ik}, \tag{5}$$

where Δ is the flat space Laplace operator.

The spin connection ω can be expressed in terms of perturbations of the metric as follows:

$$\omega_0 = \frac{1}{2}\epsilon^{jk}\delta g_{ij}\dot{g}_{ik} \qquad \omega_i = -\frac{1}{2}\epsilon^{jk}\partial_j \delta g_{ik}. \tag{6}$$

There is an additional building block describing dilatations—the trace of the metric which we denote as

$$G \equiv \delta g^i_i. \tag{7}$$

Effective action.—In the following, we present the quadratic effective action as a sum

$$S_{\text{eff}} = S^{(1)} + S^{(\eta)} + S^{(\text{geom})} + S^{(\text{em})} + S^{(g)} + S^{(\text{mix})}. \tag{8}$$

The first contribution collects all "linear" terms,

$$S^{(1)} = \int d^2x dt \sqrt{g}(-\epsilon_0 + \rho_0 A_0). \tag{9}$$

Notice that although Eq. (9) is linear in A_0, it also contains (through $\sqrt{g}$) terms quadratic in deviation from the constant background. This term encodes the properties of the unperturbed ground state: energy density ϵ_0 and density $\rho_0 = \nu/2\pi l^2$, where $l^2 = 1/B_0$ is the magnetic length and ν is the filling fraction.

The coefficients in Eq. (9) and below generally depend on the external magnetic field B_0, the filling fraction ν, and other microscopic parameters of the system such as the Coulomb gap, the cyclotron mass, etc.

The next term has a form

$$S^{(\eta)} = \int d^2x dt \eta_H \epsilon^{jk} g_{ij} \dot{g}_{ik}, \tag{10}$$

where η_H (in Fourier space) is a function of frequency. One can think about $\eta_H(\omega)$ as of frequency dependent Hall viscosity. We notice comparing to Eq. (6) that the term (10) at zero frequency has a form $2\eta_H(0)\omega_0$ which allows us to identify $2\eta_H(0)$ as the orbital spin density and $\bar{s} = 2\eta_H(0)/\rho_0$ as an average orbital spin per particle. For the conformal block states [12], the latter is given by $2\bar{s} = \nu^{-1} + 2h_\psi$, where h_ψ is the conformal weight of the electron operator in the "neutral" sector of the conformal field theory [13,14].

The next contribution contains topological and *geometric* terms [15]

$$S^{(\text{geom})} = \int \left(\frac{\sigma_H}{2} A dA + S A d\omega + C \omega d\omega\right), \tag{11}$$

known as the Chern-Simons, Wen-Zee [16], and gravitational Chern-Simons terms. These terms are special, as they are invariant with respect to gauge transformations and local rotations only up to full derivatives. In the presence of the boundary, they are related to the boundary theory and are the natural candidates for encoding universal properties. It is convenient to allow η_H, σ_H, S, and C in Eq. (11) to depend on frequency so that they coincide with their conventional values at zero frequency. In the following expressions [(12)–(14)], the coefficients $\epsilon, \sigma, \mu, \ldots$ depend on both frequency and momentum [17].

The electromagnetic response is represented by

$$S^{(\mathrm{em})} = \int d^2x dt\big(\epsilon E^2 + \sigma(\partial_i E_i)B - \mu^{-1}B^2\big). \tag{12}$$

Here ϵ and μ are electromagnetic susceptibilities and σ encodes the gradient corrections to the Hall conductivity.

Analogously, we write down the gravitational and mixed terms

$$S^{(g)} = \int d^2x dt\Big(\epsilon_g \mathcal{E}^2 + \sigma_g(\partial_i \mathcal{E}_i)R - \frac{1}{\mu_g}R^2 + \zeta_3 GR + \zeta_4 G(\partial_i \mathcal{E}_i) + \zeta_5 G^2\Big), \tag{13}$$

$$S^{(\mathrm{mix})} = \int d^2x dt\Big(\epsilon_m(E_i\mathcal{E}_i) + \sigma_{m1}(\partial_i E_i)R - \frac{1}{\mu_m}BR + \sigma_{m2}(\partial_i\mathcal{E}_i)B + \zeta_1 G(\partial_i E_i) + \zeta_2 GB\Big). \tag{14}$$

Equations (8)–(14) give the effective action expanded to the second order in fields and to an arbitrary order in gradients.

Ward identities.—Equations (8)–(14) contain all possible combinations that can enter real, rotationally, gauge, and parity-time invariant quadratic effective action of a gapped system in a transverse constant magnetic field. They define 19 different response coefficients $\eta_H, \sigma_H, S, C, \epsilon, \ldots$. The coefficients in S_{eff} encode *all* possible two point correlation functions of electric charge density, electric current, and stress tensor at finite frequency ω and momentum k. Imposing the LGI (3) will give additional relations between the coefficients.

The next step is to derive the Ward identities of LGI. We apply the transformations (3) to S_{eff} and demand the invariance of the full effective action under these transformations up to the terms quadratic in fields. This requirement imposes constraints on the linear response functions in all orders of the gradient expansion in a form of a system of linear (in response functions) equations. In full generality these relations are not enlightening and we present only several particular relations.

Hall conductivity and orbital spin.—We start with the following relations:

$$\sigma_H = \frac{\nu}{2\pi}\frac{\omega_c^2}{\omega_c^2 - \omega^2}, \qquad S = 2\eta_H l^2 \frac{\omega_c^2}{\omega_c^2 - \omega^2}, \tag{15}$$

where $\omega_c = B_0/m$ is the cyclotron frequency. These are the familiar relations for the Hall conductivity and the Wen-Zee shift [4]. Integrating the charge density ρ from Eq. (2) over the curved manifold and using Eq. (15), we obtain that the shift in the total charge on the curved manifold of the Euler character χ is given by

$$Q = \nu N_\phi + \nu\bar{s}\chi. \tag{16}$$

Zero momentum.—Here we present the Ward identities at zero momentum, $k = 0$. In order to lighten up the notations we suppress the dependence on frequency. We stress that all response functions below are evaluated at finite frequency ω and $k = 0$.

We start with relations

$$\epsilon(\omega) = \frac{\nu}{4\pi}\frac{\omega_c}{\omega_c^2 - \omega^2}, \qquad \epsilon_m(\omega) = \eta_H l^2 \frac{\omega_c}{\omega_c^2 - \omega^2}. \tag{17}$$

The first relation determines the homogeneous dielectric response function $\epsilon(\omega, k = 0)$ completely and the pole at ω_c reflects Kohn's theorem. The second relation is an elastic analogue of Kohn's theorem.

The next relation is the finite frequency version of the Hall viscosity-conductivity relation [1]:

$$\frac{\sigma}{l^2} = \frac{\omega_c^2(\omega_c^2 + \omega^2)}{(\omega_c^2 - \omega^2)^2}\left(\eta_H l^2 - \frac{\nu g_s}{16\pi}\right) - \frac{\omega_c^2}{\omega_c^2 - \omega^2}\frac{\mu^{-1}}{\omega_c l^2}. \tag{18}$$

Here we slightly generalized the relation obtained in Ref. [4] by including an arbitrary g_s factor.

We also find two elastic analogues of Eq. (18):

$$\frac{\mu_m^{-1}}{\omega_c l^2} = \frac{C}{2} - \frac{g_s}{4}\eta_H l^2 \frac{\omega_c^2 + \omega^2}{\omega_c^2 - \omega^2} - \frac{\sigma_{m1}}{l^2} + \frac{\epsilon_m^{(1)}\omega^2}{\omega_c}, \tag{19}$$

$$\frac{\sigma_{m2}}{l^2} = \frac{g_s}{2}\eta_H l^2 \frac{\omega_c^2}{\omega_c^2 - \omega^2} + (2\epsilon_m^{(1)} - \epsilon_g)\omega_c, \tag{20}$$

where we introduced $(kl)^2\epsilon_m^{(1)} = \epsilon_m(k, \omega) - \epsilon_m(0, \omega)$.

The coefficients $\zeta_1, \ldots, \zeta_5$ are completely fixed by the Galilean invariance in terms of other coefficients. Their expansions start with ω^2 and we do not list them here.

Regularity of the limit $\omega_c \to \infty$.—Let us consider the static limit $\omega = 0$ of Eq. (19),

$$m\mu_m^{-1}(0) = \frac{C}{2} - \frac{\nu\bar{s}g_s}{16\pi} - \frac{1}{l^2}\sigma_{m1}(0). \tag{21}$$

The coefficient $\sigma_{m1}(0)$ describes the contribution to the expectation value of the density proportional to the Laplacian of curvature ΔR. We introduce $b = -8\pi\sigma_{m1}(0)/l^2$, defined as a coefficient in the gradient expansion for the static density-curvature response [18]

$$\delta\rho = \frac{\nu\bar{s}}{4\pi}R + \frac{b}{8\pi}l^2\Delta R + \cdots. \tag{22}$$

For $g_s = 2$ the ground state of noninteracting electrons is degenerate even in the presence of inhomogenious background fields and it is expected that the limit $m \to 0$ (i.e., $\omega_c \to \infty$) is regular for $\nu \le 1$ [2,6]. Therefore, $\mu_m^{-1}(0)$ is finite in the limit $m \to 0$ at $g_s = 2$.

We take the limit $m \to 0$ of Eq. (21) at $g_s = 2$. The left-hand side vanishes and we find a relation between the coefficients of the Wen-Zee and gravitational Chern-Simons (gCS) terms (11) and the coefficient b:

$$C = \frac{S}{2} - \frac{b}{4\pi}. \tag{23}$$

This relation is obtained for $g_s = 2$. However, b is a response of the density to curvature and cannot depend on g_s; neither can the coefficients C or S. Therefore, the relation (23) is valid for general g_s.

Chiral central charge.—We split the geometric part of the effective action (11) as

$$S_{\mathrm{eff}}^{(\mathrm{geom})} = \int \frac{\nu}{4\pi}(A + \bar{s}\omega)d(A + \bar{s}\omega) - \frac{c}{48\pi}\omega d\omega. \tag{24}$$

Here we used Eq. (15) at zero frequency. The first contribution in Eq. (24) reflects the Wen-Zee arguments [16] (see also Ref. [2]) stating that every electron carries not only charge, but also an intrinsic orbital spin $\bar{s}$ that couples to the curvature. Thus, in any transport process the electric current will be accompanied by the "spin current." Formally, this amounts to changing the vector potential as $A_i \to A_i + \bar{s}\omega_i$. We have noted in Ref. [19], however, that even in the noninteracting case with $\nu = 1$, there is an additional contribution to the gCS term represented by the second term in Eq. (24). Comparing Eq. (11) with Eq. (24), we identify $C = (\nu/4\pi)\bar{s}^2 - (c/48\pi)$ and rewrite Eq. (23) as

$$b = \nu\bar{s}(1 - \bar{s}) + c/12. \tag{25}$$

This equation relates the coefficients of geometric terms with the static bulk density-curvature response. A relation of this kind appeared recently in Ref. [7].

We refer to c as to the chiral central charge. In relativistic physics c is related to the gravitational anomaly at the boundary [20].

Let us consider the relation (25) for a few cases where b has been computed independently. The first such case is noninteracting fermions filling the lowest Landau level $\nu = 1$. It was found in Ref. [19] that in this case $\nu = 1$, $\bar{s} = 1/2$, and $b = 8\pi\sigma_{m1}(0)/l^2 = 1/3$. Then Eq. (25) gives $c = 1$ corresponding to $C = 1/24\pi$ and is in agreement with the straightforward calculation of Ref. [19]. The coefficient b was also computed in Refs. [21,22] from the Bergman kernel expansion.

For the Laughlin states $\nu\bar{s} = 1/2$ and $b = \frac{1}{3} + ((\nu - 1)/4\nu)$ [7]. Using Eq. (25) and assuming that the results of Ref. [7] are compatible with Galilean invariance we predict,

$$C = \frac{1}{8\pi} - \frac{1}{4\pi}b = \frac{1}{24\pi} + \frac{1}{2\pi}\frac{\nu^{-1} - 1}{8}, \tag{26}$$

again corresponding to $c = 1$.

In both cases the boundary theory is the chiral boson $c = 1$ and the results given by Eq. (25) are in agreement with our expectations for the (chiral) central charge. Therefore, we conjecture that c in Eq. (24) coincides with the central charge of boundary theory for all other states of the fractional quantum Hall effect (FQHE) hierarchy.

Note that the relation (25) was derived using regularity conditions at $g_s = 2$ specific for $\nu \le 1$ and is not supposed to hold for $\nu > 1$. However, for a noninteracting case with $\nu = N$ we found using the results of Ref. [19] that Eq. (25) can still be written as a sum over filled Landau levels

$$b = \sum_{n=1}^{N} \left(\nu_n \bar{s}_n(1 - \bar{s}_n) + \frac{c_n}{12}\right). \tag{27}$$

Here $\bar{s}_n = (2n - 1)/2$, $\nu_n = 1$, and $c_n = 1$ for the nth Landau level.

The significance of Eqs. (24) and (25) is that in the nonrelativistic case, the averaging over the microscopic degrees of freedom produces two gCS terms. One originates from the coupling of the orbital spin to the curvature and the other one is related to the gravitational anomaly of the boundary.

Abelian quantum Hall states.—For general Abelian states we rewrite the geometric action (24) as

$$S_{\mathrm{eff}}^{(\mathrm{geom})} = \frac{1}{4\pi}\int (t_i A + \bar{s}_i\omega)K_{ij}^{-1}d(t_j A + \bar{s}_j\omega) - \frac{c}{12}\omega d\omega, \tag{28}$$

where the K matrix, the charge vector t_i, and the spin vector $\bar{s}_i$ characterize the state [23]. Then Eq. (23) takes the form (in matrix notations)

$$\frac{c}{12} = (\bar{s} - t)^t K^{-1}\bar{s} + b, \tag{29}$$

generalizing Eq. (25) to more general Abelian quantum Hall states. Here the parameter c counts the number of chiral propagating modes and is equal to $c = n_+ - n_-$, where $n_\pm$ is the number of positive and negative eigenvalues of the K matrix, respectively.

We conclude this section with a few examples of applications of Eq. (29) to some well-known FQHE states. For the Laughlin's state $\nu = 1/m$, $K = (m)$, $t = 1$, $\bar{s} = m/2$, and $c = 1$ and we obtain $b = \frac{1}{3} - ((m - 1)/4)$. For the corresponding particle-hole conjugated state [23] $\nu = 1 - 1/m$, $t = (1, 0)$, $\bar{s} = (\frac{1}{2}, (1 - m)/2)$, $c = 0$, and

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 - m \end{pmatrix}.$$

The relation (29) gives $b = (m - 1)/4$.

As an example of a non-Abelian state we consider the fermionic Pfaffian state [12] with $\nu = 1/2$, $t = (-1, -2)$, $\bar{s} = (-3/2, -3)$, $c = 3/2$, and

$$K = \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix}$$

[23,24]. We obtain $b = -1/4$.

Thermal Hall effect.—It has been demonstrated that the thermal Hall current (the Leduc-Righi effect) is related to the chiral central charge of edge modes via the relation [25–28]

$$K_H = \frac{\partial J_H}{\partial T} = \frac{\pi k_B^2 T}{6} c. \tag{30}$$

We use Eq. (29) in order to express the thermal Hall conductivity through other response functions:

$$\frac{K_H}{2\pi k_B^2 T} = (\bar{s} - t)^t K^{-1} \bar{s} + b. \tag{31}$$

An important remark is in order. Equation (31) allows us to obtain the thermal Hall response in terms of the bulk quantities. Of course, "measuring" b involves gradients of curvature or "tidal forces" (cf. Ref. [30]).

Conclusion.—We have explored the constraints imposed by the local Galilean invariance on linear electromagnetic and gravitational responses of gapped systems in the background of a quantizing magnetic field. Several new relations between linear response functions have been found [see, e.g., Eqs. (19) and (20)]. Using the regularity of the limit of large cyclotron frequency $\omega_c \to \infty$ in addition to the Galilean invariance, we have found a relation [Eqs. (25) and (29)] between the bulk density-curvature response coefficient b [Eq. (22)] and the chiral central charge. The relation has been tested for the cases of noninteracting electrons and for Laughlin's states using the results of Refs. [7,19]. As an application we have used the relation to predict the values of the density-curvature response b for several other quantum Hall states. It would be interesting to understand whether expression (29) is general and can be derived without the use of the Galilean invariance.

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