Fibonacci Anyons From Abelian Bilayer Quantum Hall States

Abolhassan Vaezi[1] and Maissam Barkeshli[2]
[1]Department of Physics, Cornell University, Ithaca, New York 14850, USA
[2]Station Q, Microsoft Research, Santa Barbara, California 93106-6105, USA
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The possibility of realizing non-Abelian statistics and utilizing it for topological quantum computation (TQC) has generated widespread interest. However, the non-Abelian statistics that can be realized in most accessible proposals is not powerful enough for universal TQC. In this Letter, we consider a simple bilayer fractional quantum Hall system with the $1/3$ Laughlin state in each layer. We show that interlayer tunneling can drive a transition to an exotic non-Abelian state that contains the famous "Fibonacci" anyon, whose non-Abelian statistics is powerful enough for universal TQC. Our analysis rests on startling agreements from a variety of distinct methods, including thin torus limits, effective field theories, and coupled wire constructions. We provide evidence that the transition can be continuous, at which point the charge gap remains open while the neutral gap closes. This raises the question of whether these exotic phases may have already been realized at $\nu = 2/3$ in bilayers, as past experiments may not have definitively ruled them out.

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Introduction.—There is currently intense interest in the realization of exotic quantum phases of matter that host quasiparticles with non-Abelian statistics [1,2], partially due to the possibility of topological quantum computation (TQC). While there are many promising candidate platforms for realizing non-Abelian statistics, almost all of them have the drawback that they are not powerful enough to realize universal TQC.

Recently, it has been proposed that a wide class of non-Abelian defects can be synthesized by starting with simple double-layer or single-layer fractional quantum Hall (FQH) states and properly including certain spatially nonuniform patterns of interlayer tunneling [3–5] or superconductivity [6–9]. Subsequently, it was shown that by coupling these engineered non-Abelian defects in an appropriate manner, it is possible to realize exotic non-Abelian phases that are powerful enough for universal TQC [10–12]. However, engineering the interactions of these defects in a physically realistic setup is a major challenge. Nevertheless, these studies suggest the possibility that these exotic, computationally universal non-Abelian phases might be realized in a simpler fashion, by starting with either (i) conventional double layer FQH states [such as the (330) state, which contains independent $1/3$ Laughlin states in each layer] and increasing the interlayer tunneling uniformly in space, or (ii) conventional single layer FQH states, and uniformly increasing the coupling to a superconductor.

In this Letter, we present two basic advances, mainly in the double layer context with interlayer tunneling. First, we show that the appearance of these computationally universal non-Abelian states can be understood in the thin torus limit, where the interlayer tunneling is taken to be uniform in space. In this limit we systematically derive the properties of the quasiparticles for large interlayer tunneling. These include the so-called "Fibonacci" quasiparticle, whose non-Abelian braiding statistics allow for universal TQC. Second, we find the possibility of a continuous quantum phase transition between the conventional bilayer FQH states and these exotic non-Abelian ones, as the interlayer tunneling is increased. We show that this theory is described by a $SU(3)_1 \times SU(3)_1 \to SU(3)_2$ Chern-Simons-Higgs transition, and also provides a many-body wave function for the non-Abelian state. The startling agreement between these distinct approaches, and with the earlier constructions [10,11,15], provides evidence that this non-Abelian state can be stabilized with uniform tunneling.

Several years ago [16,17] it was argued that the (330) state, in the presence of interlayer tunneling, could continuously transition to a different non-Abelian FQH state, known as the Z_4 Read-Rezayi FQH state [18], whose non-Abelian braiding statistics alone is *not* powerful enough for universal TQC. Combining the earlier results with those of the present Letter leads to a rich global phase diagram at total filling fraction $\nu = 2/3$ in bilayer systems, which we explore (see Fig. 1).

Thin torus limit.—For a wide variety of FQH states, it was found [19–23] that the wave function in the thin torus limit ($L_x/L_y \ll 1$) is smoothly connected to the fully two-dimensional wave function ($L_x/L_y \sim 1$), where L_x and L_y are the lengths of the torus in the two directions. This thin torus limit, which we review below, allows for a simple understanding of fractionalization in the FQH state in terms of one-dimensional fractionalization [24].

In the $L_x/L_y \to 0$ limit and at filling fraction $1/n$, the dominant contribution to the pseudopotential Hamiltonian for the Laughlin state is

$$H_n = \sum_i \sum_{0<r<n} U_{r,0} \hat{n}_i \hat{n}_{i+r}, \qquad U_{r,0} = g_{r,0} e^{-2\pi^2 r^2/L_x^2}. \quad (1)$$

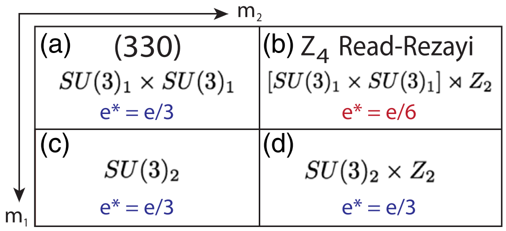

FIG. 1 (color online). $\nu = 2/3$ proposed global phase diagram, for interlayer tunneling on the order of interaction strengths. We find possible continuous transitions between four different states, as described in the main text. m_1, m_2 are phenomenological parameters in the effective theory and drive the two types of Higgs transitions. They depend on interlayer tunneling and inter- or intralayer interactions in a way which requires further study. The minimal quasiparticle charge e^* distinguishes (b) from the others. The others must be distinguished in principle through detecting phase transitions in the neutral sector, or through tunneling or interferometry measurements.

i indexes the lowest Landau level orbitals, extended in the x direction and localized in the y direction, $g_{r,0} = 1, r^2$ when $n = 2$, 3, respectively. Since H_n involves only commuting number operators ($\hat{n}_i$'s), it can be immediately diagonalized. At 1/3 filling, the following charge-density-wave patterns of electrons in the occupation basis minimize H_3: $|g\rangle_1 = |100100100\cdots\rangle$, $|g\rangle_2 = |010010010\cdots\rangle$, $|g\rangle_3 = |001001001\cdots\rangle$. In the two-dimensional limit, these three ground states evolve into the three topologically degenerate ground states of the 1/3 Laughlin state on a torus [25–27].

The fractional quasiparticles can be understood as domain walls between these different patterns. For example, there is an excitation with $q = e/3$ charge at the domain wall between the [100] and [010] patterns, i.e., $[100][010] \equiv [\ldots 1001001\mathbf{00}|\mathbf{0}10010\ldots]$, because there are three consecutive zeros, which leads to a deficit of charge $e/3$, according to the Su-Schrieffer counting [24]. The same is true for [010][001] and [001][100] patterns. In general, the domain wall between the ground states $|g\rangle_i$ and $|g\rangle_{(i+k)\%n}$ corresponds to a quasiparticle with electric charge $q = ke/n$.

Now let us consider a double layer system, consisting of two identical layers, in the presence of interlayer tunneling. In the thin torus limit,

$$H_{tt} = \sum_{\substack{i \\ 0<r<n}} (U^{\alpha,\beta}_{r,0} \hat{n}_{i\alpha} \hat{n}_{i+r\beta} - t^{\perp} c^{\dagger}_{i\alpha} \sigma^{x}_{\alpha\beta} c_{i\beta} + \mathrm{H.c.}), \quad (2)$$

where $\alpha, \beta = \uparrow(\downarrow)$ refers to the top (bottom) layer,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$U^{\uparrow,\uparrow}_{r,0} = U^{\downarrow,\downarrow}_{r,0}$, and $U^{\uparrow,\downarrow}_{r,0} = U^{\downarrow,\uparrow}_{r,0}$ parametrize the intra- and interlayer interactions, respectively. H_{tt} is invariant under the Z_2 layer exchange symmetry $c_{i\uparrow} \leftrightarrow c_{i\downarrow}$. In the two-dimensional system, as long as interlayer tunneling $t^{\perp}$ is much smaller than the bulk gap in each layer, no phase transition is expected. As interlayer tunneling is increased, at some point the bulk gap can close and reopen in a different topological phase. In order to understand the resulting phase in a tractable limit, we will study the effect of interlayer tunneling in (2).

For simplicity, let us first consider only vertical tunneling, $t^{\perp}$ and ignore the interlayer interactions, $U^{\uparrow\downarrow}_{r,0} = U^{\downarrow\uparrow}_{r0} = 0$. When $t^{\perp} = 0$, H_{tt} has 9 exactly degenerate ground states, with the degeneracy protected by the independent translation symmetries in each layer, and the layer exchange symmetry. When $t^{\perp} \neq 0$, the independent translation symmetries in each layer reduce to a single combined translation symmetry. The 3 Z_2 layer symmetric states, which we can label as

$$|D_1\rangle \equiv \begin{bmatrix} 100 \\ 100 \end{bmatrix},$$

$$|D_2\rangle = \begin{bmatrix} 010 \\ 010 \end{bmatrix},$$

$$|D_3\rangle = \begin{bmatrix} 001 \\ 001 \end{bmatrix},$$

then acquire an energy splitting relative to the remaining 6 Z_2 layer symmetry-breaking ground states. As $t^{\perp}$ is increased further, we find that the energy gap closes, and the 1D system passes through an Ising phase transition [28]. On the other side of the transition, there are 3 exactly degenerate ground states that are fully symmetric under the Z_2 layer exchange symmetry. Deep in this Z_2 symmetric phase, we can represent these states by a product over the state in each three-site unit cell: $|O_1\rangle \equiv \prod_{a=1}^{N_{\mathrm{uc}}} |\psi_1\rangle_a$, where N_{uc} is the number of unit cells. Since $t^{\perp}_r \propto \delta_{r0}$, we have

$$|\psi_1\rangle = \alpha_1 \left| \begin{matrix} 100 \\ 010 \end{matrix} \right\rangle + \alpha_2 \left| \begin{matrix} 010 \\ 100 \end{matrix} \right\rangle + \alpha_3 \left| \begin{matrix} 110 \\ 000 \end{matrix} \right\rangle + \alpha_4 \left| \begin{matrix} 000 \\ 110 \end{matrix} \right\rangle, \quad (3)$$

where the other states are related by translations: $T_y|O_i\rangle = |O_{i+1}\rangle$. Here, α_j, $j = 1, \ldots, 4$ are variational parameters, chosen to minimize the ground state energy.

Therefore, for large enough interlayer tunneling, the 9 states that we started with split into 3 degenerate Z_2 symmetric states $\{|O_i\rangle\}$, with energy E_S, 3 degenerate states $\{|D_i\rangle\}$ with energy E_D, and 3 remaining degenerate Z_2 antisymmetric states, with energy E_A. Now, we can consider two distinct possibilities as we take the two-dimensional limit: either the 6 states $\{|O_i\rangle, D_i\rangle\}$ continuously evolve into 6 topologically degenerate ground states with a gap to other excited states, or only 3 of the states (e.g., $\{|O_i\rangle\}$), evolve into 3 topologically degenerate ground states. Based on previous studies of the thin torus limit of the FQH states [19–22], we expect that the former case will likely occur when $E_S \approx E_D \ll E_A$, while the latter case will occur in the regime $E_S \ll E_D, E_A$. Depending on parameters, H_{tt} can access either regime; for example, $U_{0,0}^{\uparrow,\downarrow} < U_{1,0}^{\uparrow,\downarrow}$ or longer range tunneling can favor the former case over the latter.

In what follows, we focus on the possibility where all six states, $\{|O_i\rangle, |D_i\rangle\}$, evolve into six topologically degenerate ground states in the 2D limit. The feasibility of this depends on microscopic details of the 2D system. This appears to be a reasonable assumption because the results are in remarkable agreement with the effective field theory considerations presented below, and the earlier approach in [10,11]. Additionally, the same assumption, when applied to the case of the (331) Halperin state, or the bosonic (220) state, yields results which agree with previous work [28] [23,29–33].

It is natural to relabel the 6 ground states as follows: [200], [020], [002] denote $|D_i\rangle$, for $i = 1, 2, 3$, respectively, and [110], [011], [101], denote $|O_i\rangle$, for $i = 1, 2, 3$. Below, our goal is to identify the type of topological order associated with this phase.

First, observe that the total center of mass degeneracy (associated with translations T_y), only accounts for a degeneracy of 3 Therefore, the existence of 6 states immediately signals the existence of a non-Abelian FQH state. Recall that the quasiparticles can be understood as domain walls between the different ground state patterns. If we start with the state [200] and consider a domain wall with the state [110], then from the Su-Schrieffer counting argument we see that there is a charge $e/3$ quasihole. This can be understood as the original Laughlin $e/3$ quasihole, but inserted in either the top layer or the bottom layer, with equal weight. If instead we start with the state [110] and consider a domain wall with either [020] or [101], we see that there is again a charge $e/3$ quasihole. In general, we can ask which pairs of ground states, labeled i and j, give rise to a charge $e/3$ quasihole at their domain wall. This defines an adjacency matrix for the charge $e/3$ quasihole,

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{4}$$

where the rows and/or columns correspond to [200], [020], [002], [110], [101], [011], respectively. More generally, let us consider $n_{\rm qh}$ quasiholes with $q = e/3$ at positions $j_1, j_2, \ldots, j_{n_{\rm qh}}$ [34]. To do so, we start with a fixed ground-state pattern, say [200]. At site j_1, there is a domain wall with [110], at site j_2 there can be either [020] or [101] patterns, and so on. We see that the number of possibilities grows exponentially with $n_{\rm qh}$. It is straightforward to verify that $\mathbf{tr}(A^{n_{\rm qh}})$ gives the total number of different possibilities on the torus. Therefore, the degeneracy of the ground state in the presence of $n_{\rm qh}$ quasihole insertions grows as $\lambda_1^{n_{\rm qh}}$ where λ_1 is the dominant eigenvalue of the adjacency matrix A. Consequently, the quantum dimension of the quasihole operator with minimum electric charge is λ_1. Using the above adjacency matrix, the quantum dimension of the charge $e/3$ quasihole is the golden ratio $d_{\rm qh} = F \equiv (1+\sqrt{5})/2$.

Since there are 6 degenerate ground states on the torus, there are correspondingly 6 topologically distinct types of quasiparticles. These include the $e/3$ quasiparticle described above and it's charge $-e/3$ particle-hole conjugate. There are also charge $2e/3$ and $4e/3$ quasiparticles, which involve inserting charge $e/3$ or $2e/3$ Laughlin quasiparticles into both layers simultaneously. We will label them as V_n, with charge $q = 2ne/3$. These are inherited directly from the (330) state, with their topological properties unchanged. Finally, there is a neutral quasiparticle, which we label τ. τ can be understood as inserting an $e/3$ quasiparticle in the top layer and a $-e/3$ quasiparticle in the bottom layer, superposed with reverse process, $-e/3$ and $e/3$ in the top and bottom layers, respectively. By studying the adjacency matrix, we find that the quantum dimension of τ is also $d_\tau = F$.

The adjacency matrices A_i, for $i = 1, \ldots, 6$, encode the fusion rules of the quasiparticles, $i \times j = \sum_k (A_i)_{jk} k$, which dictates the number of ways quasiparticles i and j can fuse into k [34]. We find that the quasiparticles V_a are simple Abelian quasiparticles, with quantum dimension 1: $V_a \times V_b = V_{a+b\%3}$. Furthermore, $\tau \times \tau = 1 + \tau$; this is the fusion rule of the famous Fibonacci quasiparticle, whose braiding statistics allows for universal topological quantum computation [1]. The remaining two quasiparticles are identified with $V_a\tau$, for $a = 1, 2$.

In addition to the fusion rules, we can obtain information about the topological spins. Since the theory has a subset of quasiparticles, $\{1, \tau\}$, with a closed fusion subalgebra $\tau \times \tau = 1 + \tau$, mathematical consistency [35] requires that the topological spin of τ be $\theta_\tau = \pm 2/5$. Furthermore, the quasiparticles V_a are just the simple Abelian quasiparticles that were present in the (330) state. Since the phase transition occurs entirely within the neutral sector, the topological spins of these charged quasiparticles should remain unchanged and are given by their value in the (330) state. Therefore, $\theta_{V_a} = a^2/3$. These results are summarized in Table I.

Generalizing the above arguments to the $(nn0)$ states gives $n(n+1)/2$ quasiparticles, whose fusion rules coincide with the representation algebra of the quantum group $SU(n)_2$ [28]. Remarkably, the thin torus patterns [200], [020], [002], [110], [011], [101], and the connection to $SU(3)_2$, have appeared previously in a completely different context [36], in terms of the gapless, single-layer bosonic Gaffnian wave function. See also [37,38] for other distinct realizations of $SU(3)_2$ fusion rules.

TABLE I. The anyon content of the non-Abelian state obtained from the (330) state with strong interlayer tunneling. Plus (minus) sign denotes the two possibilities consistent with results obtained from the thin torus limit, and correspond to the chirality of the non-Abelian sector, where the full edge theory has central charge $c = 2 \pm 4/5$. The CS-Higgs theory fixes the $c = 14/5$ case. F is the golden ratio, $(1+\sqrt{5})/2$.

	Label	Charge (mod e)	Topological Spin	Quantum Dim.
1	V_0	0	0	1
2	V_1	$2e/3$	$1/3$	1
3	V_2	$e/3$	$1/3$	1
4	τ	0	$\pm 2/5$	F
5	$V_1\tau$	$2e/3$	$1/3 \pm 2/5$	F
6	$V_2\tau$	$e/3$	$1/3 \pm 2/5$	F

The above results can be understood from the perspective of the edge conformal field theory. Consider two free chiral bosons, φ_1 and φ_2, such that $e^{in\varphi_1}$, $e^{in\varphi_2}$ are considered to be local electron operators. In the $(nn0)$ state, $V_{a,b} \equiv e^{ia\varphi_1 + b\varphi_2}$, for $a, b = 0, \ldots, n-1$ correspond to the n^2 nontrivial quasiparticle operators. If we consider the $n(n+1)/2$ symmetrized operators $\Phi_{a,b} = V_{a,b} + V_{b,a}$, and continue to treat the operators $e^{in\varphi_1}$, $e^{in\varphi_2}$ as trivial, local operators, then we find the remarkable result that $\Phi_{a,b}$ satisfy the fusion rules of $SU(n)_2$: $\Phi_{a,b} \times \Phi_{a',b'} = \Phi_{a+a',b+b'} + \Phi_{a+b',b+a'}$. Recovering the topological spin from this procedure is more involved, as the stress-energy tensor in the CFT also changes through this transition.

Effective field theory.—Here, we show that there is a possible continuous phase transition between this non-Abelian FQH state and the (330) state. We show that from the point of view of the effective field theory of the (330) state, the appearance of the state we have found is quite natural in the presence of interlayer tunneling.

One way of understanding the effective field theory of the (330) state is through a parton construction [39], where we write the electron operator as $c_\sigma = f_{1\sigma} f_{2\sigma} f_{3\sigma}$, where $\sigma = \uparrow, \downarrow$ is the layer index, and $f_{i\sigma}$ are charge $e/3$ fermionic "partons." This rewriting of the electron operator introduces an $SU(3) \times SU(3)$ gauge symmetry, associated with the transformations $f_\sigma \to W_\sigma f_\sigma$, for $W_\sigma \in SU(3)$, which keep all physical operators invariant. The theory in terms of electron operators can therefore be replaced by a theory in terms of the partons $f_{a\sigma}$, coupled to an $SU(3)$ gauge field, A_σ. In the presence of a magnetic field B, the partons feel an effective magnetic field $B_{\rm eff} = B/3$. When the electrons are at filling $1/3$, the partons are then poised to form a $\nu = 1$ integer quantum Hall state at the mean-field level. Integrating out the partons then gives an $SU(3)_1 \times SU(3)_1$ CS gauge theory: $\mathcal{L} = (\epsilon^{\mu\nu\lambda}/4\pi)\sum_\sigma \mathrm{tr}(A^\sigma_\mu \partial_\nu A^\sigma_\lambda + \frac{2}{3} A^\sigma_\mu A^\sigma_\nu A^\sigma_\lambda) + j_\sigma \cdot A^\sigma$. j_σ is the current of quasiparticles, which, after integrating out the partons, appear in this theory as classical "test" charges. They correspond to the fermionic particles or holes in the parton Landau levels, and acquire fractional statistics after being dressed by the CS gauge field.

Next, let us consider the effect of interlayer tunneling, $\delta\mathcal{H}_t = -t^\perp c^\dagger_\uparrow c_\downarrow + \mathrm{H.c.} = -t^\perp (f_{1\uparrow} f_{2\uparrow} f_{3\uparrow})^\dagger f_{1\downarrow} f_{2\downarrow} f_{3\downarrow} + \mathrm{H.c.}$, on the mean-field state of the partons. For $t^\perp$ large enough, this induces a nonzero expectation value $\langle f^\dagger_\uparrow f_\downarrow \rangle \neq 0$, which breaks the gauge symmetry $SU(3) \times SU(3) \to SU(3)$, leaving a single gauge field $A \equiv A^\uparrow = A^\downarrow$ at long wavelengths. Now, integrating out the partons leads to a $SU(3)_2$ CS gauge field: $\mathcal{L}_{\mathrm{CS},\sigma} = (2/4\pi)\epsilon^{\mu\nu\lambda} \mathrm{tr}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda)$. At the critical point, only the fluctuations of the electrically neutral operator $f^\dagger_\uparrow f_\downarrow$ will be massless. Consequently, charged fluctuations remain gapped across the transition.

The edge CFT of the parton mean field states is described by a $U(6)_1$ chiral Wess-Zumino-Witten CFT. Implementing the projection onto the physical degrees of freedom yields a $U(6)_1/SU(3)_2$ coset theory, with central charge $c = 14/5$. We can systematically obtain the topological properties of the quasiparticles in this theory [28]. Remarkably, the result coincides with the $SU(3)_2$ fusion rules obtained from the thin torus limit above, and the topological spins match those of Table I exactly, with the choice $\theta_\tau = 2/5$. We conclude that there exists a continuous phase transition between these two phases, associated with the Chern-Simons-Higgs transition $SU(3)_1 \times SU(3)_1 \to SU(3)_2$. The generalization to $(nn0)$ states gives $SU(n)_1 \times SU(n)_1 \to SU(n)_2$ CS-Higgs transitions, all of which match results obtained from symmetrizing the thin torus patterns. The case $n = 2$ is related to [40]; it is closely related to, but distinct from, the theory of [31,41], since the edge theory of the non-Abelian state in this case is $U(4)_1/SU(2)_2 \neq SU(2)_2$.

The parton construction suggests wave functions that capture the universal features of this state. In a continuum system, a natural ansatz is $\mathcal{P}_{LLL}(\Phi_{\nu=2})^3$, where $\Phi_{\nu=2}$ is a wave function where the two lowest symmetric Landau levels are filled, and $\mathcal{P}_{LLL}$ is the projection onto the lowest Landau level. On a lattice, one can consider $\Phi_{C=2}(\{r_i\})^3$ [42], where $\Phi_{C=2}(\{r_i\})$ is a wave function for a band insulator with Chern number 2.

Global phase diagram.—The above field theoretic understanding helps us understand the relation of this non-Abelian state to the Z_4 Read-Rezayi (RR) state, which can also continuously transition to the (330) state [16,43]. As was shown in [17,44], the Z_4 RR state can be understood in terms of $[SU(3)_1 \times SU(3)_1] \rtimes Z_2$ CS gauge theory. Here, the meaning of the $\rtimes Z_2$ is that the symmetry of interchanging the two $SU(3)$ gauge fields is itself promoted to a local gauge symmetry. The transition from the Z_4 RR state to the (330) state can be understood as a Z_2 gauge symmetry breaking transition: $[SU(3)_1 \times SU(3)_1] \rtimes Z_2 \to SU(3)_1 \times SU(3)_1$. Combining this with the result above, we see that there are four closely related phases that are separated by continuous phase transitions (see Fig. 1).

Returning to the thin torus Hamiltonian (2), in the limit where $t^{\perp}$ is the largest energy scale, the electrons only occupy the symmetric orbitals on each site, with two electrons per unit cell. In this limit the ground state will be threefold degenerate. Since this degeneracy can be accounted for by center of mass translations T_y, the resulting state is Abelian and corresponds to the particle-hole conjugate of the $1/3$ Laughlin state. A similar result is obtained in the context of the (331) state [32].

Conclusion.—At the transition between the (330) state and the non-Abelian states, the charge gap remains open while the neutral gap closes. Past experiments [45–47], which have probed the $\nu = 2/3$ phase diagram in bilayers through resistivity measurements, were directly sensitive only to the charge gap and thus have not yet definitively ruled out these exotic non-Abelian states and transitions.

In the Supplemental Material [28], we discuss a different "coupled wire" approach [9,10,48] and show the remarkable agreement with the results presented above, we provide additional details and generalizations of our analyses, and we discuss the duality between the (nnl) bilayer state with interlayer pairing and the $(n, n, -l)$ state with interlayer tunneling.

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[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.113.236804 for the coupled wires approach and the remarkable agreement with the results presented in the main text. We also provide more details about the non-Abelian phase, and discuss various generalizations of the problem discussed in this Letter.
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