Eight Types of Symmetrically Distinct Vectorlike Physical Quantities

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The Letter draws the attention to the spatiotemporal symmetry of various vectorlike physical quantities. The symmetry is specified by their invariance under the action of symmetry operations of the nonrelativistic space-time rotation group $\mathrm{O}(3) \times \{1, 1'\} = \mathrm{O}'(3)$, where $1'$ is a time-reversal operation, the symbol $\times$ stands for the group direct product, and O(3) is a group of proper and improper rotations. It is argued that along with the canonical polar vector, there are another seven symmetrically distinct classes of stationary physical quantities, which can be—and often are—denoted as standard three-component vectors, even though they do not transform as a static polar vector under all operations of $\mathrm{O}'(3)$. The octet of symmetrically distinct "directional quantities" can be exemplified by two kinds of polar vectors (electric dipole moment **P** and magnetic toroidal moment **T**), two kinds of axial vectors (magnetization **M** and electric toroidal moment **G**), two kinds of chiral "bidirectors" **C** and **F** (associated with the so-called true and false chirality, respectively) and still another two bidirectors **N** and **L**, achiral ones, transforming as the nematic liquid crystal order parameter and as the antiferromagnetic order parameter of the hematite crystal α-Fe_2O_3, respectively.

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Physical quantities defined by a magnitude and an oriented axis in 3D space are often represented by three-component Euclidean vectors. Frequently, polar and axial (or pseudo-) vectors are distinguished, depending on whether they change their sense or not, respectively, upon the operation of spatial inversion (parity operation $\bar{1}$) [1–4]. For classification of temporal processes or magnetic phenomena of a vectorial nature, the action of the time-inversion operator ($1'$) can be used. For example, magnetization **M** and magnetic field vector **H** are "time-odd axial" vectors (preserved by the $\bar{1}$ operation but changing their sign under the $1'$ operation); electric polarization **P** or electric field **E** are "time-even polar" vectors, while other quantities, like velocity **v** or toroidal moment **T**, are "time-odd polar" vectors [1–6]. The two inversion operations generate an Abelian (commutative) group of four elements $\{1, \bar{1}, 1', \bar{1}'\}$ with four one-dimensional irreducible representations (irreps); the symmetry operations of this group allow us to classify these vectors into four categories (see Table I) [1–4].

The aim of this Letter is to emphasize that there are another four types of quantities, which are also defined by a magnitude, an axis, and a geometrical sign, and which are, also, often associated with three-component Euclidean vectors, but which possess a different spatiotemporal symmetry than the examples given in Table I. We are going to specify, here, all eight types of "directional quantities" (i) by describing their transformation properties under the action of the elements of the nonrelativistic space-time rotation group $\mathrm{O}(3) \times \{1, 1'\} = \mathrm{O}'(3)$ (where the symbol $\times$ stands for group direct product and O(3) is a group of proper and improper rotations), (ii) by enumerating the associated limiting groups defining their symmetry invariance, and (iii) by providing several examples of each case. We shall also briefly discuss possibilities and difficulties with the extension of formal algebraic operations. Simultaneous considerations about all eight different types of such directional quantities can be useful in various areas of physics.

Basic symmetry argument.—These eight symmetrically different species are presented pictographically in Fig. 1. Why do we have just eight of such quantities? Let us consider any stationary physical quantity **X** (attached to a physical object), which simultaneously defines a two-valued, geometry-related sign, a nonnegative magnitude, and a unique 1D linear subspace of 3D Euclidean space (the axis of this quantity), but nothing more. Since the quantity **X** defines a unique axis in space, the symmetry of **X** can be classified by those $\mathrm{O}'(3)$ group operations that leave this axis invariant. Such operations form an infinite subgroup of $\mathrm{O}'(3)$ that can be expressed as $\infty/\mathrm{mm} \times \{1, 1'\}$, what can be denoted as an $\infty/mm1'$ or $D'_{\infty h}$ group [5,8–11]. Moreover, it is natural to postulate that the magnitude of **X** ($|\mathbf{X}| \geq 0$) does not change under the operations of the $\mathrm{O}'(3)$ group. This implies that transformation properties of

TABLE I. Action of space ($\bar{1}$) and time ($1'$) inversion operations on selected examples of vectorial quantities: 1 stands for the invariance, -1 stands for the sign reversal[3,7,8].

1	$\bar{1}$	$1'$	$\bar{1}'$	Vectorial quantity	Symbol
1	1	1	1	Electric toroidal moment	**G**
1	−1	1	−1	Electric dipole moment	**P**
1	1	−1	−1	Magnetic dipole moment	**M**
1	−1	−1	1	(Magnetic) toroidal moment	**T**

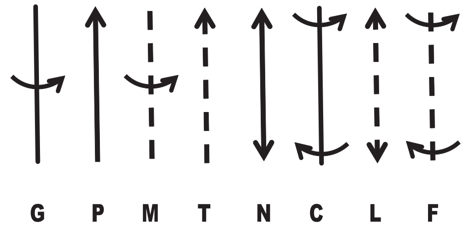

FIG. 1. Pictographs of eight kinds of quantities defined by a sign, a magnitude, and an axis. Letter symbols allow us to identify each pictograph with the symmetry assignment given in Tables I–III. Arrows in pictographs drawn with dashed lines should be considered as indicating a stationary current or motion (time inversion operation does change their sense), while arrows in the pictographs drawn with full lines are time irreversible (as, e.g., the electric polarization). Pictographs were inspired by pictures employed for similar purposes in Refs. [5,14,15].

X can be fully defined by specifying how its sign is changed when elements of $\infty/mm1'$ are applied to it. Since we restrict ourselves only to the quantities for which the sign of **X** can have only one of the two possible values, the symmetry operation can either preserve the sign or change it to the opposite one. In other words, the action of the associated $\infty/mm1'$ group operations consist of multiplication of the geometrical sign of **X** either by 1 or by -1. In terms of the theory of groups, this implies that **X** transforms as a one-dimensional (necessarily irreducible) representation of the associated $\infty/mm1'$ group. It is known that the ordinary ∞/mm ($D_{\infty h}$) group has four distinct one-dimensional irreps [12,13], so the $\infty/mm1'$ ($D_{\infty h} \times \{1, 1'\}$) one has twice as many of them. Therefore, the physical quantities defined by a sign, a magnitude, and an axis can be classified in eight symmetrically different categories.

Classification by irreps and basic examples.—The list of all one-dimensional irrep of the $\infty/mm1'$ group is given in Table II. The first column gives the irreps label following the convention used, e.g., in Refs. [12] and [13], respectively, the last column contains a letter symbol used in Table III and in Fig. 1. The remaining columns of Table II are associated with the classes of symmetry elements of the $\infty/mm1'$ group. There are various physical quantities having the listed transformation properties. For example, polarization (**P**) and magnetization (**M**) transform as $A_{1u}(\Sigma_u^+)$ and $mA_{2g}(m\Sigma_g^-)$ irreps, respectively. The symbol **T** invokes the often discussed toroidization or toroidal moment [6,16–19], even though there are many other, more frequently used, quantities that also transform as the $mA_{1u}(m\Sigma_u^+)$ representation, such as electric current, momentum or velocity of a particle, and vector potential or the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. It is clear from Table II that this "magnetic" toroidal moment **T** has a different symmetry than the "electric" toroidal moment **G**, the latter exploited, e.g., for characterization of electric polarization vortex states of small ferroelectric particles [20–22] or poloidal spin currents [23]. Recently, spontaneous magnetic toroidization **T** has been found, e.g., in

TABLE II. Characters of one-dimensional irreps for selected elements of $\infty/mm1'(D'_{\infty h})$ group. The extra dash symbol identifies the operations combined with time inversion. For example, $m'_{\parallel}$ stands for a mirror plane parallel to the axis of infinite order, combined with the time-reversal operation. Irreps are labeled similarly to that of the ∞/mm group [12,13,28], the "m" in front of the irrep label indicates the antisymmetry with respect to the time inversion, similarly to it is adopted for irrep of crystallographic grey symmetry groups [9,29] of magnetically ordered crystals [30–32].

Irrep	E ∞ $2_{\parallel}$	$\bar{1}$ $\bar{\infty}$ $m_{\perp}$	$m_{\parallel}$	$2_{\perp}$	$1'$ ∞' $2'_{\parallel}$	$\bar{1}'$ $\bar{\infty}'$ $m'_{\perp}$	$m'_{\parallel}$	$2'_{\perp}$	Symbol
$A_{1g}(\Sigma_g^+)$	1	1	1	1	1	1	1	1	**N**
$A_{2g}(\Sigma_g^-)$	1	1	−1	−1	1	1	−1	−1	**G**
$A_{1u}(\Sigma_u^+)$	1	−1	1	−1	1	−1	1	−1	**P**
$A_{2u}(\Sigma_u^-)$	1	−1	−1	1	1	−1	−1	1	**C**
$mA_{1g}(m\Sigma_g^+)$	1	1	1	1	−1	−1	−1	−1	**L**
$mA_{2g}(m\Sigma_g^-)$	1	1	−1	−1	−1	−1	1	1	**M**
$mA_{1u}(m\Sigma_u^+)$	1	−1	1	−1	−1	1	−1	1	**T**
$mA_{2u}(m\Sigma_u^-)$	1	−1	−1	1	−1	1	1	−1	**F**

$Ba_2CoGe_2O_7$ crystals [24], the **G**-type distortion has been identified, e.g., in the "ferroaxial" structures of $CaMn_7O_{12}$ and $RbFe(MoO_4)_2$ crystals [25–27].

Let us note that **G** and **M** are symmetric with respect to the perpendicular mirror plane operation $m_{\perp}$, and **P** and **T** are symmetric with respect to the parallel mirror plane $m_{\parallel}$. Thus, none of these quantities is chiral [33]. In fact, only two irreps from Table II fulfill the group theoretical condition of a chiral object (absence of improper rotation

TABLE III. List of eight symmetrically distinct "arrow" quantities and their transformation under three independent operations $\infty/mm1'(D'_{\infty h})$ group attached to the axis. ($m_{\parallel}$ stands for any mirror plane operation parallel to the axis.) Last column indicates limiting group describing the symmetry invariance of the quantity. The symbol is derived from the international (Hermann-Mauguin) symbol for the ordinary limiting group by appending additional symbol $1'$ or 1, indicating whether the $1'$ operation (alone) is or is not an element of the group, respectively. This convention is used in this Letter in order to clearly distinguish the symbols of the ordinary and the space-time symmetry groups, for example, to distinguish the ordinary limiting group ∞/mm and the (time-odd) space-time limiting group $\infty/mm1$.

		$\bar{1}$	$1'$	$m_{\parallel}$	Limiting group
G	Time-even axial	1	1	−1	$\infty/m1'$
P	Time-even polar	−1	1	1	$\infty m1'$
M	Time-odd axial	1	−1	−1	$\infty/mm'1$
T	Time-odd polar	−1	−1	1	$\infty/m'm1$
N	Time-even neutral	1	1	1	$\infty/mm1'$
C	Time-even chiral	−1	1	−1	$\infty 21'$
L	Time-odd neutral	1	−1	1	$\infty/mm1$
F	Time-odd chiral	−1	−1	−1	$\infty/m'm'1$

symmetry, such as center of inversion or mirror planes [33]): A_{2u} and mA_{2u}. They are naturally suitable for description of chiral directional quantities, as their geometrical sign can reflect the sign of their enantiomorphism. For example, a helix might be characterized by its axis, the magnitude (given by the pitch of the helix), and a geometrical sign, indicating whether the helix is right handed or left handed. Such a chiral quantity **C** transforms as an A_{2u} irrep. As a beautiful example of the mA_{2u} quantity, (**F**) can be taken as the antiferromagnetic order parameter of the linear-magnetoelectric chromite crystal Cr_2O_5 [34,35]. This latter kind of chirality, reversible upon time reversal, is sometimes called "false chirality" [33,36].

Finally, there are also two irreps symmetric with respect to both $m_\parallel$ and $m_\perp$ (**L** and **N**). The time-odd variant (**L**) can be used to describe another type of directional antiferromagnetic order parameter, e.g., in the hematite crystal α-Fe_2O_3 [35]. The fully symmetric (A_{1g}) representation is perhaps the most singular one. It can be associated with the so-called director, exploited in the theory of liquid crystals to characterize the spontaneously parallel spatial orientation of rodlike molecules in nematic phases [37]. In this particular case, there is no reason to define its geometrical sign. However, there are other **N**-like quantities that do have a sign. For example, a consistently defined Frank vector of a wedge disclination [38–40] should allow us to distinguish whether the disclination can be formed by removing or inserting a material body adjacent to the plane of the cut. At the same time, this disclination itself is invariant against all operations of the $\infty/mm1'(D'_{\infty h})$ group [38].

Classification in terms of symmetry invariance groups.— Table II fully defines transformation properties of various uniaxial quantities discussed above. For many purposes, it is enough to consider only those symmetry operations, which leave the quantity invariant [8,11,41]. Such operations form infinite subgroups of the $\infty/mm1'$ group. They are listed for each irrep in Table III. The content of these invariance groups can be easily figured out from the international (Hermann-Mauguin-type) symbol [9], but also from the pictographic symbols shown in Fig. 1.

In addition, each pictograph shows a segment indicating the magnitude of the quantity and an arrow associated with its geometric sign (see Figs. 1 and 2). Arrows in pictographs drawn with dashed lines should be considered as indicating a stationary current or motion (time inversion operation does change their sense). This is the case of time-odd quantities (**L**, **M**, **T**, **F**). In contrast, the arrows in pictographs drawn with full lines should be considered as time-even (time inversion operation does not change them, as they have a grey-group [29] symmetry). These pictographs stands for the time-even quantities **N**, **G**, **P**, **C**. Let us note that the **P**, **T**, **N**, **L** quantities, symmetric with respect to the parallel mirror plane operation $m_\parallel$, have arrows only in the radial direction, while $m_\parallel$-antisymmetric quantities, **G**, **M**, **C**, **F**, all have only tangential arrows (bend arrows should be understood as

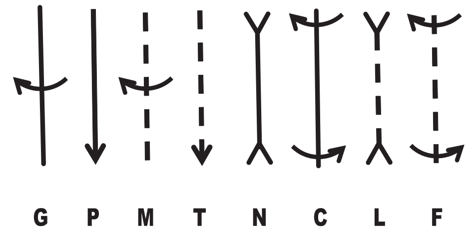

FIG. 2. Pictographs of same eight kinds of quantities as in Fig. 1, but with an opposite sense.

drawn on a visible part of the outer lateral surface of a coaxial circular cylinder.) One can also easily distinguish the single-arrow pictographs of $2_\perp$-antisymmetric quantities **G**, **P**, **M**, **T** (vectors) from all the double-arrow graphical symbols standing for $2_\perp$-invariant quantities **N**, **C**, **L**, **F**, which we call bidirectors.

Meaning of the geometric parity signs, bidirectors.—The fact that the parity sign can be represented in this way emphasizes its geometrical nature. Obviously, the strict meaning of the parity sign of a physical quantity relies on some convention, too. For example, the vector of the electric dipole moment is taken as pointing towards the center of the positive charge (and not the opposite), the arrow associated with the velocity of a particle is drawn towards its future position (and not the opposite), the sense of the electric current refers normally to the velocity of the positive charges, and the arrow in the pictograph standing for the magnetic dipole moment is that of the equivalent positive stationary electric current circulating around the indicated axis.

Another set of conventions is needed to facilitate the algebraic representation of such quantities. Typically, a polar vector is represented by three coordinates defined by its scalar-product projections to an oriented set of three orthonormal basis vectors. It is so practical that we tend to represent all other quantities in a similar way.

In the case of vector quantities (those of Table I), such algebraic representation is usually defined through the time derivatives and vectorial products or equivalent rules. In fact, this representation justifies the common usage of the simple **P**-arrow pictograph for all other vector quantities of Table I. For example, magnetic moment **m** of a current turn is defined as a vector perpendicular to the turn and directed so that the current observed from the end of vector **m** envelops the turn counterclockwise [42]. Therefore, the pictograph for **M** (as well as for **G** and **T**) can formally be replaced by that of **P**, and so, we often do that—even though these quantities actually do have quite different symmetry (in fact, Fig. 1 could conveniently serve as a replacement table). Moreover, this algebraic representation allows us to calculate any scalar and vectorial products in the usual way. Interestingly, vectorial products of vectors are vectors, and scalar products of vectors transform as one of the four possible scalar species [4] (time-even scalar σ, time-even pseudoscalar ϵ, time-odd scalar τ, and time-odd pseudoscalar μ, see Table IV).

TABLE IV. Four scalar types [4] specified according to their invariance under space-inversion and time-reversal operations (time-even scalar σ, time-even pseudoscalar ϵ, time-odd scalar τ, and time-odd pseudoscalar μ).

	$\bar{1}$	$1'$	Examples
σ	1	1	**G**.**G**, **T**.**T**, **P**.**P**, **M**.**M**, ∇**P**, mass, charge
ϵ	−1	1	**P**.**G**, **T**.**M**
τ	1	−1	**M**.**G**, **T**.**P**, time
μ	−1	−1	**T**.**G**, **M**.**P**, magnetic monopole

TABLE V. Look-up table of transformation properties of vectorial products and scalar multiplications. The symbol ~ has a meaning of "transforms as…," the operations involving bidirector quantities **N**, **C**, **L**, **F** are defined in the text.

A~			**G**	**P**	**M**	**T**	**N**	**C**	**L**	**F**
[**G** × **A**]	or	[σ**A**]~	**G**	**P**	**M**	**T**	**N**	**C**	**L**	**F**
[**P** × **A**]	or	[ϵ**A**]~	**P**	**G**	**T**	**M**	**C**	**N**	**F**	**L**
[**M** × **A**]	or	[τ**A**]~	**M**	**T**	**G**	**P**	**L**	**F**	**N**	**C**
[**T** × **A**]	or	[μ**A**]~	**T**	**M**	**P**	**G**	**F**	**L**	**C**	**N**

In the case of bidirectors, none of the SO(3) operations can reverse their geometrical sign. It indicates the fundamental difficulty with representation of bidirectors by three-component algebraic vectors. In fact, each of these bidirector quantities transforms as an "antitandem" arrangement of two vectors—as a couple ("dipole") of two opposite vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ ($|\mathbf{X}_1| = |\mathbf{X}_2|$) arranged on a common axis at some nonzero distance $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$. Obviously, **N** transforms as an antitandem of two **P** vectors, **C** as an antitandem of two **G** vectors, **L** as a **T**-vector antitandem, and **F** as a **M**-vector antitandem. Therefore, a bidirector can be represented by a simple "two-body" term $\mathbf{a}_{12} = \mathbf{X}_2 - \mathbf{X}_1$. Here, it is assumed that the symmetry operations act on both the vectors and their positions: operations that change $\mathbf{r}_{21}$ to the opposite are actually interchanging sites 1 and 2. The geometrical parity sign of such antitandem quantities could be denoted as inward or outward, depending on whether the vector $\mathbf{X}_2$ is parallel or antiparallel to the vector $\mathbf{r}_{21}$, and so, its evaluation actually requires knowing two quantities at a time, $\mathbf{X}_2$ and $\mathbf{r}_{21}$. Having this in mind, a range of algebraic operations can, nevertheless, be extended to all the above vectors and bidirectors. For the sake of convenience, types of the quantities obtained as vectorial cross products or as multiplication by a scalar are given in Table V. Let us also note that, from a symmetry point of view, the time derivative acts, here, as a multiplication by the time-odd scalar τ, so that, e.g., the time derivative of the bidirector **L** transforms as the bidirector **N** and vice versa.

Classification of axes and concluding remarks.—In general, an object may have a physical property transforming as one the eight discussed cases only if the symmetry invariance group of this object is a subgroup of the limiting group of the corresponding quantity. For example, macroscopic magnetization can exist only in crystals belonging to 31 different Heesch-Shubnikov point groups that are subgroups of $\infty/mm'1$ group [3,41,43]. If the axis of the limiting supergroup coincides with the symmetry axis of the object, it is often named according to the associated property (ferromagnetic axis, polar axis). Other axes could be similarly labeled as toroidal, truly chiral, falsely chiral, **G** axis, fully symmetric and so on.

The term vector is sometimes employed to describe phenomena that have a bidirector symmetry. For example, the so-called Burgers vector is widely used to characterize screw dislocations, which are obviously nonpolar, truly chiral (C-type) objects. Similarly, the antiferromagnetic vector [34,35,44] is often used to describe the falsely chiral (F-type) antiferroelectric order. On the contrary, the so-called "chiral vector" or "vector chirality" [45–47] is sometimes used to characterize cyclic spin arrangements on spin loops, for example, in triangular antiferromagnetic lattices, even if the spin arrangement happens to have toroidal symmetry, which is "unidirectorial" but achiral (similar to spin cycloids and Néel domain walls [48]).

Finally, it is well known that axial vector **G** can be represented as a polar antisymmetric second-order tensor. The bidirector quantities can also be classified within the established tensorial calculus [1,4,49,50]. They correspond to a special kind of second rank tensors, that was once coined in Russian literature as the "simplest tensor" (i.e., a symmetrical second rank tensor having in its canonical form only a single nonzero element) [14]. In particular, an **N**-type bidirector could be considered as dual to the simplest time-even polar tensor, a **C**-type bidirector transforms as the simplest time-even axial tensor, an **L**-type bidirector as the simplest time-odd polar tensor, and an **F**-type bidirector as the simplest time-odd axial tensor.

Nevertheless, we think that the unifying classification via irreps of the limiting dihedral group $\infty/mm1'$ still provides a very practical concept, applicable in various areas of physics. In solid state physics, at least, the simple perspective, where vectors and bidirectors have equal legitimacy, might be useful when dealing with problems where several such quantities are interacting, for classification of long-wavelength excitations or structural components of magnetoelectric multiferroic crystals [11,51–53], for description of macroscopic properties of chiral objects [54,55], or for symmetry classification of topological defects in vectorial fields (for example, domain walls, vortices or Skyrmions) [56,57]. In fact, we would like to offer a more complete discussion of possible applications of this concept in the future, and so, we would be grateful to learn about other cases where this perspective could bring some useful insight.

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[1] R. R. Birss, *Symmetry and Magnetism* (North-Holland, Amsterdam, 1964).
[2] H. Grimmer, Ferroelectrics **161**, 181 (1994).
[3] E. Ascher, Int. J. Magn. **5**, 287 (1974).
[4] V. Kopský, Z. Kristallogr. **221**, 51 (2006).
[5] Yu. Sirotin and M. P. Shaskolskaya, *Osnovy Kristallophiziki* (Nauka, Moscow, 1975) [*Fundamentals of Crystal Physics* (Mir Publishers, Moscow, 1982)].
[6] V. M. Dubovik and V. V. Tugushev, Phys. Rep. **187**, 145 (1990).
[7] D. B. Litvin, Acta Crystallogr. Sect. A **64**, 316 (2008).
[8] V. M. Dubovik, S. S. Krotov, and V. V. Tugushev, Kristallografiya **32**, 540 (1987) [Sov. Phys. Crystallogr. **32**, 314 (1987)].
[9] V. K. Wadhawan, *Introduction to Ferroic Materials* (Gordon and Breach, New York, 2000).
[10] A. S. Borovik-Romanov and H. Grimmer, in *International Tables for Crystallography*, edited by A. Authier (Kluwer Academic, Dordrecht, 2003), Vol. D, p. 137.
[11] H. Schmid, J. Phys. Condens. Matter **20**, 434201 (2008).
[12] M. Hamermesh, *Group Theory And its Applications to Physical Problems* (Addison-Wesley, Reading, MA, 1964).
[13] S. L. Altmann and P. Herzig, *Point-Group Theory Tables* (Oxford University Press, New York, 1994).
[14] I. S. Zheludev, *Fizika Kristallov i Simmetria* (Nauka, Moscow, 1987).
[15] I. S. Zheludev, Acta Crystallogr. Sect. A **42**, 122 (1986).
[16] C. Ederer and N. A. Spaldin, Phys. Rev. B **76**, 214404 (2007).
[17] A. A. Gorbatsevich and Yu. V. Kopaev, Ferroelectrics **161**, 321 (1994).
[18] N. A. Spaldin, M. Fiebig, and M. Mostovoy, J. Phys. Condens. Matter **20**, 434203 (2008).
[19] Yu. V. Kopaev, Phys. Usp. **52**, 1111 (2009).
[20] S. Prosandeev, I. Ponomareva, I. Kornev, I. Naumov, and L. Bellaiche, Phys. Rev. Lett. **96**, 237601 (2006).
[21] S. Prosandeev, A. R. Akbarzadeh, and L. Bellaiche, Phys. Rev. Lett. **102**, 257601 (2009).
[22] S. Prosandeev and L. Bellaiche, J. Mater. Sci. **44**, 5235 (2009).
[23] A. A. Gorbatsevich and Yu. V. Kopaev, Pis'sma Zh. Eksp. Teor. Fiz. **39**, 558 (1984) [J. Exp. Theor. Phys. **39**, 684 (1984)].
[24] P. Toledano, D. D. Khalyavin, and L. C. Chapon, Phys. Rev. B **84**, 094421 (2011).
[25] R. D. Johnson, L. C. Chapon, D. D. Khalyavin, P. Manuel, P. G. Radaelli, and C. Martin, Phys. Rev. Lett. **108**, 067201 (2012).
[26] M. Mostovoy, Physics **5**, 16 (2012).
[27] A. J. Hearmon, F. Fabrizi, L. C. Chapon, R. D. Johnson, D. Prabhakaran, S. V. Streltsov, P. J. Brown, and P. G. Radaelli, Phys. Rev. Lett. **108**, 237201 (2012).
[28] M. Lax, *Symmetry Principles in Solid State and Molecular Physics* (Wiley-Interscience, New York, 1974).
[29] A. P. Cracknell, Rep. Prog. Phys. **32**, 633 (1969).
[30] W. Slawinski, R. Przenioslo, I. Sosnowska, and V. Petricek, Acta Crystallogr. Sect. B **68**, 240 (2012).
[31] J. M. Perez-Mato, J. L. Ribeiro, V. Petricek, and M. I. Aroyo, J. Phys. Condens. Matter **24**, 163201 (2012).
[32] H. T. Stokes, D. M. Hatch, and B. J. Campbell, computer code ISOTROPY (2007), http://stokes.byu.edu/isotropy.html.
[33] L. D. Barron, *Molecular Light Scattering and Optical Activity* (Cambridge University Press, New York, 2004).
[34] A. M. Kadomtseva, A. K. Zvezdin, Yu. F. Popov, A. P. Pyatakov, and G. P. Vorobev, Pis'sma Zh. Eksp. Teor. Fiz. **79**, 705 (2004) [JETP Lett. **79**, 571 (2004)].
[35] P. Tolédano, Ferroelectrics **161**, 257 (1994).
[36] L. D. Barron, Nature (London) **405**, 895 (2000).
[37] S. T. Lagerwall, *Ferroelectric and Antiferroelectric Liquid Crystals* (Wiley-VCH, Weinheim, 1999).
[38] A. E. Romanov and V. I. Vladimirov, Phys. Status Solidi A **78**, 11 (1983).
[39] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific, Singapore, 2008), Chap. 9.
[40] F. C. Frank, Discuss. Faraday Soc. **25**, 19 (1958).
[41] T. S. G. Krishnamurty and P. Gopalakrishnamurty, Acta Crystallogr. Sect. A **25**, 333 (1969).
[42] G. E. Zilberman, *Electricity and Magnetism* (Mir Publishers, Moscow, 1973).
[43] W. Opechowski, *Crystallographic and Metacrystallographic Groups* (North-Holland, Amsterdam, 1986).
[44] C. Ederer and N. A. Spaldin, Phys. Rev. B **71**, 060401 (2005).
[45] H. Kawamura, J. Appl. Phys. **63**, 3086 (1988).
[46] D. Grohol, K. Matan, J.-H. Cho, S.-H. Lee, J. W. Lynn, D. G. Nocera, and Y. S. Lee, Nat. Mater. **4**, 323 (2005).
[47] K. Matan, B. M. Bartlett, J. S. Helton, V. Sikolenko, S. Matas, K. Prokes, Y. Chen, J. W. Lynn, D. Grohol, T. J. Sato, M. Tokunaga, D. G. Nocera, and Y. S. Lee, Phys. Rev. B **83**, 214406 (2011).
[48] B. M. Tanygin, J. Magn. Magn. Mater. **323**, 616 (2011).
[49] D. B. Litvin, Acta Crystallogr. Sect. A **50**, 406 (1994).
[50] V. Kopský, Acta Crystallogr. Sect. A **35**, 95 (1979).
[51] M. Fiebig, J. Phys. D **38**, R123 (2005).
[52] A. Saxena and T. Lookman, Phase Transit. **84**, 421 (2011).
[53] A. B. Harris, Phys. Rev. B **76**, 054447 (2007).
[54] P. Jungwirth, L. Skála, and R. Zahradník, Chem. Phys. Lett. **161**, 502 (1989).
[55] V. Gopalan and D. B. Litvin, Nat. Mater. **10**, 376 (2011).
[56] B. M. Tanygin, Physica (Amsterdam) **407B**, 868 (2012).
[57] Spatiotemporal symmetry of a topological defect in a given vectorial field depends, simultaneously, on the nature of the topological defect as well as on the nature of the field itself. For example, a vortexlike defect in a magnetic field has a symmetry the **T**-type moment, while the equivalent vortex defect in a polarization field has a symmetry of the **G**-type moment. Similarly, the core of a Skyrmion-like defect in a magnetic field allows, simultaneously, for **T**-, **M**-, and **C**-type quantities, while the core of a Skyrmion-like defect in a polarization field allows, simultaneously, for **G**-, **P**-, and **C**-type quantities.