Generalized Radially Self-Accelerating Helicon Beams

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We report, in theory and experiment, on a new class of optical beams that are radially self-accelerating and nondiffracting. These beams continuously evolve on spiraling trajectories while maintaining their amplitude and phase distribution in their rotating rest frame. We provide a detailed insight into the theoretical origin and characteristics of radial self-acceleration and prove our findings experimentally. As radially self-accelerating beams are nonparaxial and a solution to the full scalar Helmholtz equation, they can be implemented in many linear wave systems beyond optics, from acoustic and elastic waves to surface waves in fluids and soft matter. Our work generalized the study of classical helicon beams to a complete set of solutions for rotating complex fields.

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Self-accelerating wave packets freely accelerate without any external potential present. This intriguing phenomenon is of rapidly growing interest since its advent in optics in 2007 [1–6]. The most prominent example of self-accelerating waves has been introduced by Siviloglou and co-workers [1,2]. They demonstrated, that an Airy-type wave packet exhibits a linear transversal acceleration and is therefore following a parabolic trajectory.

Enlarging the scope and the versatility of Airy beams, nonparaxial generalizations in terms of full vectorial solutions of Maxwell's equations [7] and by the method of caustics [8] were investigated. Their curved trajectories render classical Airy beams a powerful tool in many areas of application. For instance, in the field of particle manipulation, microbeads have been guided in a new fashion [9] beyond the scope of classical optical tweezers. Moreover, it was shown that curved plasma channels have many advantages over their straight counterparts [10]. Additionally, Airy wave packets inspired fundamental research in the field of nonlinear optics [11–14] and boosted the study of waves with intensity maxima that propagate along almost arbitrary trajectories [15]. Broadening the range of influence beyond the scope of optics, Airy beams have been utilized in electron beam shaping [16] as well.

One common feature of the aforementioned waves—even in two-dimensional settings—is that they accelerate along a specific Cartesian coordinate axis [see Fig. 1(a)]. This obvious limitation brings about a number of fundamental questions. Is it possible to generate optical beams that show a self-accelerating behavior along different trajectories? Could those be shape invariant or even non-diffractive? Based on the numerous already existing applications of Airy-type beams discussed in the previous paragraph, it should be obvious that beams for which the aforementioned questions can be answered affirmatively would enrich the optical toolbox in many areas of application and research. Moreover, from a fundamental point of view, it is essential to determine under which kind of approximations analytical solutions for such beams can be found. In other words, will those solutions be restricted to the paraxial case or do they obey the scalar Helmholtz equation or even Maxwell's equations?

In the present Letter, we report on a new class of self-accelerating diffraction-free waves that move along three-dimensional spiraling trajectories. As such, they behave as if they were influenced by a radially symmetric external potential even though the propagation takes place in free space. Observed from a rotating, comoving frame of reference like the one depicted in Fig. 1(b), the beams we present are propagation invariant. While rotating diffraction-free beams are of great scientific interest for a long time [17–25], with helicon beams being the most prominent ones, so far no generalized theory has been presented providing a complete set of solutions to the Helmholtz equation that exhibit the described behavior. Moreover, most investigations content themselves with rotating intensity distributions and do not include complex fields. Since our Letter covers those points, it has profound implications on many linear wave systems in nature beyond optics where time-harmonic waves obey the Helmholtz equation, ranging from sound and elastic waves to surface

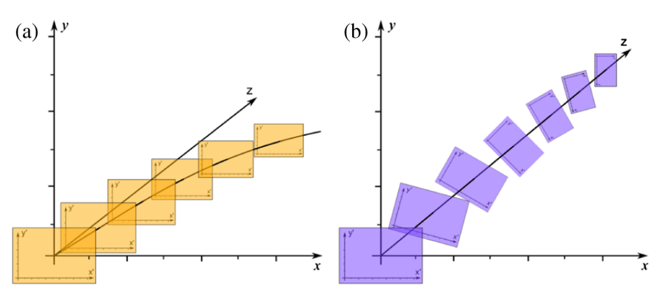

FIG. 1 (color online). Illustration showing the accelerative behavior of (a) Airy and (b) radially self-accelerating beams.

waves in fluids and soft matter. Within a thorough theoretical discourse, we will derive an analytical expression for this new class of beams, for which—while outperforming Airy-type and classical helicon beams—the transverse cross section is highly tunable. In addition, we are going to verify our theoretical findings on an experimental basis utilizing an intuitive understanding borrowed from Fourier optics. In this regard, we present a simple yet powerful setup that enables one to address the entire parameter range.

Our theoretical section is comprised of two parts. The first one consists of an extensive theoretical derivation regarding the most general expression of a beam which exhibits a field pattern that is invariant in a rotating frame of reference. As it will be discussed later, of more practical interest might be the implementation of beams with a rotating intensity distribution. For this reason, in the second part, we pose conditions on the beam intensity only finding a more general class of beams.

First, we model a complex field that is invariant in a comoving, rotating frame. This wave is supposed to be a solution to the scalar Helmholtz equation $\Delta E + k^2 E = 0$, where E is the electric field and $k = 2\pi/\lambda$ the corresponding wave number. Since we are dealing with rotating solutions, it is a natural choice to work in cylindrical coordinates. Then, the most general solution of the scalar Helmholtz equation can be written as

$$E(r,\varphi,z) = \sum_{n=-\infty}^{\infty} \int_0^\infty d\alpha\, C_n(\alpha) J_n(\alpha r) e^{i(n\varphi+\beta z)}, \quad (1)$$

which is essentially a superposition of fundamental eigenmodes given in terms of diffraction-free Bessel waves. The spatial structure of each eigenmode is determined by $J_n(\alpha r)e^{i(n\varphi+\beta z)}$, where $J_n(\alpha r)$ represents the Bessel function of order n and $\beta = \sqrt{k^2-\alpha^2}$ is the longitudinal component of the wave vector or propagation constant. For an arbitrary beam, the expansion coefficients $C_n(\alpha)$ are arbitrary as well. In the following, we will derive conditions for $C_n(\alpha)$ to obtain rotating solutions to the scalar Helmholtz equation. Note, that we restrict our analysis to beams that are propagating in the positive z direction.

For a beam that is self-accelerating, three major requirements need to be fulfilled. First, no external potential or nonlinear optical effect should be present. Second, the beam is diffraction free in a certain frame of reference. Finally, an observer resting in the aforementioned frame would experience a fictitious force. The first condition is fulfilled immediately as we start our analysis from the linear and time-independent scalar Helmholtz equation. For the second one, a coordinate transformation needs to exist for which the field distribution no longer depends on the propagation direction. An electric field of the form,

$$E(r,\varphi,z) \overset{!}{=} E(r,\varphi+\omega z), \quad (2)$$

fulfills this condition. Obviously, with the substitution $\varphi' = \varphi + \omega z$, the field is no longer dependent on the longitudinal position z and, thus, remains unchanged for every z. Moreover, the aforementioned transformation describes a reference frame that is rotating with angular velocity ω. Consequently, the last requirement is satisfied, as an observer resting in this frame experiences a centrifugal force.

Since Eq. (2) has to hold for every φ and z, from Eq. (1), it immediately follows that

$$\beta/n \overset{!}{=} \omega. \quad (3)$$

As β was restricted to be positive, the first conclusion from Eq. (3) is that the signs of n and ω have to be equal. Moreover, since β is a function of α, Eq. (3) can be rewritten

$$\alpha \overset{!}{=} \alpha_n = \sqrt{k^2 - \omega^2 n^2}. \quad (4)$$

Obviously, this restriction can only be fulfilled for the specific choice of coefficients

$$C_n(\alpha) = \tilde{C}_n \delta(\alpha - \alpha_n). \quad (5)$$

Applying the restrictions on $\mathrm{sgn}(n)$ as well as on $C_n(\alpha)$, Eq. (1) becomes

$$E(r,\varphi,z) = \sum_{n=1}^{n_{\max}} \tilde{C}_n J_n(\alpha_n r) e^{i[\mathrm{sgn}(\omega) n(\varphi+\omega z)]}. \quad (6)$$

This is the most general expression of a beam that rotates in a shape-invariant fashion with an angular velocity ω. Note that $n_{\max} = \max\{n \in \mathbb{N} : k^2 > \omega^2 n^2\}$ in order to ensure that evanescent waves are excluded from the sum. To give an intuitive description of this finding, it is helpful to consider its Fourier transformation. In essence, the Fourier transform is a discrete superposition of concentric rings with radius α_n, whereas the amplitude of these rings is given by the coefficients $\tilde{C}_n$. Note that for a given ω, Eq. (5) states that for each order n there is exactly one ring of radius α_n. Moreover, each ring carries a helical phase pitch of $2\pi n$.

The field pattern described by Eq. (6) gives rise to screw-shaped trajectories, that exhibit a nondegenerate periodicity in azimuthal and propagation direction. Figure 2 shows an appropriate example using four Bessel waves with $\tilde{C}_n = 1$ for $1 \le n \le 4$ and $\tilde{C}_n = 0$ for $n > 4$. The insets demonstrate that amplitude and phase are rotating synchronously as predicted. The angular frequency spectrum consists of four concentric rings with radii determined by Eq. (4).

At the beginning of the theory section, we indicated that for certain applications only the intensity distribution of a

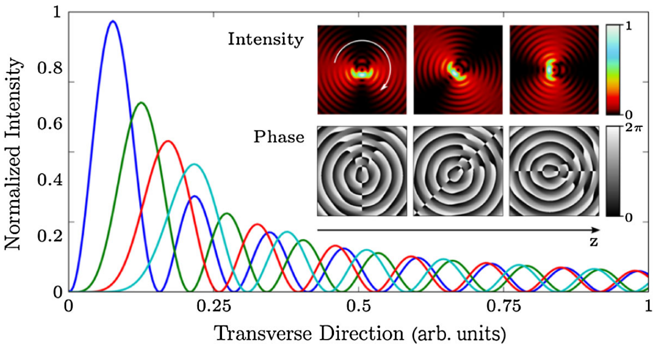

FIG. 2 (color online). Exemplary illustration of a radially self-accelerating field distribution with $\tilde{C}_n = 1$ for $1 \leq n \leq 4$ and $\tilde{C}_n = 0$ for $n > 4$. The figure consists of one-dimensional representation of the superimposed Bessel functions (main plot), resulting intensity distribution (upper inset row), and resulting phase pattern (lower inset row).

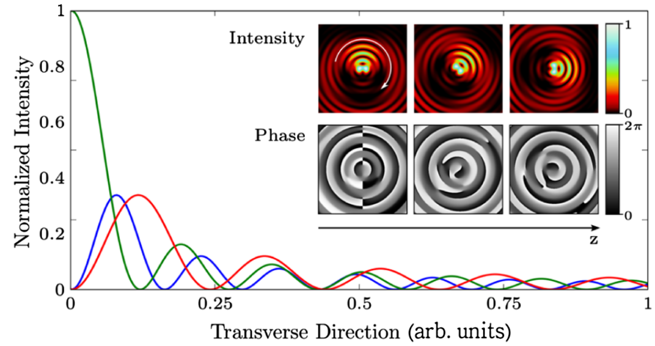

FIG. 3 (color online). Exemplary illustration of a radially self-accelerating intensity distribution with $\tilde{C}_n = 1$ for $-1 \leq n \leq +1$ and $\tilde{C}_n = 0$ for $|n| > 1$. The figure consists of one-dimensional representation of the superimposed Bessel functions (main plot), resulting intensity distribution (upper inset row), and resulting phase pattern (lower inset row).

beam might be of interest. For this reason, we want to state how the requirements for the beam profile change if $I(r,\varphi,z) = |E(r,\varphi,z)|^2 = I(r,\varphi + \omega z)$. It will be shown that this scenario is more general and offers a larger degree of freedom. Consequently, in our subsequent discussion (including our experimental section), we will concentrate on this case. As the derivation of the following results does not convey much additional physical insight, it is contained in the Supplemental Material [26]. One arrives at a constrain similar to Eq. (4), which reads

$$\alpha_n = \sqrt{k^2 - (\omega n + \beta_0)^2}. \tag{7}$$

Moreover, under these conditions, the field is given by

$$E(r,\varphi,z) = e^{i\beta_0 z} \sum_{n\in\mathcal{N}} \tilde{C}_n J_n(\alpha_n r) e^{i[n(\varphi+\omega z)]}. \tag{8}$$

Note that there are two main differences between Eq. (8) and Eq. (6). First, Eq. (8) contains the global phase factor $e^{i\beta_0 z}$ with the propagation constant β_0, which can be regarded as a free parameter. Consequently, Eq. (8) does not fulfill the requirement of a rotation invariant field anymore as only the intensity is rotation invariant. It is important to be aware of the fact that β_0 does not only determine the global phase factor but poses an important degree of freedom for scaling the transverse beam properties. This becomes apparent when comparing Eq. (4) and Eq. (7). The second difference is that the sum in Eq. (8) contains also negative n, whereas Eq. (6) covers only positive ones. To be specific, the set $\mathcal{N} = \{n \in \mathbb{Z} : k^2 > (\omega n + \beta_0)^2\}$ contains all integer numbers n for which Eq. (7) yields real values. Figure 3 shows an exemplary beam with $\tilde{C}_n = 1$ for $-1 \leq n \leq +1$ and $\tilde{C}_n = 0$ for $|n| > 1$. The depicted insets demonstrate that the intensity distribution is indeed rotating during propagation while the corresponding phase is no longer synchronized.

An important, yet open, question is how versatile the transverse cross section of beams described by Eq. (8) can be tailored. To answer this question, consider Eq. (8) in the initial plane. For a given distance R from the origin of the coordinate system, Eq. (8) can be written

$$E(R,\varphi,0) = \sum_{n\in\mathcal{N}} D_n e^{in\varphi}, \tag{9}$$

where $D_n = \tilde{C}_n J_n(\alpha_n R)$. If the distance R is chosen such that $J_n(\alpha_n R) \neq 0$, Eq. (9) represents a Fourier series. As an important consequence, the beam profile can be tailored such that the field distribution on a circle with radius R can be chosen arbitrarily, i.e., $E(R,\varphi,z=0) = f(\varphi)$, where the complex function $f(\varphi)$ can be set without any restriction. It follows that a cooking recipe to tailor these beams could be to fix β_0 as well as $f(\varphi)$. Then Eq. (7) determines α_n and the expansion coefficients are given by

$$\tilde{C}_n = \frac{1}{2\pi J_n(\alpha_n R)} \int_0^{2\pi} f(\varphi) e^{-in\varphi} d\varphi. \tag{10}$$

Note that as soon as the field is specified for one radius R, the entire field in the transverse plane is determined. This is due to the fixed radial dependence of the Bessel functions.

In order to experimentally implement our findings, we exploit that the presented beams show a multi-ring pattern with distinct helical phase pitch in the angular frequency domain. Fourier transforming this pattern by means of a conventional lens will match the previously discussed theory. For the experimental setup, different approaches are conceivable ranging from the use of axicons (conical lenses), ring slit apertures, and phase plates to the exclusive use of spatial-light modulators (SLMs). Here, we followed the last approach as it provides the highest amount of flexibility. Our setup is presented in Fig. 4 and uses a technique introduced in Ref. [27]. This enables simultaneous amplitude and phase modulation with a single

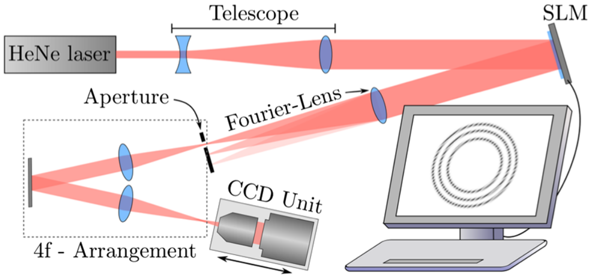

FIG. 4 (color online). Experimental setup containing a telescope for beam expansion, SLM (Holoeye Pluto VIS) for amplitude and phase modulation, lens for Fourier transformation ($f = 300$ mm), aperture and $4f$ arrangement ($f_1 = f_2 = 200$ mm) for signal cleaning, and a movable CCD unit for data acquisition.

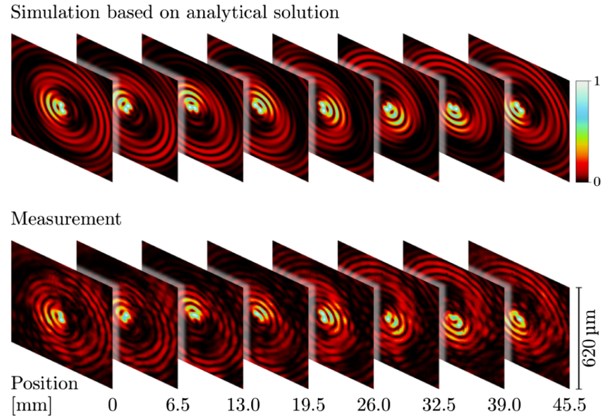

FIG. 5 (color online). Propagation dynamics of a radially self-accelerating beam identical to the one in Fig. 3. Radii on the SLM were $R_1 = 2.328$, $R_2 = 1.958$, and $R_3 = 1.501$ mm.

phase-only SLM by multiplying the desired amplitude distribution with a blazed grating. In our case, this desired amplitude distribution is given by concentric rings—in other words, we implement the Fourier transform of the desired beam in the SLM plane. After the Fourier-transforming lens, undesired grating orders are filtered by a pinhole, and the primary signal is imaged by an additional $4f$ setup. Finally, a movable CCD camera allows us to measure the change of the intensity profile in propagation direction.

With the proposed setup, we are able to cover almost the entire parameter range provided by our theory. This, of course, would go beyond the scope of this Letter. We will therefore present an exemplary set of parameters upon which we show the practicability of radially self-accelerating beams and discuss the experimental limitations. One of these limitations is already given by the fact that Bessel beams cannot be created to their full extend as they would carry an infinite amount of energy and require a nonfinite aperture. This is also true for the presented radially self-accelerating beams, since they are a specific discrete superposition of Bessel beams. Another limitation arises from the fact that the Fourier transform of a Bessel wave is a ring with infinitesimal thickness. In an experimental setting, only rings with finite width can be generated. Consequently, the range over which the generated beam resembles the theoretical prediction will be limited.

For experimental realization, the set of parameters presented in Fig. 3 was used. Hence, a superposition of three rings was implemented in the SLM plane. The subsequent Fourier lens with focal length f connects the ring radii R_n on the SLM with the transverse components of the wave vector α_n via

$$\alpha_n = \frac{kR_n}{f}. \tag{11}$$

Figure 5 shows an experimental scan along the propagation direction together with a simulation based on the analytical solution. In total, we were able to observe about two rotations over a length of 101.5 mm. See Supplemental Material [26] for an animated representation of the full scan. The rotation rate was found to be $\omega_{\mathrm{exp}} = (123.2 \pm 2.4)$ rad/m and is therefore in very good agreement with the intended value of $\omega_{\mathrm{th}} = 125$ rad/m.

In conclusion, we demonstrated a new class of self-accelerating waves. Theoretically, those waves, which accelerate freely on spiraling trajectories, were derived from the scalar Helmholtz equation. It was pointed out that they can be generated as a discrete superposition of Bessel waves with well-defined properties. As such, they are quasi-nondiffractive, meaning that they are diffraction free in a rotating, comoving frame of reference. With the proposed experimental setup, the study of beam properties was shown to be possible with great flexibility. In a first proof of principle experiment, it was verified that the beam shows indeed the desired rotating behavior—yielding excellent agreement with the theoretical predictions.

We foresee a broad range of applications for this new class of self-accelerating beams ranging from particle manipulation, e.g., as tractor beams and for sorting, mixing, and cell extraction applications, to material processing, and photolithography of three-dimensional chiral structures. To widen the range of possible applications even further, it is also of interest to study the properties of these beams on a fundamental basis. For instance, the self-healing behavior or the propagation dynamics in random media might be a worthwhile field of research. Moreover, in the field of material processing the dynamics in nonlinear environments or the behavior under strong focusing conditions might be of interest.

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