States of Carbon-12 in the Skyrme Model

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The Skyrme model has two Skyrmion solutions of baryon number 12, with D_{3h} and D_{4h} symmetries. The first has an equilateral triangular shape and the second an extended linear shape, analogous to the triangle and linear chain structures of three alpha particles. We recalculate the moments of inertia of these Skyrmions, and deduce the energies and spins of their quantized rotational excitations. There is a good match with the ground-state band of carbon-12, and with the recently established rotational band of the Hoyle state. The ratio of the root mean square matter radii also matches the experimental value.

Introduction.—Static Skyrmion solutions of the Skyrme model with $B = 12$, where B is the baryon number, have been established for some time [1]. The solutions can be interpreted as three $B = 4$ Skyrmions bound together. The $B = 4$ Skyrmion, which has cubic symmetry [2,3], is particularly stable, and is a building block for many further Skyrmions with B a multiple of 4. The $B = 4$ solution is illustrated in Fig. 1, together with a deformed configuration of slightly higher energy that is a tetrahedral arrangement of four $B = 1$ Skyrmions, analogous to the conventional picture of two protons and two neutrons in an alpha particle. $B = 12$ Skyrmions can be found numerically by allowing a symmetric arrangement of three $B = 4$ Skyrmions to relax to a minimal energy solution. The initial arrangement may have the D_{3h} symmetry of an equilateral triangle, or a straight chain structure, with D_{4h} symmetry. In both cases, neighboring cubes are oriented so as to strongly attract. The relaxed solutions retain the initial symmetries and have almost identical final energies. They are shown in Figs. 2 and 3. There is an energy barrier between these solutions, because the cubes have to be moved apart a little, and rotated, to pass from one solution to the other. The chain solution is one member of a family of chain solutions made from any number of $B = 4$ cubes [4]. The limiting, infinitely long chain has a 45° twist symmetry, and can be split into cubes in two independent ways.

Details of the Skyrme model [6] have been reviewed elsewhere [7–9]. Briefly, it is a nonlinear field theory of pions where baryons are topological solitons. It is a type of nonlinear sigma model, with three pion fields $\boldsymbol{\pi}(\mathbf{x}, t)$ combined into an SU(2)-valued Skyrme field

$$U(\mathbf{x}, t) = \sigma(\mathbf{x}, t)\mathbf{1}_2 + i\boldsymbol{\pi}(\mathbf{x}, t) \cdot \boldsymbol{\tau}. \qquad (1)$$

The field σ is not independent, because of the constraint $\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1$.

The Lagrangian in Skyrme units is

$$L = \int \left\{ -\frac{1}{2}\text{Tr}(R_\mu R^\mu) + \frac{1}{16}\text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right.$$
$$\left. + m^2\text{Tr}(U - \mathbf{1}_2) \right\} d^3x, \qquad (2)$$

where $R_\mu = (\partial_\mu U)U^\dagger$, and m is the dimensionless pion mass. Physical units are obtained by fixing an energy and length scale appropriate to nuclear physics. The value of m has been suggested to be in the range of 0.68 to 1.15 [9]. This range of values gives a reasonable match to a range of nuclei. Here, we fix $m = 1$. At the end of this Letter, we will discuss the effect of changing m to 0.7. The Lagrangian splits into a kinetic part, quadratic in time derivatives of U, and a static potential part. Skyrmions are minima of the potential energy and are labeled by their baryon number B, the topological degree of the field $U : \mathbb{R}^3 \to \text{SU}(2)$ at a given time, which is well defined for fields satisfying the boundary condition $U \to \mathbf{1}_2$ at spatial infinity. The baryon density is
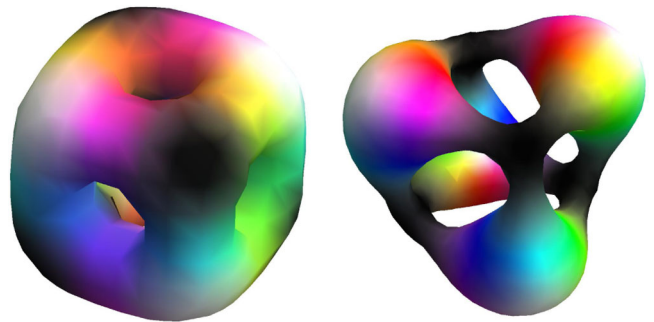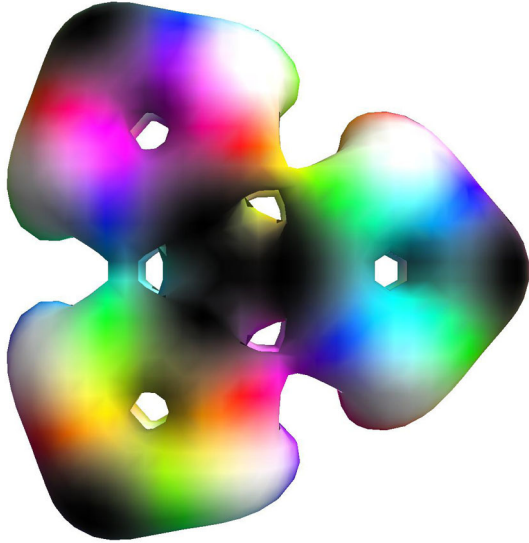

FIG. 1 (color online). $B = 4$ Skyrmion (left) and deformed configuration (right). The figures show a constant baryon density surface and are colored using Runge's color sphere. The colors indicate the value of the unit pion field $\hat{\boldsymbol{\pi}} = \boldsymbol{\pi}/|\boldsymbol{\pi}|$ [5].

FIG. 2 (color online). $B = 12$ Skyrmion with D_{3h} symmetry.

$$\mathcal{B} = -\frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}(R_iR_jR_k), \qquad (3)$$

and B is the spatial integral of this. Here, μ, ν are spacetime indices and i, j, k are spatial indices.

To model nucleons and nuclei, one quantizes the Skyrmions as rigid bodies [10]. Some vibrational excitations may also be included, but it is not practicable to treat the Skyrme model as a quantum field theory. In any case, one hopes that in the low energy regime of nuclear physics, where free pion particles are not produced, a finite dimensional truncation of the Skyrme model is sufficient. The Skyrme Lagrangian is invariant under rotations in space (it has full Poincaré invariance but this is not needed here), and isorotations $U \to A(t)UA(t)^\dagger$, where $A \in \mathrm{SU}(2)$. Isorotations mix the pion fields among themselves. A Skyrmion's mass is its field potential energy, and it has a moment of inertia tensor that arises from the field kinetic energy when the Skyrmion rotates and isorotates. The formulas for the inertia tensors are now well known, but are rather complicated [11].
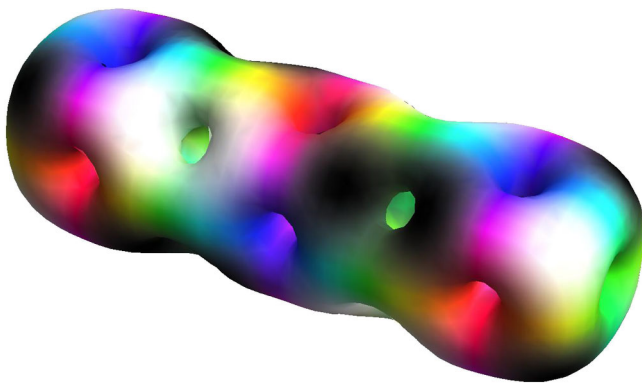
FIG. 3 (color online). $B = 12$ Skyrmion with D_{4h} symmetry.

A quantized Skyrmion acquires spin and isospin. For even baryon numbers, these are integral. For the D_{3h}-symmetric $B = 12$ Skyrmion, the rotational and isorotational motions are weakly coupled, but we ignore this, as it has a negligible effect on energy levels. There is no such coupling for the D_{4h}-symmetric Skyrmion. We are mainly interested here in the states of isospin 0, corresponding to carbon-12, and just present the quantum Hamiltonian for purely rotational motion. The quantized D_{3h}-symmetric Skyrmion models the 0^+ ground state of carbon-12 and its rotational excitations. The quantized D_{4h}-symmetric Skyrmion is identified with the 0^+ Hoyle state and its rotational excitations [12–14]. We discuss below whether this identification is reasonable.

Quantizing the $B = 12$ Skyrmions.—The two Skyrmions of interest, shown in Figs. 2 and 3, have moment of inertia tensors of symmetric-top type, with distinct eigenvalues $V_{11} = V_{22}$, and V_{33}. The 3-axis is the C_3 or C_4 symmetry axis. V_{33} is larger than V_{11} for the oblate, triangular solution, and smaller for the prolate, chain solution.

In both cases, the quantum Hamiltonian for rotational motion is

$$H = \frac{1}{2V_{11}}\mathbf{J}^2 + \left(\frac{1}{2V_{33}} - \frac{1}{2V_{11}}\right)K^2, \qquad (4)$$

where $\mathbf{J}$ is the quantum spin operator and K its projection along the (body-fixed) 3-axis. The energy eigenvalues are simply

$$E(J,k) = C\left\{\frac{1}{2V_{11}}J(J+1) + \left(\frac{1}{2V_{33}} - \frac{1}{2V_{11}}\right)k^2\right\}, \qquad (5)$$

where J is the total spin label, and k the eigenvalue of K, with the projection chosen so that $0 \leq k \leq J$. C is a conversion factor from Skyrme units to physical units.

The moments of inertia for the D_{3h} triangular Skyrmion were calculated before [9], and we have confirmed them. They are $V_{11} = 5039$ and $V_{33} = 7689$. The numerical errors are of order ± 3. The allowed rotational states for the D_{3h} Skyrmion are exactly the same as for an equilateral triangle of three identical, bosonic alpha particles. k must be zero or a multiple of 3. Up to spin 6 the allowed states have spin and parities $J^P = 0^+, 2^+, 3^-, 4^-, 4^+, 5^-, 6^+, 6^-, 6^+$, where the negative parity states have $k = 3$ and the second 6^+ state has $k = 6$. The first two of these represent the ground state and first excited state of carbon-12. The D_{3h} symmetry was fairly clear from the experimentally observed states up to spin 4, after the existence of the 4^- state was clarified [15,16], and has been further confirmed by the recent observation of the 5^- state [17]. Higher energy states of carbon-12 with isospin 0 have been seen, but none are yet established as having spin 6 [13]. In Fig. 4 we plot the energies and J^P values of the observed states and the best fit of the formula (5) to the 0^+, 2^+, and 4^+
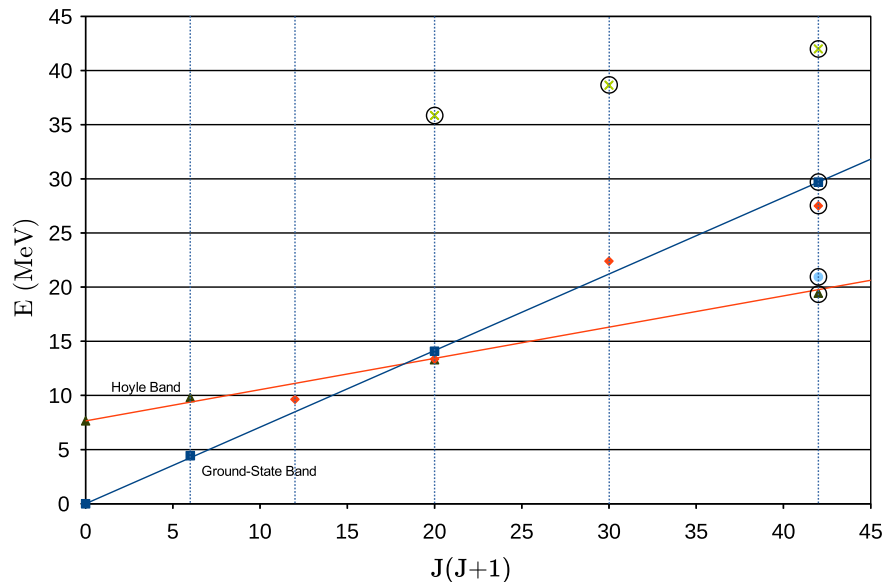
FIG. 4 (color online).   Experimental states of carbon-12. New states predicted by the Skyrme model are encircled. The symbols square, diamond, and circle denote the $k = 0, 3, 6$ states of the ground-state band, and triangle and cross denote the $k = 0, 4$ states of the Hoyle band. The 0^+, 2^+, 4^+ states have energies 0, 4.4, 14.1 MeV for the ground-state band and 7.65, 9.8, and 13.3 MeV for the Hoyle band [13].

states of the ground-state band. C is the fitted parameter, and has value 7130 MeV. Because $V_{33}/V_{11} = 1.53$ in the Skyrme model, the states in the $k = 3$ band are predicted to be about 2 MeV below those of the same spin in the $k = 0$ band. The data are marginally consistent with this. The three spin 6 states are characteristic of any model involving rigid rotations of an equilateral triangle, although their precise energy ratios depend on V_{33}/V_{11}.

Hoyle states.—Because the ground state and Hoyle state of carbon-12 are both 0^+ states, their energies are simply the classical Skyrmion energies in our approach to the Skyrme model. Our best estimate is that the energies of the D_{3h} and D_{4h} Skyrmions are, respectively, 1816 and 1812 in Skyrme units, with numerical errors that may be as large as 0.2%. The difference of these large numbers is very uncertain. As a result, we cannot confirm with any accuracy at all the 7.65 MeV energy difference between the Hoyle state and the carbon-12 ground state, which is just 0.07% of the total rest energy of carbon-12. Figures 2 and 3 show that the D_{4h} Skyrmion has two very strong bonds between the three $B = 4$ cubes, whereas the D_{3h} Skyrmion has three rather weaker bonds, because the cubes are differently oriented. As the $B = 4$ Skyrmion has energy 613, the classical bond energy is about 10, so it is not unreasonable that the ground and Hoyle states are close in energy, but we cannot be more precise, even about their ordering. Altogether, there is a problem in the Skyrme model concerning nuclear binding energies, which we will not address here.

However, we can with some confidence study the slope of the rotational band based on the Hoyle state, and

compare it to the slope of the band based on the carbon-12 ground state, because these depend on the moments of inertia. The moments of inertia for the D_{4h} chain Skyrmion were only estimated previously [18]. The estimate used the parallel axis theorem applied to three separated, undeformed $B = 4$ Skyrmions. We have calculated these moments of inertia properly for the first time, and find $V_{11} = 12699$ and $V_{33} = 2106$. V_{11}, in particular, is rather greater than what was estimated (after extrapolating to $m = 1$). For each of the $k = 0$ bands there are observed $J^P = 0^+$, 2^+, 4^+ states. The experimental energies of the Hoyle state and its excitations are included in Fig. 4. The Skyrme model prediction for the ratio of the slopes is just the ratio of the V_{11} values for the D_{4h} and D_{3h} Skyrmions, which is $12699/5039 = 2.52$. The dimensional conversion factor C cancels. We estimate the ratio of the experimental slopes from the best linear fit to the 0^+, 2^+, and 4^+ states in Fig. 4. For the ground-state band the best fit slope is 0.707 MeV, and for the Hoyle band it is 0.289 MeV. The ratio is 2.45, agreeing with the prediction.

The ratio 2.52 reflects the extended structure and separation of the $B = 4$ Skyrmion subunits. For three ideal point alpha particles with a fixed bond length separating them, arranged as an equilateral triangle or as a linear chain, the ratio of the V_{11} values is 4. In the Skyrme model the ratio is smaller, partly because the bond is less tight in the D_{3h} Skyrmion, but mainly because of the extended form of the $B = 4$ cubes.

In total, we predict five states of spin 6, three in the carbon-12 ground-state band (with $k = 0, 3, 6$) and two in the Hoyle band (with $k = 0, 4$). Only the $k = 3$ state has

negative parity. The predicted energies are, respectively, $E = 29.7, 27.5, 20.9, 19.4$, and 42.0 MeV. The Hoyle state excitations with $k = 4$, starting with 4^+ and 5^+ states, are of considerably higher energy because of the strongly prolate nature of the D_{4h} Skyrmion.

Matter radii and vibrational modes.—The root mean square matter radius $\langle r^2 \rangle^{\frac{1}{2}}$ provides a further test of the Skyrme model. For each Skyrmion we can calculate

$$\langle r^2 \rangle^{\frac{1}{2}} = \left(\frac{\int \rho(\mathbf{x}) r^2 d^3 x}{\int \rho(\mathbf{x}) d^3 x} \right)^{\frac{1}{2}}, \qquad (6)$$

where $\rho(\mathbf{x})$ is the static energy density, interpreted as a mass density, and $r = |\mathbf{x}|$. We find that the D_{3h} and D_{4h} Skyrmions have $\langle r^2 \rangle^{\frac{1}{2}}$ values 2.29 and 2.80, respectively, in Skyrme units. The ratio is 1.22. The experimental matter radius for the ground state is 2.43 fermi [19] and for the Hoyle state it is inferred to be 2.89 fermi [20]. Their ratio is 1.19. Again the prediction of the Skyrme model is very good.

Replacing ρ by the baryon density $\mathcal{B}$ gives radii 1% smaller, as the baryon density has a more compact tail than the energy density. These radii are the predictions for the root mean square charge radii of the carbon-12 ground state and Hoyle state, because, for isospin 0 states, the charge density is half the baryon density in the Skyrme model. The ratio is still 1.22. Including Coulomb energy effects may change the ratio a little.

In the point alpha particle model the ratio of matter radii is $\sqrt{2}$, and other models predict larger ratios [21]. The smaller experimental ratio has been obtained in lattice calculations of 12-nucleon states, using interactions derived from chiral effective field theory, where the Hoyle state is found to be an obtuse triangular structure [22]. The Skyrme model seems to favor a straight chain of alpha particles. The straight chain, with its large D_{4h} symmetry group, admits far fewer low-lying rotational states than an obtuse triangle, just the $J^P = 0^+$, 2^+, 4^+ states for which there is clear experimental evidence. So, arguing from the Skyrme model, we expect that Hoyle state excitations with $J^P = 3^-$ and 4^-, suggested in [17], will not be seen.

Vibrational spectra of the $B = 12$ Skyrmions have not been studied, but the lowest vibrational modes of the D_{3h} Skyrmion are likely to be the degenerate pair of triangle-deforming modes. These should give rise to the observed $J^P = 1^-$, 2^- states of carbon-12, but we do not have an estimate for their energies. The potential energy landscape is rather flat for this vibrational excitation, particularly in the direction where the triangle becomes obtuse, so the vibrations may not be close to harmonic. The first excited state of this vibrational mode could be quite similar to a state based on the obtuse triangle in a rotational excitation. The D_{3h}-symmetric breathing vibrational mode will give rise to a 0^+ state, probably of higher energy [23], and not to be identified as the Hoyle state.

Effect of changing m.—Properties of Skyrmions such as the moments of inertia and energy depend on the value of m. It is reasonable to ask how sensitive our results are to the value of m. We repeated our analysis using a value of $m = 0.7$, the choice preferred in [9]. In Skyrme units, a Skyrmion gets larger and less massive when m is reduced. But recalibration of the length and energy scale compensates for this, and the nuclear properties change rather little. We calibrated the Skyrme parameters using the rest mass and the slope of the ground-state band of carbon-12, so these quantities are the same for $m = 0.7$ as for $m = 1$. For $m = 1$ our calibration gives a pion mass of 210 MeV; for $m = 0.7$ it gives 162 MeV, which is closer to the experimental value 138 MeV.

In Skyrme units, when m is reduced from 1 to 0.7, the matter radius of the D_{3h} Skyrmion increases by 10% while the matter radius of the D_{4h} Skyrmion increases by 7%. The ratio of these radii decreases from 1.22 to 1.19, a decrease of only 3%. The new ratio matches exactly with the experiments. In Skyrme units, the moments of inertia of the Skyrmions also increase because of the increase in size. The moment of inertia V_{11} of the D_{3h} Skyrmion increases by 15% while the moment of inertia V_{11} of the D_{4h} Skyrmion increases by 7%. This gives a ratio of these moments of inertia, and hence of the slopes of the rotational bands, of 2.34 which is 7% lower than before. This ratio is about 4% lower than the experimental value, whereas the ratio for $m = 1$ is about 3% higher.

Conclusions.—We have seen that the Skyrme model gives a quantitative understanding of the spectrum of rotational excitations of carbon-12, including the excitations of the Hoyle state. The ground state is interpreted as an equilateral triangle of $B = 4$ Skyrmions, modeling three alpha particles, and the Hoyle state is a linear chain of these. Calculations of the moments of inertia and the ratio of the root mean square matter radii support the linear chain interpretation. It would be good to calculate electromagnetic transition strengths between the states, using the Skyrme model, but the technology for this needs to be developed.

We end by recalling that within the Skyrme model, isospin excitations are treated in the same way as spin excitations. The allowed isospin and spin combinations are determined by the topological Finkelstein–Rubinstein constraints [24], which encode the Pauli principle for nucleons within the Skyrme model, combined with symmetry considerations [25]. Calculations for the isospin 1 and isospin 2 excitations of the D_{3h} Skyrmion have been performed before [9]. The four lowest lying states with isospin 1 have spins $J^P = 1^+$, 1^+, 2^-, 2^+, and have energies close to the lowest observed states of boron-12, nitrogen-12, and the isospin 1 states of carbon-12. They have a total energy separation of about 2 MeV, which is correct, but the 2^+, 2^- states are in the wrong order, and two 1^+ states have not yet been resolved experimentally.

The D_{4h} Skyrmion also has isospin excitations. Formulas for the quantum Hamiltonian and its energies were found by Wood [18], but now that we have calculated the inertia tensors accurately, these energies should be recalculated. In addition to the values V_{11} and V_{33} given above, we find for $m = 1$ isospin moments of inertia $U_{11} = 437$, $U_{22} = 449$, and $U_{33} = 472$, and for $m = 0.7$, $U_{11} = 496$, $U_{22} = 529$, and $U_{33} = 537$. There are no off-diagonal terms or spin/isospin cross terms here. A key prediction is that the lowest isospin 1 excitation of the Hoyle state has $J^P = 0^-$, with energy a few MeV above the four isospin 1 states of the D_{3h} Skyrmion mentioned above.

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