Joint Measurability of Generalized Measurements Implies Classicality

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The fact that not all measurements can be carried out simultaneously is a peculiar feature of quantum mechanics and is responsible for many key phenomena in the theory, such as complementarity or uncertainty relations. For the special case of projective measurements, quantum behavior can be characterized by the commutator but for generalized measurements it is not easy to decide whether two measurements can still be understood in classical terms or whether the already show quantum features. We prove that a set of generalized measurements which does not satisfy the notion of joint measurability is nonclassical, as it can be used for the task of quantum steering. This shows that the notion of joint measurability is, among several definitions, the proper one to characterize quantum behavior. Moreover, the equivalence allows one to derive novel steering inequalities from known results on joint measurability and new criteria for joint measurability from known results on the steerability of states.

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Introduction.—Quantum theory is formulated in the language of Hilbert spaces, where states correspond to vectors or density matrices, and measurements are described by Hermitian matrices, the so-called observables. As realized by M. Born and P. Jordan, two observables A and B do not necessarily commute, which means, in the first place, that the corresponding measurements cannot be carried out simultaneously in a direct way [1,2]. This intuition can be made precise by formulating uncertainty relations, where the commutator $[A,B]=AB-BA$ quantifies the degree of uncertainty about the values of A and B [2–4]. Consequently there is the widespread opinion that sets of noncommuting observables are central for many quantum effects, while commuting observables are considered to be classical.

It has turned out, however, that the notion of observables is far too narrow to describe all measurements procedures in quantum mechanics. This has led to the formulation of generalized measurements or positive operator valued measures (POVMs). Mathematically, a POVM consists of a collection of operators $E=\{E(i), i\in I\}$ which are positive, $E(i)\geq 0$, and sum up to the identity, $\sum_i E(i)=\mathbb{1}$. The POVM elements $E(i)$ describe the measurement outcomes and the probability of an outcome i is given by $p(i)=\mathrm{tr}[\varrho E(i)]$. Physically, any POVM can be realized by first letting the physical system interact with an auxiliary system and then measuring an ordinary observable on the auxiliary system. Finally, any observable A is also a POVM if one identifies the $E(i)$ with the projectors onto the eigenspaces of A, in which case the measurement is also called a projection valued measure (PVM).

Given the notion of generalized measurements the question arises, when two or more POVMs can be considered to be nonclassical. One possibility is to require the commutativity of all the POVM elements, but more refined notions are useful. Indeed, several notions such as "nondisturbance," "joint measurability," and "coexistence" have been introduced and their investigation is an active area of research [5–9].

In this Letter, we argue that the notion of joint measurability is the proper one to describe the classical behavior of two or more generalized measurements. To do so, we establish a connection between joint measurability and the task of quantum steering. Quantum steering refers to the scenario, where one party, usually called Alice, wishes to convince the other party, called Bob, that she can steer the state at Bob's side by making measurements on her side. This task was introduced by E. Schrödinger to demonstrate the puzzling effects of quantum correlations [10] and recently it has attracted increasing attention again [11–16].

More precisely, we show that a set of POVMs in the finite dimensional case is nonjointly measurable if and only if the set can be used for Alice to show the steerability of some quantum state. This allows one to derive new steering inequalities from results known for joint measurability, and we will also find new criteria for joint measurability from results on steering. Finally, we demonstrate that other possible extensions of commutativity to generalized measurements, such as coexistence, lead to nonclassical effects and we explore the relation of joint measurability to Bell inequality violations.

Joint measurability.—The notion of joint measurability is most conveniently introduced with an example. The Pauli spin matrices σ_x and σ_z are noncommuting and cannot be measured jointly. However, one can consider the smeared or unsharp measurements S_x and S_z, defined by the POVM elements $S_x(\pm)=\frac{1}{2}(\mathbb{1}\pm(1/\sqrt{2})\sigma_x)$ and

$S_z(\pm) = \frac{1}{2}(\mathbb{1} \pm (1/\sqrt{2})\sigma_z)$. It was shown in Ref. [17] that these are jointly measurable: One can consider the joint observable

$$G(i,j) = \frac{1}{4}\left(\mathbb{1} + \frac{i}{\sqrt{2}}\sigma_x + \frac{j}{\sqrt{2}}\sigma_z\right), \qquad i,j \in \{-1,+1\}, \tag{1}$$

and since $S_x(\pm) = \sum_j G(\pm, j)$ and $S_z(\pm) = \sum_i G(i, \pm)$, one can jointly determine the probabilities of the generalized measurements S_x and S_z by measuring G.

More precisely, joint measurability of the set $\{E_k\}$ of POVMs can be formulated as the existence of a set of positive operators $\{G(\lambda)\}$ from which the original observables can be attained as

$$\sum_\lambda D_\lambda(x|k)G(\lambda) = E_k(x) \quad \text{for all } x, k, \tag{2}$$

with $\sum_\lambda G(\lambda) = \mathbb{1}$ and where $D_\lambda(x|k)$ are positive constants with $\sum_x D_\lambda(x|k) = 1$ [18]. In practice, this means that the probabilities of the results $E_k(x)$ can be determined by measuring the operators $G(\lambda)$ and classically postprocessing the data.

Quantum steering.—The essence of steering can also be described by an example. Let us assume that two parties, Alice and Bob, share a maximally entangled two-qubit state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. If Alice measures the Pauli operators σ_x or σ_z, the state on Bob's side will be an eigenstate $|x^\pm\rangle$ or $|z^\pm\rangle$ depending on Alice's measurement and result. Since all these states are pure, Bob cannot explain this by assuming that he has a fixed marginal state ϱ_B which is only modified due to the additional knowledge from Alice's measurements. So Bob must conclude that Alice can steer the state in his lab by making measurements on her side. The question arises whether the same phenomenon occurs if Alice uses the smeared measurements S_x and S_z introduced above. This will be answered in full generality in the following.

First, we label Alice's and Bob's POVMs by $\{A_k\}$ and $\{B_l\}$ and the system's state by ϱ_{AB}. Clearly, the scenario is nonsteerable if the probabilities of possible events can be written in the form

$$\mathrm{tr}[\varrho_{AB} A_k(x) \otimes B_l(y)] = \sum_\lambda p(\lambda)p(x|k,\lambda)\mathrm{tr}[\varrho_\lambda B_l(y)] \tag{3}$$

because then Bob can assume that he has the collection of states ϱ_λ with probabilities $p(\lambda)$ which is only modified by additional information from Alice's measurements quantified by conditional probability distributions $p(x|k,\lambda)$. We can write the left-hand side of this equation as

$$\mathrm{tr}\big(\mathrm{tr}_A\{[A_k(x) \otimes \mathbb{1}]\varrho_{AB}\}B_l(y)\big) =: \mathrm{tr}[\varrho_{x|k} B_l(y)] \tag{4}$$

and if Bob's measurements are tomographically complete it follows that $\varrho_{x|k} = \sum_\lambda p(\lambda)p(x|a,\lambda)\varrho_\lambda$. If, on the other hand, the quantities $\varrho_{x|k}$ admit this kind of a decomposition (also called a hidden state model) we conclude that the scenario is nonsteerable.

This can be reformulated as suggested in Refs. [12,13]: Steering is equivalent to the nonexistence of a set of positive operators $\{\sigma_\lambda\}$ such that

$$\sum_\lambda p(x|k,\lambda)\sigma_\lambda = \varrho_{x|k} \quad \text{for all } x, k, \tag{5}$$

with $\mathrm{tr}(\sum_\lambda \sigma_\lambda) = 1$ and where $\varrho_{x|k} = \mathrm{tr}_A\{[A_k(x) \otimes \mathbb{1}]\varrho_{AB}\}$ are Bob's not-normalized conditional states. The formal similarity between Eq. (2) and Eq. (5) is appealing and, as we will see now, no coincidence.

Steering and joint measurements.—Consider the case where Alice has observables $\{A_k\}$ which are jointly measurable. Using Eq. (2) we can write for any steering scenario the conditional states of Bob as

$$\varrho_{x|k} = \sum_\lambda D_\lambda(x|k)\mathrm{tr}_A\{[G(\lambda) \otimes \mathbb{1}]\varrho_{AB}\}, \tag{6}$$

which is a decomposition as in Eq. (5). Therefore, if Alice's observables are jointly measurable then the scenario is nonsteerable.

Conversely, if the measurements are nonjointly measurable, one can always find a state which can be used for steering: For the maximally entangled state $|\phi^+\rangle = 1/\sqrt{d}\sum_{i=1}^d |ii\rangle$ one can write Bob's conditional states as

$$\varrho_{x|k} = \mathrm{tr}_A[(A_k(x) \otimes \mathbb{1})|\phi^+\rangle\langle\phi^+|] = \frac{1}{d}[A_k(x)]^T. \tag{7}$$

If the scenario is not steerable then one can find a set of positive operators $\{\sigma_\lambda\}$ and a set of positive numbers $p(x|k,\lambda)$ such that

$$A_k(x) = d\sum_\lambda p(x|k,\lambda)\sigma_\lambda^T =: \sum_\lambda D_\lambda(x|k)G(\lambda), \tag{8}$$

where $G(\lambda) = d\sigma_\lambda^T$. This is just the joint measurability condition from Eq. (2). Note that by summing over x in Eq. (8) we see that G is properly normalized. We now state the main result of this article.

Observation 1: Generalized measurements are nonjointly measurable if and only if they can be used for quantum steering.

Let us note that the reasoning prior to Observation 1 was done for the maximally entangled state. Steering is, however, invariant under stochastic local operations and classical communication [19] on the characterized (Bob's) side. This means that any state which is obtained from the maximally entangled one by stochastic local operations and classical communication can be used to show steering for a set of nonjointly measurable observables. Therefore, any pure Schmidt rank d state (possibly having an arbitrarily small amount of entanglement) reveals steering.

We exploit the connection by giving a generic incompatibility criteria for sharp observables, deriving a steering inequality based on the Fermat-Torricelli point, and pointing out two interesting notes on different formulations of simultaneous measurability.

From steering to incompatibility.—We show that there exists a threshold value of white noise [that is, adding the identity as in Eq. (11)] that one needs to add in order to get any set of PVMs jointly measurable. For this purpose we need the following connection between noisy states and noisy observables:

$$\mathrm{tr}_A[A_k(x) \otimes \mathbb{1}\varrho^{\lambda}_{AB}] = \mathrm{tr}_A[A^{\lambda}_k(x) \otimes \mathbb{1}\varrho_{AB}], \tag{9}$$

where

$$\varrho^{\lambda}_{AB} = \lambda\varrho_{AB} + \frac{1-\lambda}{d}\mathbb{1} \otimes \mathrm{tr}_A[\varrho_{AB}], \tag{10}$$

$$A^{\lambda}_k(x) = \lambda A_k(x) + \frac{1-\lambda}{d}\mathrm{tr}[A_k(x)]\mathbb{1}. \tag{11}$$

In order to obtain the threshold value we take the known result from Ref. [11] stating that the maximally entangled state is steerable with PVMs up to the amount $\lambda^* \coloneqq (H_d - 1)/(d-1)$ of white noise, where $H_d = \sum_{n=1}^{d}(1/n)$. Using Eq. (9) and Observation 1 one obtains that for any smearing parameter $\lambda \geq \lambda^*$ there must exist a set of PVMs which is noise resistant up to the amount λ of white noise; i.e., one can add this amount of white noise to the PVMs without making them jointly measurable. On the other hand, the maximally entangled state reveals steering for nonjointly measurable observables, so all PVMs must be jointly measurable with the amount λ^* of white noise. Thus, we arrive at the following result.

Observation 2: In a d-dimensional Hilbert space, any set of sharp observables is jointly measurable with the amount λ^* of white noise. Moreover, for any amount of smearing above this limit there exists a set of PVMs which remains nonjointly measurable.

Note that this is formerly known to be sufficient for $d = 2$ [20]. The result leads to an interesting open question: Are there sets of POVMs which remain nonjointly measurable with the amount λ^* of white noise? If this is the case then PVMs are not enough for concluding steerability of a state and if it is not the case then this directly leads to new local hidden variable models for POVMs.

Fermat-Torricelli steering inequality.—There are many results of joint measurability known in terms of white noise resistance [17,21,22]. As an example, consider that Alice has three dichotomic unbiased [i.e., $p(\pm|k) = \frac{1}{2}$] measurements while Bob's conditional (normalized) qubit states are characterized by the Bloch vector $\vec{x}_k$, $k = 1, 2, 3$. Using the joint measurability criterion of Ref. [23] we see steering iff

$$||\vec{x}_1 + \vec{x}_2 + \vec{x}_3 - \vec{x}_{FT}|| + ||\vec{x}_1 - \vec{x}_2 - \vec{x}_3 - \vec{x}_{FT}|| + ||\vec{x}_1 - \vec{x}_2 + \vec{x}_3 + \vec{x}_{FT}|| + ||\vec{x}_1 + \vec{x}_2 - \vec{x}_3 + \vec{x}_{FT}|| > 4, \tag{12}$$

where $\vec{x}_{FT}$ denotes the Fermat-Torricelli point of the vectors $\vec{x}_1 + \vec{x}_2 + \vec{x}_3$, $\vec{x}_1 - \vec{x}_2 - \vec{x}_3$, $-\vec{x}_1 + \vec{x}_2 - \vec{x}_3$, and $-\vec{x}_1 - \vec{x}_2 + \vec{x}_3$; i.e., it is the vector that minimizes the sum in Eq. (12).

Coexistence leads to a nonclassical effect.—Coexistence of POVMs A_1 and A_2 means the possibility of making a measurement G of which statistics include the statistics of A_1 and A_2. To be more precise, A_1 and A_2 are coexistent if their POVM elements are contained in the range (i.e., all possible sums of POVM elements) of a third POVM G. Note that contrary to joint measurements, the statistics do not need to originate from a postprocessing scheme as in Eq. (2). To clarify the notion we present an example given in Ref. [5] which was originally used to show that coexistence is more general than joint measurability; for a similar example, see Ref. [8].

In $\mathbb{C}^3$ define $|\varphi\rangle = 1/\sqrt{3}(|1\rangle + |2\rangle + |3\rangle)$ and a POVM G by the elements $\{\frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|2\rangle\langle 2|, \frac{1}{2}|3\rangle\langle 3|, \frac{1}{2}|\varphi\rangle\langle\varphi|, \frac{1}{2}(\mathbb{1} - |\varphi\rangle\langle\varphi|)\}$. One sees straightforwardly that the measurement statistics of a three-valued POVM A_1 defined as $A_1(i) = \frac{1}{2}(\mathbb{1} - |i\rangle\langle i|)$ and a two-valued POVM A_2 defined as $A_2(1) = \frac{1}{2}|\varphi\rangle\langle\varphi|$, $A_2(2) = \mathbb{1} - A_2(1)$ are contained in the measurement statistics of G; hence, they are coexistent. In Ref. [5] it was shown that these measurements are nevertheless nonjointly measurable due to the lack of a postprocessing relation. By Observation 1 we conclude the following.

Observation 3: As coexistence is more general than joint measurability it can reveal steering; i.e., it can lead to nonclassical effects in the distributed scenario.

Disturbing measurements can be useless for steering.—One way to define the classicality of two measurements, say A_1 and A_2, is to say that the measurement of A_1 does not disturb the measurement of A_2. This means that a measurement of A_1 updates the state in such a way that a subsequent measurement of A_2 has the same statistics for both the updated and the original state. It was shown in Ref. [9] that there exists pairs of observables that can be measured jointly even though they do not admit a nondisturbing sequential measurement. Using this together with Observation 1 we conclude that disturbing measurements do not necessarily lead to steering.

Joint measurability and nonlocality.—From the previous discussion we know that any nonjointly measurable set of POVMs can reveal its "quantumness" in a strictly nonclassical, nonlocal effect, more precisely, in the form of steering. Steering is, however, not the ultimate strongest form of nonlocality since one still needs a quantum description on one side. Thus, it is of course a natural question whether this connection can even be strengthened, so whether it also holds that any nonjointly measurable set of POVMs can show nonclassicality in a Bell-type scenario.

This is indeed the case for two dichotomic measurements as has been shown by Wolf *et al.* in Ref. [24]. It also holds for an arbitrary number of PVMs. In the following, we argue that it would be very surprising if this connection were to hold in general, since via a very simple example one encounters already large difficulties.

Consider the three dichotomic spin measurements of a qubit $A_k^\lambda(\pm) = (\mathbb{1} \pm \lambda\sigma_k)/2$ with $k \in \{x, y, z\}$. As already mentioned, the additional parameter λ characterizes the noise on these measurements. For $\lambda = 1$ the measurements $A_k := A_k^{\lambda=1}$ are noncommuting projectors, while for $\lambda \leq 1/\sqrt{3} \approx 0.5774$ the set of POVMs becomes jointly measurable. Suppose that joint measurability and nonlocality are as strongly connected as steering. This would mean that for any noisy, but nonjointly measurable set of these POVMs, i.e., for all $1/\sqrt{3} < \lambda$, it is possible to find a respective bipartite state ϱ_{AB} and corresponding measurements for Bob $B_l(k)$, such that the obtained data $P(\pm, y|k, l) = \mathrm{tr}[\varrho_{AB} A_k^\lambda(\pm) \otimes B_l(y)]$ violate a Bell inequality.

In the search for such an appropriate state, first note that pure states $\varrho_{AB} = |\psi\rangle\langle\psi|$ are sufficient, since any mixed state can only violate a Bell inequality if at least one pure state from its range does so. Using the Schmidt decomposition together with the fact that $\dim(\mathcal{H}_A) = 2$ we can write the most general pure state as $|\psi\rangle = U_A \otimes U_B|\psi_s\rangle$ with $|\psi_s\rangle = s|00\rangle + \sqrt{1 - s^2}|11\rangle$ where $1/\sqrt{2} \leq s \leq 1$. Since we optimize Bob's measurements we can additionally assume $U_B = \mathbb{1}$, meaning that Bob similarly holds a qubit. Next we also wish to transfer the noise of the measurements into the state, as given by Eq. (9). Thus, rather than looking for a pure state which violates a Bell inequality using the noisy measurements A_k^λ, we can equivalently search for a mixed state that violates a Bell inequality with perfect measurements A_k. To sum up, we would need to show that for any parameter $\lambda > 1/\sqrt{3}$, a state of the form

$$\varrho_{AB}(s; U_A) = \lambda U_A \otimes \mathbb{1}|\psi_s\rangle\langle\psi_s|U_A^\dagger \otimes \mathbb{1} + (1 - \lambda)\mathbb{1}/2 \otimes \mathrm{tr}_A[|\psi_s\rangle\langle\psi_s|] \quad (13)$$

with appropriately chosen $1/\sqrt{2} \leq s \leq 1$ and U_A violates a Bell inequality using the three perfect spin measurements on system A, and arbitrary measurements for system B.

Let us start with the maximally entangled state, $s = 1/\sqrt{2}$, for which it is known that it does not violate a Bell inequality using projective measurements if $\lambda \leq 0.6595$ [25]. Hence, for the given noisy nonjointly measurable set of POVMs within $1/\sqrt{3} < \lambda \leq 0.6595$, the data of the maximally entangled state, using also projective measurements for Bob, will not display any nonlocality. For nonmaximally entangled states the situation is much less analyzed, especially under the influence of nonwhite noise as in Eq. (13). The statement extends, however, to $1/\sqrt{3} < \lambda \leq 0.6009$ [25] for arbitrary, nonmaximally entangled states if one wants to reproduce the full correlations. Thus, the only Bell inequalities that remain are the ones with marginals.

A different way to prove that certain states do not violate a Bell inequality is to write them as a convex combination of states known to possess a local hidden variable model for the considered configuration

$$\varrho_{AB}(s; U_A) = \sum_i p_i \varrho_i^{\mathrm{LHV}}. \quad (14)$$

Generic states that we consider in this decomposition include (i) noisy Bell states with $\lambda \leq 0.6595$ and (ii) states with two symmetric extensions for system A [26]. States of class (ii) are known to have a local hidden variable model for three generic measurements for system A [27], such that we exploit the fact that Alice has only a restricted number of measurements. Such a search for symmetric extensions can be easily done with semidefinite programming [28]. Figure 1 shows, depending on the Schmidt coefficient s (and for all U_A), the respective maximal values of λ when such a decomposition is possible. As can be seen for $s \leq 0.835$, there is always a noise parameter $\lambda > 1/\sqrt{3}$ such that the given set of POVMs is nonjointly measurable, but the measured state will not violate a Bell inequality using an arbitrary number of projective measurements for Bob. Finally, if one additionally constrains Bob to perform only n different dichotomic measurements then one can further add (iii) the class of states that have $n - 1$ symmetric extensions for system B. As shown in Fig. 1 for $n \leq 6$, such a decomposition is possible for all values of s. Thus, there

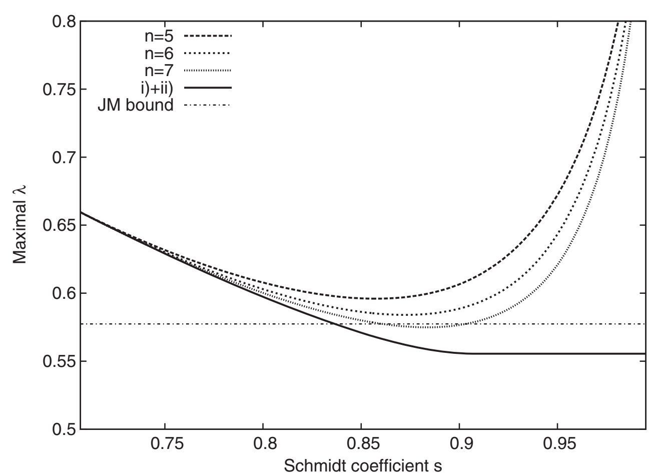

FIG. 1. Maximal values of λ when a decomposition as given by Eq. (14) is possible for all U_A depending on the Schmidt coefficient s. It shows that a pure state with $s \leq 0.835$ is never able to reveal Bell nonlocality for an arbitrary number of projective measurements, while for $n \leq 6$ projective measurements it is not possible for any state.

exists a parameter $\lambda > 1/\sqrt{3}$ such that the corresponding set of POVMs is nonjointly measurable but no quantum state will display nonlocality if Bob only carries out 6 dichotomic measurements.

These observations give strong hints that there are sets of POVMs which are nonjointly measurable, but which are nevertheless useless to certify nonlocality.

Conclusions.—We have shown that joint measurability and quantum steering are intrinsically connected: A collection of different measurements are nonjointly measurable if and only if they can reveal its "nonclassicality" as a violation of a steering inequality. This connects the abstract notion of joint measurability to an explicit nonlocality task, and thereby singles out nonjoint measurability as a special nonclassical property among other peculiar quantum features of measurements.

Since measurements are as relevant as quantum states, we believe that this connection will spur the resource theory of measurements, i.e., which kind of measurements are required for certain tasks. This investigation could provide some operational meaning to other quantum properties of measurements such as disturbance or noncoexistence in the distributed scenario.

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Note added.—After finishing this work we noticed that similar results were obtained in Ref. [29].

[1] M. Born and P. Jordan, Z. Phys. **34**, 858 (1925).
[2] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[3] E. H. Kennard, Z. Phys. **44**, 326 (1927).
[4] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[5] D. Reeb, D. Reitzner, and M. M. Wolf, J. Phys. A **46**, 462002 (2013).
[6] P. Lahti, Int. J. Theor. Phys. **42**, 893 (2003).
[7] E. Haapasalo, J.-P. Pellonpää, and R. Uola, arXiv:1404.4172.
[8] J.-P. Pellonpää, J. Phys. A **47**, 052002 (2014).
[9] T. Heinosaari and M. M. Wolf, J. Math. Phys. (N.Y.) **51**, 092201 (2010).
[10] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935).
[11] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[12] M. F. Pusey, Phys. Rev. A **88**, 032313 (2013).
[13] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Phys. Rev. Lett. **112**, 180404 (2014).
[14] T. Moroder, O. Gittsovich, M. Huber, and O. Gühne, Phys. Rev. Lett. **113**, 050404 (2014).
[15] J. Bowles, T. Vértesi, M. T. Quintino, and N. Brunner, Phys. Rev. Lett. **112**, 200402 (2014).
[16] M. Piani and J. Watrous, arXiv:1406.0530.
[17] P. Busch, Phys. Rev. D **33**, 2253 (1986).
[18] S. Twareque Ali, C. Carmeli, T. Heinosaari, and A. Toigo, Found. Phys. **39**, 593 (2009).
[19] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[20] T. Heinosaari and M. Ziman, *Mathematical Language of Quantum Theory* (Cambridge University Press, Cambridge, 2012) Example 3.7.
[21] C. Carmeli, T. Heinosaari, and A. Toigo, Phys. Rev. A **85**, 012109 (2012).
[22] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. **506**, 1 (2011).
[23] S. Yu and C. H. Oh, arXiv:1312.6470.
[24] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
[25] A. Acín, N. Gisin, and B. Toner, Phys. Rev. A **73**, 062105 (2006).
[26] By two symmetric extensions we mean the existence of an operator acting on $\mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, which is invariant under the swap of the first three systems and which gives the original state of $\mathcal{H}_A \otimes \mathcal{H}_B$ as its partial trace over any two of the $\mathcal{H}_A$ systems.
[27] B. M. Terhal, A. C. Doherty, and D. Schwab, Phys. Rev. Lett. **90**, 157903 (2003).
[28] L. Vandenberghe and S. Boyd, SIAM Rev. **38**, 49 (1996).
[29] M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014).