Stronger Uncertainty Relations for All Incompatible Observables

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The Heisenberg-Robertson uncertainty relation expresses a limitation in the possible preparations of the system by giving a lower bound to the product of the variances of two observables in terms of their commutator. Notably, it does not capture the concept of incompatible observables because it can be trivial; i.e., the lower bound can be null even for two noncompatible observables. Here we give two stronger uncertainty relations, relating to the sum of variances, whose lower bound is guaranteed to be nontrivial whenever the two observables are incompatible on the state of the system.

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In his seminal paper [1,2], Heisenberg analyzes various notions of uncertainties for measurement of noncommuting observables in quantum theory. Here we deal with Robertson's formalization [3] that implies a restriction on the possible preparations of the properties of a system. Indeed, the Heisenberg-Robertson uncertainty relation quantitatively expresses the impossibility of jointly sharp preparation of incompatible observables. However, in practice, the conventional uncertainty relations cannot achieve this, because the lower bound in the uncertainty relation inequalities can be null and hence trivial even for observables that are incompatible on the state of the system (namely, the state is not a common eigenstate of both observables). This is due to the fact that the uncertainty relations are expressed in terms of the product $\Delta A^2 \Delta B^2$ of the variances of the measurement results of the observables A and B, and the product can be null even when one of the two variances is different from zero. Here we provide a different uncertainty relation, based on the sum $\Delta A^2 + \Delta B^2$, that is guaranteed to be nontrivial whenever the observables are incompatible on the state.

Uncertainty relations are useful for a wide range of applications that span from the foundations of physics all the way to technological applications: they are useful for formulating quantum mechanics [4] (e.g., to justify the complex structure of the Hilbert space [5] or as a fundamental building block for quantum mechanics and quantum gravity [6]), for entanglement detection [7,8], for the security analysis of quantum key distribution in quantum cryptography (e.g., see [9]), etc. Previous uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables [10], a lower bound based on the entropic uncertainty relations [11], and a sum uncertainty relation for angular momentum observables [12]. In contrast to the last, our bound applies to general observables, and in contrast to the previous ones, it is built to be strictly positive if the observables are incompatible on the state of the system.

Stronger uncertainty relations.—The Heisenberg-Robertson uncertainty relation [3] bounds the product of the variances through the expectation value of the commutator

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle [A,B]\rangle\right|^2, \tag{1}$$

where the expectation value and the variances are calculated on the state of the quantum system $|\psi\rangle$. It was strengthened by Schrödinger [13] who pointed out that one can add an anticommutator term, obtaining

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle [A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle \{A,B\}_+\rangle - \langle A\rangle\langle B\rangle\right|^2. \tag{2}$$

Both these inequalities can be trivial even in the case in which A and B are incompatible on the state of the system $|\psi\rangle$, e.g., if $|\psi\rangle$ is an eigenstate of A, all terms in (1) and (2) vanish. Both relations can be derived through an application of the Cauchy-Schwarz inequality.

A simple lower bound for the sum of the variances can be obtained from these, by noticing that $(\Delta A - \Delta B)^2 \geq 0$, so that, using (1), we find $\Delta A^2 + \Delta B^2 \geq 2\Delta A \Delta B \geq |\langle [A,B]\rangle|$. This inequality is still not useful, as the lower bound can be null even if A and B are incompatible on $|\psi\rangle$ so that the sum is trivially bounded as $\Delta A^2 + \Delta B^2 > 0$. Instead, the following two inequalities (which are the main result of this Letter) have lower bounds which are nontrivial. The first inequality is

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle [A,B]\rangle + |\langle \psi | A \pm iB | \psi^\perp\rangle|^2, \tag{3}$$

which is valid for arbitrary states $|\psi^\perp\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A,B]\rangle$ (a real quantity) is positive. The lower bound in (3) is nonzero for almost any choice of $|\psi^\perp\rangle$ if $|\psi\rangle$ is not a common eigenstate of A and B (Fig. 1): just choose $|\psi^\perp\rangle$ that is orthogonal to $|\psi\rangle$ but not orthogonal to the state $(A\pm iB)|\psi\rangle$. Such a choice is always possible unless $|\psi\rangle$ is a joint eigenstate of A and B.

For illustration, we give an example of how one can choose $|\psi^\perp\rangle$: if $|\psi\rangle$ is an eigenstate of A one can choose $|\psi^\perp\rangle=(B-\langle B\rangle)|\psi\rangle/\Delta B\equiv|\psi_B^\perp\rangle$ (see below), or $|\psi^\perp\rangle=(A-\langle A\rangle)|\psi\rangle/\Delta A\equiv|\psi_A^\perp\rangle$, if $|\psi\rangle$ is an eigenstate of B. If $|\psi\rangle$ is not an eigenstate of either and $|\psi_A^\perp\rangle\neq|\psi_B^\perp\rangle$, one can choose $|\psi^\perp\rangle\propto(\mathbb{1}-|\psi_B^\perp\rangle\langle\psi_B^\perp|)|\psi_A^\perp\rangle$, or $|\psi^\perp\rangle=|\psi_A^\perp\rangle$ if $|\psi_A^\perp\rangle=|\psi_B^\perp\rangle$. An optimization of $|\psi^\perp\rangle$ (namely, the choice that maximizes the lower bound), will saturate the inequality (3): it becomes an equality.

A second inequality with nontrivial bound even if $|\psi\rangle$ is an eigenstate either of A or of B is

$$\Delta A^2+\Delta B^2\geq\frac{1}{2}|\langle\psi_{A+B}^\perp|A+B|\psi\rangle|^2, \tag{4}$$

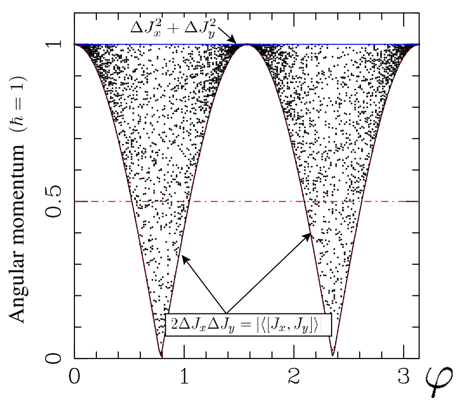

FIG. 1 (color online). Example of comparison between the Heisenberg-Robertson uncertainty relation (1) and the new ones (3), (4). We choose $A=J_x$ and $B=J_y$, two components of the angular momentum for a spin-1 particle, and a family of states parametrized by φ as $|\psi\rangle=\cos\varphi|+\rangle+\sin\varphi|-\rangle$, with $|\pm\rangle$ eigenstates of J_z corresponding to the eigenvalues ±1. None of these is a joint eigenstate of J_x and J_y, nonetheless the Heisenberg-Robertson relation can be trivial for $\varphi=\pi/4$ and $\varphi=3\pi/4$. The lower curves are the product of the uncertainties and the expectation value of the commutator (this is a favorable case for the Heisenberg-Robertson relation since the product of uncertainties and its lower bound coincide). The upper curve is $\Delta J_x^2+\Delta J_y^2=1$ (it is constant for this family of states). The dash-dotted line is the bound (4), the black points are the calculation of the bound (3) for 20 randomly chosen states $|\psi^\perp\rangle$ for each of the 200 values of the phase φ depicted. It is clear that the bound (3) well outperforms the Heisenberg-Robertson relation for almost all choices of $|\psi^\perp\rangle$. [The random $|\psi^\perp\rangle$ are generated by generating a random unitary U (uniform in the Haar measure) using the procedure detailed in [14], applying it to the $|+\rangle$ state, projecting on the orthogonal subspace to $|\psi\rangle$, and renormalizing the resulting state. Namely, $|\psi^\perp\rangle\propto(\mathbb{1}-|\psi\rangle\langle\psi|)U|+\rangle$.]

where $|\psi_{A+B}^\perp\rangle\propto(A+B-\langle A+B\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$ (with $\langle O\rangle$ denoting the expectation value of O). The form of $|\psi_{A+B}^\perp\rangle$ implies that the right-hand side of (4) is nonzero unless $|\psi\rangle$ is an eigenstate of $A+B$.

Clearly, both inequalities (3) and (4) can be combined in a single uncertainty relation for the sum of variances:

$$\Delta A^2+\Delta B^2\geq\max(\mathcal{L}_{(3)},\mathcal{L}_{(4)}), \tag{5}$$

with $\mathcal{L}_{(3),(4)}$ the right-hand side of (3) and (4), respectively.

Some comments on (3) and (4) follow: (i) they involve the sum of variances, so one must introduce some dimensional constants in the case in which A and B are measured with different units; (ii) removing the last term in (3), we find the inequality $\Delta A^2+\Delta B^2\geq|\langle[A,B]\rangle|$ implied by the Heisenberg-Robertson relation, as shown above; (iii) using the same techniques employed to derive (3), one can also obtain an amended Heisenberg-Robertson inequality:

$$\Delta A\Delta B\geq\pm\frac{i}{2}\langle[A,B]\rangle/\left(1-\frac{1}{2}\left|\langle\psi|\frac{A}{\Delta A}\pm i\frac{B}{\Delta B}|\psi^\perp\rangle\right|^2\right), \tag{6}$$

which reduces to (1) when minimizing the lower bound over $|\psi^\perp\rangle$ and becomes an equality when maximizing it.

Proofs of the results.—In this section we provide two proofs of the proposed uncertainty relations (3), (4), and (6). The first proof, based on the parallelogram law, was communicated to us by an anonymous referee, while the second (independent) proof was our original argument. While the first proof is preferable because of its simplicity, we retain also the second for completeness.

To prove (3), define $C\equiv A-\langle A\rangle$, $D\equiv B-\langle B\rangle$ so $\Delta A=\|C|\psi\rangle\|$, $\Delta B=\|iD|\psi\rangle\|$, where the imaginary unit i is introduced for later convenience. We have

$$\|(C\mp iD)|\psi\rangle\|^2=\Delta A^2+\Delta B^2\mp i\langle[A,B]\rangle, \tag{7}$$

where the left-hand side can be lower bounded through the Schwarz inequality as

$$\begin{aligned}|\langle\psi|(A\pm iB)|\psi^\perp\rangle|^2&=|\langle\psi|A\pm iB-\langle A\pm iB\rangle|\psi^\perp\rangle|^2\\&=|\langle\psi|C\pm iD|\psi^\perp\rangle|^2\leq\|(C\mp iD)|\psi\rangle\|^2,\end{aligned} \tag{8}$$

valid for all $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$, whence (3) follows by joining (7) and (8). The equality condition for (3) follows from the equality condition of the Schwarz inequality, namely, if and only if $|\psi^\perp\rangle\propto(A\mp iB-\langle A\mp iB\rangle)|\psi\rangle$.

Up to now we have considered only a pure state $|\psi\rangle$ of the system. This relation can be extended to the case of mixed states $\rho=\sum_j p_j|\psi_j\rangle\langle\psi_j|$ at least in the case in which it is possible to choose a $|\psi^\perp\rangle$ that is orthogonal to all states $|\psi_j\rangle$ (in the other cases, it is still possible to use the inequality, but it cannot be expressed as an expectation value for the density matrix). For each state $|\psi_j\rangle$ we can write (3) as

$$\Delta A_j^2 + \Delta B_j^2 \geq \mp i\mathrm{Tr}([A,B]|\psi_j\rangle\langle\psi_j|) + \mathrm{Tr}[(-A \pm iB)|\psi^\perp\rangle\langle\psi^\perp|(-A \mp iB)|\psi_j\rangle\langle\psi_j|], \quad (9)$$

where ΔA_j^2 and ΔB_j^2 are the variances calculated on $|\psi_j\rangle$. By multiplying both members by p_j and summing over j, we obtain the mixed-state extension of (3):

$$\Delta A^2 + \Delta B^2 \geq \mp i\langle[A,B]\rangle + \langle(-A \pm iB)|\psi^\perp\rangle\langle\psi^\perp|(-A \mp iB)\rangle. \quad (10)$$

To prove (4) we use the parallelogram law in Hilbert space to obtain

$$2\Delta A^2 + 2\Delta B^2 = \|C + \alpha D|\psi\rangle\|^2 + \|C - \alpha D|\psi\rangle\|^2, \quad (11)$$

for $C = A - \langle A\rangle$, $D = B - \langle B\rangle$, and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Since $\Delta(A+B) = \|(C+D)|\psi\rangle\|$, $\Delta(A-B) = \|(C-D)|\psi\rangle\|$, Eq. (11) for $\alpha = 1$ is equal to

$$\Delta A^2 + \Delta B^2 = \frac{1}{2}[\Delta(A+B)^2 + \Delta(A-B)^2] \geq \frac{1}{2}\Delta(A+B)^2, \quad (12)$$

which is equivalent to (4) since $\Delta(A+B)^2 = |\langle\psi_{A+B}^\perp|A+B|\psi\rangle|^2$. The equality condition for (4) is immediate from (12): $|\psi\rangle$ must be an eigenstate of $A-B$. Also, note that the lower bound in (4) is nonzero unless $|\psi\rangle$ is an eigenstate of $A+B$. Clearly $|\psi\rangle$ can be an eigenstate of $A+B$ without being an eigenstate of either A or B, but in the interesting case when $|\psi\rangle$ is an eigenstate of one of the two (which trivializes both Heisenberg's and Schrödinger's uncertainty relations), the lower bound must be nonzero unless $|\psi\rangle$ is an eigenstate of both. It is also easy to use (12) to modify the inequality (4) so that it has always a nontrivial lower bound except when $|\psi\rangle$ is a joint eigenstate of A and B, namely,

$$\Delta A^2 + \Delta B^2 \geq \max\left(\frac{1}{2}|\langle\psi_{A+B}^\perp|A+B|\psi\rangle|^2, |\langle\psi_A^\perp|A|\psi\rangle|^2, |\langle\psi_B^\perp|B|\psi\rangle|^2\right). \quad (13)$$

[Note that one can also obtain (3) from the parallelogram law (11) for $\alpha = \pm i$.]

We now provide a second proof of (3) and (4), and a proof of (6). They use the square-modulus inequality and follow a procedure analogous to the one employed by Holevo to derive the following useful relation [15]:

$$\Delta A + \Delta A' \geq (a - a')|\langle\psi|\psi'\rangle|/\sqrt{2(1 - |\langle\psi|\psi'\rangle|)}, \quad (14)$$

where a, a' are the expectation values of A on the states $|\psi\rangle$ and $|\psi'\rangle$, respectively, ΔA^2 and $\Delta A'^2$ are the variances on the same states.

To derive (3), start from the inequality

$$\|c_A\epsilon(A-a)|\psi\rangle \pm ic_B(B-b')|\psi'\rangle + c(\epsilon|\psi\rangle - |\psi'\rangle)\|^2 \geq 0, \quad (15)$$

with $a = \langle\psi|A|\psi\rangle$, $b' = \langle\psi'|B|\psi'\rangle$, $\epsilon \equiv \langle\psi|\psi'\rangle/|\langle\psi|\psi'\rangle|$, and c_A, c_B, and c real constants. Calculating the square modulus, we find

$$c_A^2\Delta A^2 + c_B^2\Delta B'^2 \geq -c^2\gamma - c_A c_B c\delta \mp ic_A c_B\kappa, \quad (16)$$

with ΔA^2 and $\Delta B'^2$ the variances of A and B on $|\psi\rangle$ and $|\psi'\rangle$, respectively, and where $\gamma \equiv 2(1 - |\langle\psi|\psi'\rangle|)$, $\delta = 2\mathrm{Re}[\epsilon^*\langle\psi|(a-A)/c_B \pm i(B-b')/c_A|\psi'\rangle]$, and $\kappa \equiv 2i\mathrm{Im}(\epsilon^*\langle\psi|(A-a)(B-b')|\psi'\rangle)$. Now choose the value of c that maximizes the right-hand side of (16) (assuming that one chooses the sign so the last term is positive), namely, $c = -c_A c_B\delta/(2\gamma)$. Whence, inequality (16) becomes

$$c_A^2\Delta A^2 + c_B^2\Delta B'^2 \geq (c_A c_B\delta)^2/(4\gamma) \mp ic_A c_B\kappa. \quad (17)$$

Depending on the choice of c_A and c_B one can prove (3) or (6). Start with the former by taking $c_A = c_B = 1$, we find

$$\Delta A^2 + \Delta B'^2 \geq \frac{\delta^2}{4\gamma} \mp i\kappa = \frac{[\mathrm{Re}(\epsilon\langle\psi'|(-\bar{A} \mp i\bar{B}')|\psi\rangle)]^2}{2(1 - |\langle\psi|\psi'\rangle|)} \mp i(\epsilon^*\langle\psi|\bar{A}\bar{B}'|\psi'\rangle - \epsilon\langle\psi'|\bar{B}'\bar{A}|\psi\rangle), \quad (18)$$

where $\bar{A} \equiv A - a$ and $\bar{B}' \equiv B - b'$. This inequality, which may be of independent interest, is a two-observable extension of the Holevo inequality (14), and reduces to it by choosing $\bar{B} = \pm i(A - a')$ and recalling that $(\Delta A + \Delta A')^2 \geq \Delta A^2 + \Delta A'^2$. To obtain (3), take the limit $|\psi'\rangle \to |\psi\rangle$. This can be calculated by writing $|\psi'\rangle = \cos\alpha|\psi\rangle + e^{i\lambda}\sin\alpha|\psi^\perp\rangle$, where $|\psi^\perp\rangle$ is orthogonal to $|\psi\rangle$ and taking the limit $\alpha \to 0$. The arbitrariness of $|\psi'\rangle$ ensures the arbitrariness of $|\psi^\perp\rangle$ and of the phase λ. In the limit, the last term of (18) yields the expectation value of the commutator and the other term on the right-hand side tends to $[\mathrm{Re}(e^{i\lambda}\langle\psi|(-A \pm iB)|\psi^\perp\rangle)]^2$. For either sign in this expression, we can choose λ so that the term in parenthesis is real, so that this expression can be written also as $|\langle\psi|(-A \pm iB)|\psi^\perp\rangle|^2$. This implies that the limit $|\psi'\rangle \to |\psi\rangle$ of (18) gives (3) (with the above choice of λ).

To prove the second proposed uncertainty relation (6), we can choose $c_A = \Delta B'$ and $c_B = -\Delta A$ in (17), which then becomes

$$\Delta A\Delta B' \geq \pm\frac{i}{2}(\epsilon^*\langle\psi|\bar{A}\bar{B}'|\psi'\rangle - \epsilon\langle\psi'|\bar{B}'A|\psi\rangle) + \frac{\Delta A\Delta B'}{4(1-|\langle\psi|\psi'\rangle|)}\left[\mathrm{Re}\left(\epsilon^*\langle\psi|\frac{\bar{A}}{\Delta A}\pm i\frac{\bar{B}'}{\Delta B'}|\psi'\rangle\right)\right]^2. \quad (19)$$

We can now take the limit $|\psi'\rangle \to |\psi\rangle$ using the same procedure described above. Again the first term tends to the expectation value of the commutator, while the second term tends to $\Delta A\Delta B[\mathrm{Re}(e^{-i\lambda}\langle\psi^\perp|A/\Delta A \mp iB/\Delta B|\psi\rangle)]^2/2$. Again the phase λ can be chosen so that this last term is real and (19) becomes

$$\Delta A\Delta B \geq \pm\frac{i}{2}\langle[A,B]\rangle + \frac{\Delta A\Delta B}{2}\left|\langle\psi^\perp|\frac{A}{\Delta A}\mp i\frac{B}{\Delta B}|\psi\rangle\right|^2,$$

which is equivalent to (6).

Finally, the second proof of (4) is obtained by noting that $(\Delta A+\Delta B)^2 \leq 2(\Delta A^2+\Delta B^2)$. Therefore, we have

$$\Delta A^2+\Delta B^2 \geq \frac{1}{2}[\Delta(A+B)]^2, \quad (20)$$

where we have used the sum uncertainty relation of [10], namely, $\Delta A+\Delta B \geq \Delta(A+B)$ with $[\Delta(A+B)]^2$ the variance of $(A+B)$ in the state $|\psi\rangle$. The meaning of the sum uncertainty relation is that mixing different operators always decreases the uncertainty. The lower bound in (20) can be rewritten using Vaidman's formula [16]

$$O|\psi\rangle = \langle O\rangle|\psi\rangle + \Delta O|\psi_O^\perp\rangle, \quad (21)$$

(the expectation value $\langle O\rangle$ and the variance ΔO^2 of the observable O are calculated on $|\psi\rangle$), obtaining

$$\Delta O = |\langle\psi_O^\perp|\Delta O|\psi_O^\perp\rangle| = |\langle\psi_O^\perp|(O-\langle O\rangle)|\psi\rangle| = |\langle\psi_O^\perp|O|\psi\rangle|,$$

which, inserted into (20) with $O=(A+B)$ gives [10]. Using the results of [10] it is also easy to extend this inequality to more than two observables.

Possible choices of $|\psi^\perp\rangle$.—We now show that the optimization over $|\psi^\perp\rangle$ of both inequalities (3) and (6) makes them tight. Start with (3): the lower bound is clearly maximized if we choose $|\psi^\perp\rangle$ as close as possible to the state $|\chi\rangle = (A\pm iB)|\psi\rangle$, for example, projecting such state into the orthogonal subspace to $|\psi\rangle$ as $|\psi^\perp\rangle = (\mathbb{1}-|\psi\rangle\langle\psi|)|\chi\rangle/\mathcal{N}$, with $\mathcal{N}$ a normalization. With this choice, we find

$$\langle\psi^\perp|(A\pm iB)|\psi\rangle = \langle\psi|[A-a\mp i(B-b)](A\pm iB)|\psi\rangle/\mathcal{N} = (\Delta A^2+\Delta B^2\pm i\langle[A,B]\rangle)/\mathcal{N}, \quad (22)$$

where the normalization constant is $\mathcal{N}=(\Delta A^2+\Delta B^2\pm i\langle[A,B]\rangle)^{1/2}$. Substituting (22) into (3), we see that the inequality is indeed saturated. Analogous considerations hold for (6): In this case, we should choose $|\psi^\perp\rangle \propto (\mathbb{1}-|\psi\rangle\langle\psi|)(\frac{A}{\Delta A}\mp i\frac{B}{\Delta B}|\psi\rangle$. With this choice, $\langle\psi^\perp|(\frac{A}{\Delta A}\mp i\frac{B}{\Delta B}|\psi\rangle = 2\mp i\langle[A,B]\rangle/(\Delta A\Delta B)$, which is also equal to the square of the normalization constant for $|\psi^\perp\rangle$. Hence, substituting this value in (6), we see that it is saturated for this choice of $|\psi^\perp\rangle$. [It is also clear that the choice of $|\psi^\perp\rangle$ that minimizes the lower bounds transforms (3) into $\Delta A^2+\Delta B^2 \geq |\langle[A,B]\rangle|$ that is a consequence of (1) as shown above, and it transforms (6) into (1).]

A simple prescription for how to choose an expression for $|\psi^\perp\rangle$ uses (21), namely, $|\psi^\perp\rangle = (O-\langle O\rangle)|\psi\rangle/\Delta O$.

Here we have focused on extending the Heisenberg-Robertson uncertainty relation (1), but it is also possible to give an extension to the Schrödinger relation (2), by choosing an arbitrary phase factor $e^{i\theta}$ in place of the imaginary constant i in (15).

Uncertainty relations and uncertainty principle.—Recently, there has been an interesting and lively debate on how to interpret the uncertainty principle [17,18]. To elucidate the relation between these results and ours, we introduce Peres' nomenclature that distinguishes between the uncertainty *relation* and the uncertainty *principle* [19]. The former refers solely to the preparation of the system which induces a spread in the measurement outcomes, and does not refer to the disturbance induced by the measurement or to joint measurements [20]. The latter entails also the measurement disturbance by the apparatus and the impossibility of joint measurements of incompatible observables. From Robertson's derivation [3], it is clear [19] that the Heisenberg-Robertson inequalities are uncertainty *relations* (the ones typically taught in textbooks). In contrast, Heisenberg, in his Letter [1,2], does not give a clear distinction between the two concepts, and both can be applied depending on the systems he analyzes there. The recent literature [17,18] discusses the uncertainty *principle*: measurement-induced disturbance and joint measurability. Our result instead refers to uncertainty *relations*: it can be seen as a quantitative expression for the nonexistence of common eigenstates in incompatible observables.

Conclusions.—The Heisenberg-Robertson (1) or Schrödinger (2) uncertainty relations do not fully capture the incompatibility of observables on the system state. In this Letter, we have presented a stronger uncertainty relation (5) based on two lower bounds (3) and (4) for the sum of the variances that are nontrivial if the two observables are incompatible on the state of the system. We also derived (6), a strengthening of the Heisenberg-Robertson uncertainty relation (1). There exists alternate formulations of uncertainty relations in terms of bounds on the sum of *entropic* quantities [21,22], but our new relations capture the notion of incompatibilty in terms of experimentally measured error bars, as they refer to variances.

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[20] A good definition of "uncertainty *relation*" is given in [19], pg. 93: "The only correct interpretation of [the uncertainty relations for x and p] is the following: If the same preparation procedure is repeated many times, and is followed either by a measurement of x, or by a measurement of p, the various results obtained for x and for p have standard deviations, Δx and Δp, whose product cannot be less than $\hbar/2$. There never is any question here that a measurement of x 'disturbs' the value of p and vice versa, as sometimes claimed. These measurements are indeed incompatible, but they are performed on *different* particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way. The uncertainty relation […] only reflects the intrinsic randomness of the outcomes of quantum tests." We emphasize that the uncertainty *relation* must not be confused with the uncertainty *principle*.

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