Interplay of Driving and Frequency Noise in the Spectra of Vibrational Systems

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We study the spectral effect of the fluctuations of the vibration frequency. Such fluctuations play a major role in nanomechanical and other mesoscopic vibrational systems. We find that, for periodically driven systems, the interplay of the driving and frequency fluctuations results in specific spectral features. We present measurements on a carbon nanotube resonator and show that our theory allows not only the characterization of the frequency fluctuations but also the quantification of the decay rate without ring-down measurements. The results bear on identifying the decoherence of mesoscopic oscillators and on the general problem of resonance fluorescence and light scattering by oscillators.

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The spectrum of response and the power spectrum of an oscillator is a textbook problem that goes back to Lorentz and Einstein [1–3]. It has attracted much attention recently in the context of nanomechanical systems. Here, the spectra are a major source of information about the classical and quantum dynamics [4–11]. This is the case also for mesoscopic oscillators of a different nature, such as superconducting cavity modes [12–15] and optomechanical systems [16]. Mesoscopic oscillators experience comparatively large fluctuations. Along with dissipation, these fluctuations determine the shape of the vibrational spectra.

A well-understood and most frequently considered [3] source of fluctuations is thermal noise that comes from the coupling of an oscillator (vibrational system) to a thermal reservoir and is related to dissipation by the fluctuation-dissipation theorem. Dissipation leads to the broadening of the oscillator power spectrum and the spectrum of the response to external driving.

Spectral broadening can also come from fluctuations of the oscillator frequency, which play an important role in mesoscopic oscillators. For nanomechanical resonators, frequency fluctuations can be caused by tension and mass fluctuations, fluctuations of the charge in the substrate, or dispersive intermode coupling [7–11,17–20], whereas for electromagnetic cavity modes they can come from fluctuations of the effective dielectric constant [12,13]. Identifying different broadening mechanisms is a delicate task that has been attracting much attention [9,11,12,19,21,22].

In this Letter, we study the combined effect of periodic driving and frequency fluctuations on the power spectra of nanomechanical vibrational systems. For a linear oscillator with no frequency fluctuations, driving leads to a δ-like peak at the driving frequency ω_F [3,4], because here the only effect of the driving is forced vibrations linearly superimposed on thermal motion. Frequency fluctuations make forced vibrations random. As we show, this qualitatively changes the spectrum, leading to characteristic new spectral features. We observe these features in a carbon-nanotube resonator and use them to separate the energy relaxation rate from the overall broadening of the power spectrum in the absence of driving as well as reveal and explore the narrow-band frequency noise. We predict that for fluctuating nonlinear vibrational systems, too, even weak driving leads to a very specific extra spectral structure.

For a linear oscillator, the spectral features resulting from the interplay of driving and frequency noise are sketched in Fig. 1. The two limiting cases shown in Fig. 1 correspond to the long and short correlation time of the frequency noise t_c

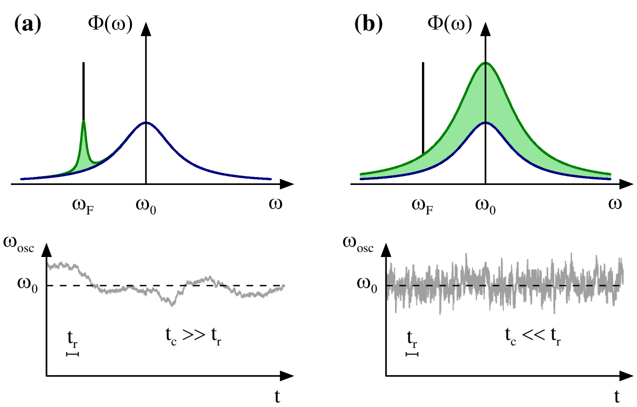

FIG. 1 (color online). Top: sketches of the power spectra of a driven linear oscillator $\Phi(\omega)$. Panels (a) and (b) refer to large and small correlation time of the frequency noise t_c compared to the oscillator relaxation time t_r, respectively, i.e., to narrow- and broadband frequency noise. The blue (lower) line shows the spectrum of thermal fluctuations in the absence of driving; it is centered at the oscillator eigenfrequency $\omega_0 = \langle\omega_{\rm osc}(t)\rangle$. In the presence of driving there is added a δ peak at the driving frequency ω_F. The green areas show the spectral features from the interplay of the driving and fluctuations of $\omega_{\rm osc}(t)$. Bottom panels: $\omega_{\rm osc}(t)$ for $t_c \gg t_r$ (a) and $t_c \ll t_r$ (b).

compared to the oscillator relaxation (decay) time t_r. For $t_c \gg t_r$ [Fig. 1(a)], the oscillator frequency $\omega_{\rm osc}(t)$ slowly fluctuates about what can be called the eigenfrequency $\omega_0 = \langle\omega_{\rm osc}(t)\rangle$. One can then think of slow fluctuations of the oscillator susceptibility χ, which depends on the detuning of the driving frequency ω_F from $\omega_{\rm osc}(t)$. The associated slow fluctuations of the amplitude and phase of forced vibrations at frequency ω_F lead to a finite-width spectral peak centered at ω_F. This is a frequency-domain analog of the Einstein light scattering due to spatial susceptibility fluctuations [23].

For $t_c \ll t_r$ [Fig. 1(b)], driving-induced random vibrations quickly lose the memory of the driving frequency. They become similar to thermal vibrations. However, their amplitude is determined by the driving, not the temperature. This leads to a spectral peak centered at the oscillator eigenfrequency ω_0, with the height quadratic in the driving amplitude.

In the quantum picture, one can think that, as a result of pumping by a driving field, the oscillator emits energy quanta. For the familiar example of an oscillating charge driven by an electromagnetic field, these quanta can be photons, and one can speak of light scattering and fluorescence by an oscillator. A quantum is emitted over time t_r after the absorption event. For $t_c \ll t_r$, the frequency of the quantum is uncorrelated with the excitation frequency ω_F. This is a fluorescence-type process. The energy difference $\hbar(\omega_F - \omega_0)$ comes from the frequency noise. For $t_c \gg t_r$, emission occurs at frequencies close to ω_F. In the both cases the spectrum is qualitatively different from just a δ-like peak in the absence of frequency fluctuations [3].

To describe the power spectrum of a nonlinear system, one has to go beyond the approximation implied above, where only the linear susceptibility is fluctuating. If driving is described by the term $-qF(t)$ in the oscillator Hamiltonian, where q is the oscillator coordinate and $F(t) = F\cos\omega_F t$ is the driving force, to obtain terms $\propto F^2$ in the power spectrum one should keep terms $\propto F$ and $\propto F^2$ in the response,

$$q(t) \approx q_0(t) + \int_{-\infty}^{t} dt'\chi_1(t,t')F(t') + \iint_{-\infty}^{t} dt'dt''\chi_2(t,t',t'')F(t')F(t''). \quad (1)$$

Here, $q_0(t)$ is thermal displacement in the absence of driving. Equation (1) does not include averaging; χ_1 and χ_2 are the fluctuating linear and nonlinear susceptibilities. The standard linear susceptibility is $\langle\chi_1(t,t')\rangle$; it is a function of $t - t'$. For a harmonic oscillator, which is the central topic of this Letter, $\chi_2 = 0$.

The conventionally measured oscillator power spectrum is $\Phi(\omega) = 2{\rm Re}\int_0^\infty dt e^{i\omega t}\langle\langle q(t+t')q(t')\rangle\rangle$, where $\langle\langle\cdot\rangle\rangle$ indicates statistical averaging and averaging with respect to t' over the driving period $2\pi/\omega_F$. For weak driving,

$$\Phi(\omega) \approx \Phi_0(\omega) + \frac{\pi}{2}F^2|\chi(\omega_F)|^2\delta(\omega - \omega_F) + F^2\Phi_F(\omega). \quad (2)$$

This spectrum is sketched in Fig. 1. Function Φ_0 is the power spectrum in the absence of driving, a resonant peak associated with thermal vibrations of the oscillator. The δ peak at the driving frequency in Eq. (2) and in Fig. 1 describes average forced oscillator vibrations; $\chi(\omega)$ is the Fourier transform of $\langle\chi_1(t,t')\rangle$ over $t - t'$.

Of primary interest to us is the term $\Phi_F(\omega)$, shown by the envelope of the green area in Fig. 1. It describes the interplay of frequency fluctuations and the driving. We consider it for ω close to ω_F assuming a high quality factor, $\omega_0 t_r \gg 1$, typical for mesoscopic systems, and resonant driving, $|\omega_F - \omega_0| \ll \omega_F$.

For a harmonic oscillator with fluctuating frequency, $\omega_{\rm osc}(t) = \omega_0 + \xi(t)$, where $\xi(t)$ is zero-mean noise. We assume that the noise is weak compared to ω_0 and that its correlation time $t_c \gg \omega_0^{-1}$. The noise then does not cause parametric excitation of the oscillator [24,25].

The most simple model of the oscillator dynamics is described by equation $\ddot{q} + 2\Gamma\dot{q} + [\omega_0^2 + 2\omega_0\xi(t)]q = F\cos\omega_F t + f(t)$, where $f(t)$ is thermal noise and $\Gamma = t_r^{-1}$ is the relaxation rate. Both $f(t)$ and the direct frequency noise $\xi(t)$ lead to fluctuations of the oscillator phase. Separating their contributions by measuring the commonly used Allan variance (cf. Ref. [4]) is complicated. However, these two types of noise have different physical origin, and our results show how they can be separated using the power spectrum; a different approach, which however, may not be implemented with a standard spectrum analyzer, was proposed in Ref. [26].

In the standard rotating wave approximation (Supplemental Material [27]), the fluctuating linear susceptibility of a damped harmonic oscillator is

$$\chi_1(t,t') = \frac{i}{2\omega_0}e^{-(\Gamma+i\omega_0)(t-t')-i\int_{t'}^{t}dt''\xi(t'')} + \text{c.c.} \quad (3)$$

($\chi_2 = 0$). Equation (3) often applies even where the oscillator dynamics in the lab frame is non-Markovian.

Explicit expressions for χ_1 and $\Phi_F(\omega)$ can be obtained from Eq. (3) in the limiting cases. For weak frequency noise, one can expand χ_1 in $\xi(t)$. To the leading order, the spectrum Φ_F is proportional to the noise power spectrum $\Xi(\Omega) = \int_{-\infty}^{\infty} dt\langle\xi(t)\xi(0)\rangle\exp(i\Omega t)$,

$$\Phi_F(\omega) \approx \frac{1}{16\omega_0^2[\Gamma^2 + (\omega_F - \omega_0)^2]}\frac{\Xi(\omega - \omega_F)}{\Gamma^2 + (\omega - \omega_0)^2}. \quad (4)$$

This expression provides a direct means for measuring the frequency noise spectrum. It already shows the peculiar features qualitatively discussed above. If $\Xi(\Omega)$ peaks at zero frequency and is narrow on the scale Γ (as for $1/f$-type noise, for example), $\Phi_F(\omega)$ has a peak at ω_F, cf. Fig. 1(a).

The shape of this peak coincides with that of $\Xi(\Omega)$. If, on the other hand, $\Xi(\Omega)$ is almost flat on the frequency scale Γ, $|\omega_F - \omega_0|$ (broadband noise), $\Phi_F(\omega)$ has a Lorentzian peak at ω_0, cf. Fig. 1(b).

To describe the effect of a narrow-band but not necessarily weak frequency noise, one can replace $\xi(t'')$ in Eq. (3) with $\xi(t)$. This corresponds to the "instantaneous" slowly fluctuating susceptibility $i/2\omega_0[\Gamma - i(\omega_F - \omega_0 - \xi(t))]$. The resulting narrow spectrum $\Phi_F(\omega)$ is determined by the spectrum and statistics of the frequency noise. The simple relation (4) between $\Phi_F(\omega)$ and $\Xi(\omega)$ follows from this analysis for $\langle \xi^2 \rangle \ll \Gamma^2 + (\omega_F - \omega_0)^2$. Importantly, this condition can be achieved by tuning ω_F somewhat away from ω_0.

The case of flat $\Xi(\Omega)$, i.e., of $\xi(t)$ being δ correlated on time scale t_r, can be analyzed for an arbitrary noise strength using the fact that the characteristic functional of a δ-correlated noise is $\mathcal{P}[k(t)] = \langle \exp[i \int dt k(t) \xi(t)] \rangle = \exp[-\int \mu(k(t)) dt]$, where function $\mu(k)$ is determined by the noise statistics. Then from Eq. (3)

$$\Phi_F(\omega) = \frac{[\mathrm{Re}\,\mu(1)]/\Gamma}{8\omega_0^2[\tilde{\Gamma}^2 + (\omega_F - \tilde{\omega}_0)^2]} \frac{\tilde{\Gamma}}{\tilde{\Gamma}^2 + (\omega - \tilde{\omega}_0)^2}. \quad (5)$$

The spectrum given by Eq. (5) and the spectrum $\Phi_0(\omega)$ in the absence of periodic driving have the same shape described by the last factor in Eq. (5): a Lorentzian centered at the noise-renormalized oscillator eigenfrequency $\tilde{\omega}_0 = \omega_0 - \mathrm{Im}\,\mu(1)$ with half-width $\tilde{\Gamma} = \Gamma + \mathrm{Re}\,\mu(1)$. However, in contrast to $\Phi_0(\omega)$, the area of $F^2\Phi_F(\omega)$ is independent of the intensity ($\propto k_B T$) of the dissipation-related noise. Instead, it is proportional to the frequency-noise characteristic $\mathrm{Re}\,\mu(1)$. Equation (5) suggests how to separate the noise-induced broadening of the oscillator spectrum from the decay-induced broadening; see below.

Function Φ_F can be found in a closed form for a Gaussian noise $\xi(t)$. The results are shown in Fig. 2 for the noise power spectrum with bandwidth λ, $\Xi(\Omega) = 2D\lambda^2/(\lambda^2 + \Omega^2)$. They illustrate how the shape of $\Phi_F(\omega)$ changes from a peak at ω_F for a narrow-band noise ($\lambda \ll \Gamma$) to a peak at ω_0 for a broadband noise ($\lambda \gg \Gamma$). The overall area of the spectrum Φ_F nonmonotonically depends on the frequency noise intensity: it is linear in the noise intensity for weak noise, cf. Eq. (4), but for a large noise intensity it decreases, since the decoherence rate of the oscillator increases.

To corroborate the theory, we measured the spectrum of a modulated carbon nanotube resonator at $T = 1.2$ K. For such T and weak driving, low-lying flexural modes of the resonator are well described by harmonic oscillators. The driving was applied as an ac voltage δV_g on the gate electrode, and the power spectrum of the mechanical vibrations was probed by measuring the noise in the current flowing through the nanotube [32].

Figure 3(a) compares the power spectra of the nanotube vibrations obtained with and without the oscillating force.

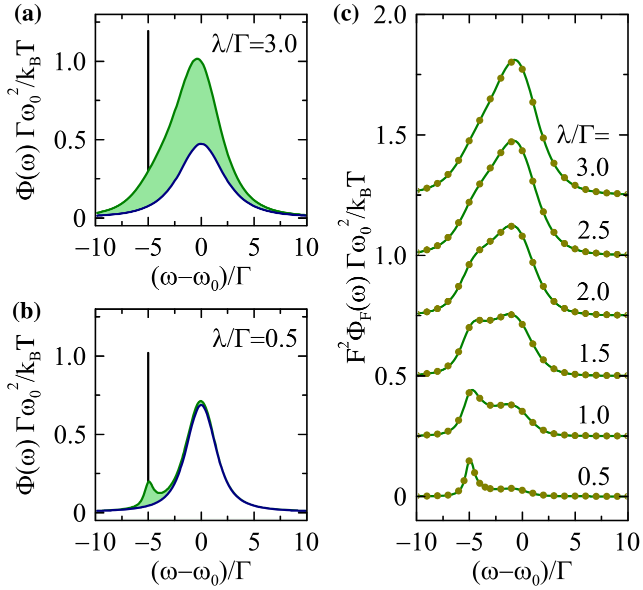

FIG. 2 (color online). The power spectrum of the oscillator with a Gaussian frequency noise with the spectrum $\Xi(\Omega) = 2D\lambda^2/(\lambda^2 + \Omega^2)$. The noise intensity is $D/\Gamma = 2$. Panels (a) and (b) show the full spectrum. The color coding is the same as in Fig. 1, $F^2/16\Gamma^2 = 20 k_B T$. Panel (c) shows the driving-induced term. The solid lines and dots show the analytic theory and simulations; the consecutive curves are shifted by 0.25 along the ordinate.

The spectrum without modulation [lower trace (blue)] is close to a Lorentzian, as expected for thermal vibrations. The spectrum with modulation [upper trace (green)] displays a narrow peak centered at the modulation frequency and a much broader peak of the same shape as the spectrum without modulation. The areas of the both modulation-induced parts of the spectrum scale as δV_g^2 [Figs. 3(b), 3(c)], in agreement with Eq. (2). The separation between the parts is subject to some uncertainty because of the measurement noise in Fig. 3(a) (Supplemental Material [27]). The resulting uncertainty in the slopes in Figs. 3(b), 3(c) is $\lesssim 10\%$. The change of the spectrum is not a heating effect associated with the modulation, since we estimate temperature increases as $\lesssim 10^{-8}$ K. The spectral feature at ω_F is not related to the phase noise of the source used for the modulation; indeed, the phase noise of our source ≈ 10 Hz away from ω_F could only account for $\approx 0.01\%$ of the measured power spectrum.

The driving-induced spectral change provides a simple means for estimating the intrinsic relaxation rate of the resonator Γ from our experimental data. This can be done using the areas $F^2 S_{\mathrm{nb}}$ and $F^2 S_{\mathrm{bb}}$ of the narrow and broad peaks in Fig. 3, respectively. Comparing them to the area under the driving-induced δ-peak $F^2 S_\delta$, we eliminate F and obtain from Eq. (5)

$$\tilde{\Gamma}/\Gamma \approx 1 + (S_{\mathrm{bb}}/S_\delta)[1 - (S_{\mathrm{nb}}/S_\delta)]; \quad (6)$$

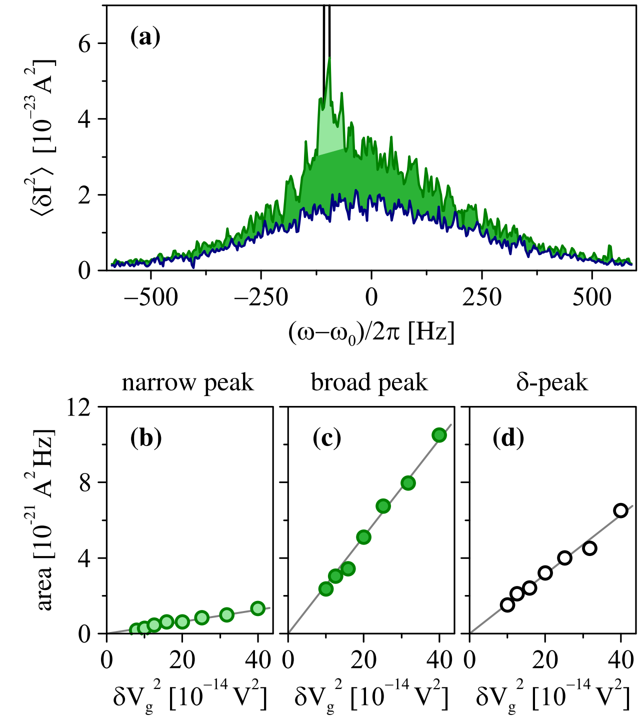

FIG. 3 (color online). (a) The power spectrum of the fluctuating current $\delta I(t)$ through a driven carbon nanotube. The measurement bandwidth is 4.7 Hz. The eigenfrequency of the studied flexural mode is 6.3 MHz. The driving frequency is 100 Hz below the resonance frequency. The lower jagged line (blue) refers to the power spectrum without driving; the green area above it shows the driving-induced spectral change. This change is separated into the broad peak (darker filling), narrow peak (lighter filling), and a delta spike at the modulation frequency. This spike lies within three bins, within our experimental resolution, and is represented by the black vertical lines. The separation of the broad and narrow peaks is done by the straight line that interpolates the broad peak. Shown in the lower panels is the dependence of the lighter green area (b), the darker green area (c), and the area under the δ peak (d) on the squared amplitude of the modulating gate voltage; as expected from the theory, it is close to linear.

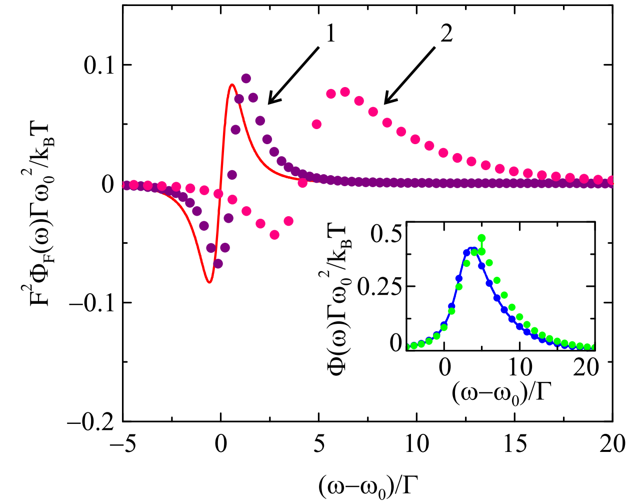

FIG. 4 (color online). The driving-induced part of the spectrum of a nonlinear oscillator. The dots show the results of simulations. The red solid line shows the analytical results for $\Phi_F(\omega)$ for small $\Delta\omega/\Gamma$ for the parameters of curve 1. The scaled values of the nonlinearity parameter, the detuning, and the driving strength on the curves 1 and 2 are, respectively, $\Delta\omega/\Gamma = 0.125$ and 1.25, $(\omega_F - \omega_0)/\Gamma = 0.5$ and 5, and $3\gamma F^2/32\omega_0^3(\omega_F - \omega_0)^3 = 0.64$ and 0.01. The inset shows the full spectrum for the parameters of curve 2 (light green dots, simulations); the spectrum without driving for the same $\Delta\omega/\Gamma$ is shown by the solid line (analytical) and blue dots on top of it (simulations).

we used $S_{\rm nb} \ll S_\delta$. With $\tilde{\Gamma}/(2\pi) \simeq 230$ Hz read out from the collected spectra [such as the one in Fig. 3(a)], along with $S_{\rm bb}/S_\delta$ and $S_{\rm nb}/S_\delta$ measured from Figs. 3(b)–3(d), we obtain $\tilde{\Gamma}/\Gamma \simeq 2.1$, which gives $\Gamma/(2\pi) \simeq 110$ Hz. Therefore, the broadband fluctuations of the resonant frequency account for $\gtrsim 50\%$ of the measured mechanical linewidth. Because of the noise in the measurement in Fig. 3(a), the uncertainty in Γ is $\lesssim 10\%$.

The narrow-band frequency noise can also be characterized from the measurements. Its power spectrum is $\propto 1/|\omega - \omega_F|^\alpha$ with $\alpha \approx 1/2$ (Supplemental Material [27]). Obtaining the power spectra Φ_F for several values of $\omega_F - \omega_0$ should allow separating the low-frequency part of the frequency noise spectrum $\Xi(\omega)$ even where it is not weaker than the broadband part and reading it directly off the data on the power spectrum using Eq. (4).

An important contribution to spectral broadening of mesoscopic oscillators can come from their nonlinearity [33]. Because the vibration frequency of a nonlinear oscillator depends on the vibration amplitude, thermal fluctuations of the amplitude lead to frequency fluctuations. This makes the power spectrum non-Lorentzian and asymmetric even in the absence of driving [34]. The shape of the spectrum is determined by the interrelation between the frequency uncertainty Γ due to the oscillator decay and the width $\Delta\omega$ of the frequency distribution due to thermal distribution of the vibration amplitude. In the presence of driving, superimposed on this effect is the frequency shift due to the driving-induced vibrations.

The strong dependence of the driving-induced spectral change on the ratio $\Delta\omega/\Gamma$ is illustrated in Fig. 4 for the important model where the nonlinear term in the oscillator energy is $\gamma q^4/4$ so that $\Delta\omega \approx 3|\gamma| k_B T/8\omega_0^3$. For small $\Delta\omega/\Gamma$, the major effect of the driving is the shift of the spectrum; then $\Phi_F(\omega) \propto \partial_\omega \Phi_0(\omega) \propto (\omega - \omega_0)\Phi_0(\omega)$ has a characteristic dispersive shape, changing sign at ω_0. With increasing $\Delta\omega/\Gamma$ the shape of $\Phi_F(\omega)$ becomes more complicated. Generally, it still has positive and negative parts, a dramatic difference from the case of a harmonic oscillator with fluctuating frequency. Also, in contrast to a harmonic oscillator, keeping terms $\propto F^2$ in the power spectrum of a nonlinear oscillator is justified only for weak modulating fields. A detailed theory of Φ_F for nonlinear oscillators based on Eq. (1) will be presented elsewhere.

The above results show that the interplay of driving and frequency noise qualitatively changes oscillator spectra compared to the spectra with no frequency fluctuations [3]. The change sensitively depends on the frequency noise intensity and power spectrum. The possibility to separate contributions from different parts of the frequency-noise spectrum is of significant interest, as they may come from physically different sources, such as two-level fluctuators and dispersive coupling to other modes, to mention but a few.

The results suggest a way of discriminating between three major factors of the broadening of the oscillator spectra: decay (energy relaxation), frequency fluctuations induced directly by the noise that modulates the eigenfrequency, and frequency fluctuations due to the oscillator nonlinearity. For linear oscillators, our simple procedure yields the decay rate without the need of an actual ring-down measurement that is often difficult to implement, in particular, for nanotube mechanical resonators. We also find that the nonlinearity-induced spectral change has a qualitatively different shape from that due to frequency noise in linear oscillators.

The analysis of driven linear oscillators with a fluctuating frequency immediately extends to the quantum regime, which is attracting much interest in nano- and optomechanics [16,22,35–38]. For nonlinear oscillators, the non-equidistance of the energy levels can bring in new features compared to the classical limit.

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