Joint Measurability, Einstein-Podolsky-Rosen Steering, and Bell Nonlocality

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We investigate the relation between the incompatibility of quantum measurements and quantum nonlocality. We show that a set of measurements is not jointly measurable (i.e., incompatible) if and only if it can be used for demonstrating Einstein-Podolsky-Rosen steering, a form of quantum nonlocality. Moreover, we discuss the connection between Bell nonlocality and joint measurability, and give evidence that both notions are inequivalent. Specifically, we exhibit a set of incompatible quantum measurements and show that it does not violate a large class of Bell inequalities. This suggests the existence of incompatible quantum measurements which are Bell local, similarly to certain entangled states which admit a local hidden variable model.

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The correlations resulting from local measurements on an entangled quantum state cannot be explained by a local theory. This aspect of entanglement, termed quantum nonlocality, is captured by two inequivalent notions, namely, Bell nonlocality [1,2] and Einstein-Podolsky-Rosen (EPR) steering [3–5]. The strongest form of this phenomenon is Bell nonlocality, witnessed via the violation of Bell inequalities. Steering represents a strictly weaker form of quantum nonlocality [4], witnessed via the violation of steering inequalities [6]. Both aspects have been extensively investigated in recent years, as they play a central role in the foundations of quantum theory and in quantum information processing.

Interestingly, quantum nonlocality is based on two central features of quantum theory, namely, entanglement and incompatible measurements. Specifically, performing (i) arbitrary local measurements on a separable state, or (ii) compatible measurements on an (arbitrary) quantum state can never lead to any form of quantum nonlocality. Hence, the observation of quantum nonlocality implies the presence of both entanglement and incompatible measurements. It is interesting to explore the converse problem. Two types of questions can be asked here (see Fig. 1): (a) Do all entangled states lead to quantum nonlocality? (b) Do all sets of incompatible measurements lead to quantum nonlocality?

An intense research effort has been devoted to question (a). First, it was shown that all pure entangled states violate a Bell inequality [7,8], hence also demonstrating EPR steering. For mixed states, the situation is much more complicated. There exist entangled states which are local, in the sense that no form of quantum nonlocality can be demonstrated with such states when using nonsequential measurements [9,10]. These issues become even more subtle when more sophisticated measurement scenarios are considered [11–14].

Question (b) has received much less attention so far. In the case of projective measurements, it was shown that incompatible measurements can always lead to Bell nonlocality [15,16]. Note that in this case, compatibility is uniquely captured by the notion of commutativity [17]. However, for general measurements, i.e., positive-operator-valued measures (POVMs), no general result is known. In this case, there are several inequivalent notions of compatibility. Here we focus on the notion of joint measurability (see, e.g., [18]) as this represents a natural choice in the context of quantum nonlocality. Several works discussed question (b) for POVMs [19,20]. The strongest result is due to Wolf *et al.* [16], who showed that any set of two incompatible POVMs with binary outcomes can always lead to violation of the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality. However, this result may not be extended to the general case (of an arbitrary number of POVMs with arbitrarily many outcomes), since pairwise joint measurability does not imply full joint measurability in general [21].

Here we explore the relation between compatibility of general quantum measurements and quantum nonlocality.

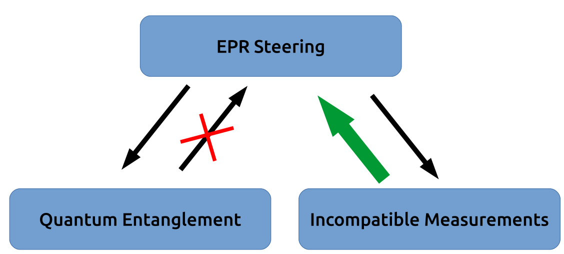

FIG. 1 (color online). The observation of EPR steering, a form of quantum nonlocality, implies the presence of both entanglement and incompatible measurements. Whether the converse links hold is an interesting question. Here we make progress in this direction by showing that any set of incompatible measurements can be used to demonstrate EPR steering (green arrow).

We start by demonstrating a direct link between joint measurability and EPR steering. Specifically, we show that for any set of POVMs that are incompatible (i.e., not jointly measurable), one can find an entangled state, such that the resulting statistics violates a steering inequality. Hence, the use of incompatible measurements is a necessary and sufficient ingredient for demonstrating EPR steering.

This raises the question of how joint measurability relates to Bell nonlocality. Specifically, the question is whether, for any set of incompatible POVMs (for Alice), one can find an entangled state and a set of local measurements (for Bob), such that the resulting statistics violates a Bell inequality. Here we give evidence that the answer is negative. In particular, we exhibit sets of incompatible measurements which can provably not violate a large class of Bell inequalities (including all full correlation Bell inequalities, also known as XOR games, see [2]). We therefore conjecture that nonjoint measurability and Bell nonlocality are inequivalent. Hence, similarly to local entangled states, there may exist incompatible quantum measurements which are Bell local.

Steering vs joint measurability.—We start by defining the relevant scenario and notations. We consider two separated observers, Alice and Bob, performing local measurements on a shared quantum state ρ_{AB}. Alice's measurements are represented by operators $M_{a|x}$ such that $\sum_a M_{a|x} = \mathbb{1}$, where x denotes the choice of measurement and a its outcome. Upon performing measurement x, and obtaining outcome a, the (un-normalized) state held by Bob is given by

$$\sigma_{a|x} = \mathrm{tr}_A(\rho_{AB} M_{a|x} \otimes \mathbb{1}). \tag{1}$$

The set of un-normalized states $\{\sigma_{a|x}\}$, referred to as an *assemblage*, completely characterizes the experiment, since $\mathrm{tr}(\sigma_{a|x})$ is the probability of Alice getting the output a (for measurement x) and given that information, Bob's state is described by $\sigma_{a|x}/\mathrm{tr}(\sigma_{a|x})$. Importantly, one has that $\sum_a \sigma_{a|x} = \sum_a \sigma_{a|x'}$ for all measurements x and x', ensuring that Alice cannot signal to Bob.

In a steering test [4], Alice wants to convince Bob that the state ρ_{AB} is entangled, and that she can steer his state. Bob does not trust Alice, and thus wants to verify Alice's claim. Asking Alice to perform a given measurement x, and to announce the outcome a, Bob can determine the assemblage $\sigma_{a|x}$ via local quantum tomography. To ensure that steering did indeed occur, Bob should verify that the assemblage does not admit a decomposition of the form

$$\sigma_{a|x} = \sum_\lambda \pi(\lambda) p(a|x,\lambda) \sigma_\lambda, \tag{2}$$

where $\sum_\lambda \pi(\lambda) = 1$. Clearly, if a decomposition of the above form exists, then Alice could have cheated by sending the (unentangled) state σ_λ to Bob and announce outcome a to Bob according to the distribution $p(a|x,\lambda)$. Note that here λ represents a local variable of Alice, representing her choice of strategy.

Assemblages of the form (2) are termed "unsteerable" and form a convex set [22,23]. Hence any "steerable" assemblage can be detected via a set of linear witnesses called steering inequalities [6]. By observing violation of a steering inequality, Bob will therefore be convinced that Alice can steer his state.

For a demonstration of steering, it is necessary for the state ρ_{AB} to be entangled. However, not all entangled states can be used to demonstrate steering [4,10,24], at least not when nonsequential measurements are performed on a single copy of ρ_{AB}.

Moreover, steering also requires that the measurements performed by Alice are incompatible. To capture the compatibility of a set of quantum measurements, we use here the notion of joint measurability; see, e.g., [18]. A set of m POVMs $M_{a|x}$ is called jointly measurable if there exists a measurement $M_{\vec{a}}$ with outcome $\vec{a} = [a_{x=1}, a_{x=2}, \ldots, a_{x=m}]$, where a_x gives the outcome of measurement x, that is

$$M_{\vec{a}} \geq 0, \qquad \sum_{\vec{a}} M_{\vec{a}} = \mathbb{1}, \qquad \sum_{\vec{a}\backslash a_x} M_{\vec{a}} = M_{a|x}, \tag{3}$$

where $\vec{a}\backslash a_x$ stands for the elements of $\vec{a}$ except for a_x. Hence, all POVM elements $M_{a|x}$ are recovered as marginals of the *joint observable* $M_{\vec{a}}$. Importantly, the joint measurability of a set of POVMs does not imply that they commute [25]. Hence, joint measurability is a strictly weaker notion of compatibility for POVMs. Moreover, joint measurability is not transitive. For instance, pairwise joint measurability does not imply full joint measurability in general [21] (see below).

Our main result is to establish a direct link between joint measurability and steering. Specifically, we show that a set of POVMs can be used to demonstrate steering if and only if it is not jointly measurable. More formally we prove the following result.

Theorem 1. The assemblage $\{\sigma_{a|x}\}$, with $\sigma_{a|x} = \mathrm{tr}_A(\rho_{AB} M_{a|x} \otimes \mathbb{1})$, is unsteerable for any state ρ_{AB} acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ if and only if the set of POVMs $\{M_{a|x}\}$ acting on $\mathbb{C}^d$ is jointly measurable.

Proof 1. The "if" part is straightforward. Our goal is to show that $\{\sigma_{a|x}\}$ admits a decomposition of the form of Eq. (2), when $\{M_{a|x}\}$ is jointly measurable, for any state ρ_{AB}. Consider $M_{\vec{a}}$, the joint observable for $\{M_{a|x}\}$, and define Alice's local variable to be $\lambda = \vec{a}$, distributed according to $\Pi(\vec{a}) = \mathrm{tr}(M_{\vec{a}}\rho_A)$, where $\rho_A = \mathrm{tr}_B(\rho_{AB})$. Next Alice sends the local state $\sigma_{\vec{a}} = \mathrm{tr}_A(M_{\vec{a}} \otimes \mathbb{1}\rho_{AB})/\Pi(\vec{a})$. When asked by Bob to perform measurement x, Alice announces an outcome a according to $p(a|x,\vec{a}) = \delta_{a,a_x}$.

We now move to the "only if" part. Consider an arbitrary pure state $\rho_{AB} = |\psi\rangle\langle\psi|$ with Schmidt number d. Notice that we can always write $|\psi\rangle = (D \otimes \mathbb{1})|\Phi\rangle$, where

$|\Phi\rangle = \sum_i |ii\rangle$ is an (un-normalized) maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$, and D is diagonal matrix that contains only strictly positive numbers. The assemblage resulting from a set of POVMs $\{M_{a|x}\}$ on ρ_{AB} is given by

$$\sigma_{a|x} = \mathrm{tr}_A(M_{a|x} \otimes \mathbb{1}|\psi\rangle\langle\psi|) = DM^T_{a|x}D, \tag{4}$$

where $M^T_{a|x}$ is the transpose of $M_{a|x}$. Our goal is now to show that if $\sigma_{a|x}$ is unsteerable then $\{M_{a|x}\}$ is jointly measurable. As $\sigma_{a|x}$ is unsteerable, we have that

$$\sigma_{a|x} = \sum_\lambda \pi(\lambda)p(a|x,\lambda)\sigma_\lambda, \tag{5}$$

which allows us to define the positive definite operator

$$\sigma_{\vec{a}} = \sum_\lambda \pi(\lambda)\sigma_\lambda \prod_x p(a_x|x,\lambda), \tag{6}$$

from which we can recover the assemblage $\{\sigma_{a|x}\}$ as marginals, i.e., $\sigma_{a|x} = \sum_{\vec{a}\backslash a_x}\sigma_{\vec{a}}$. Since the diagonal matrix D is invertible, we can define $M_{\vec{a}} := D^{-1}\sigma^T_{\vec{a}}D^{-1}$. It is straightforward to check that $M_{\vec{a}}$ is a joint observable for $\{M_{a|x}\}$: (i) it is positive, (ii) it sums to identity, and (iii) it has POVM elements $M_{a|x}$ as marginals. Hence, $\{M_{a|x}\}$ is jointly measurable, which concludes the proof. Note, finally, an interesting point that follows from the above. Considering a set of incompatible measurements acting on $\mathbb{C}^d$, any pure entangled state of the Schmidt number d can be used to demonstrate EPR steering. □

Bell nonlocality vs joint measurability.—It is natural to ask whether the above connection, between joint measurability and steering, can be extended to Bell nonlocality. Recall that in a Bell test, both observers Alice and Bob are on the same footing, and test the strength of the shared correlations. Specifically, Alice chooses a measurement x (Bob chooses y) and gets outcome a (Bob gets b). The correlation is thus described by a joint probability distribution $p(ab|xy)$. The latter can be reproduced by a predetermined classical strategy if it admits a decomposition of the form

$$p(ab|xy) = \sum_\lambda \pi(\lambda)p(a|x,\lambda)p(b|y,\lambda), \tag{7}$$

where λ represents the shared local (hidden) variable, and $\sum_\lambda \pi(\lambda) = 1$. Any distribution that does not admit a decomposition of the above form is said to be Bell nonlocal. The set of local distributions, i.e., of the form of Eq. (7) is convex, and can thus be characterized by a set of linear inequalities called Bell inequalities [2]. Hence, violation of a Bell inequality implies Bell nonlocality.

In quantum theory, Bell nonlocal distributions can be obtained by performing suitably chosen local measurements, $M_{a|x}$ and $M_{b|y}$, on an entangled state, ρ_{AB}. In this case, the resulting distribution $p(ab|xy) = \mathrm{tr}(\rho_{AB}M_{a|x} \otimes M_{b|y})$ does not admit a decomposition of the form of Eq. (7). Bell nonlocality is, however, not a generic feature of entangled quantum states. That is, there exist mixed entangled states which are local, in the sense that the statistics resulting from arbitrary nonsequential local measurements can be reproduced by a local model [9,10,12].

Given the above, we investigate now how joint measurability relates to Bell nonlocality. First, the above theorem implies that if the set of POVMs $\{M_{a|x}\}$ used by Alice is jointly measurable, then the statistics $p(ab|xy)$ can always be reproduced by a local model, for any state ρ_{AB} and measurements of Bob $\{M_{b|y}\}$. The converse problem is much more interesting. The question is whether for any set of POVMs $\{M_{a|x}\}$ that is not jointly measurable, there exists a state ρ_{AB} and a set of measurements $\{M_{b|y}\}$ such that the resulting statistics $p(ab|xy)$ violates a Bell inequality. This was shown to hold true for the case of sets of two POVMs with binary outcomes [16]. In this case, joint measurability is equivalent to a violation of the CHSH Bell inequality. Here we give evidence that this connection does not hold in general. Specifically, we exhibit a set of POVMs which is not jointly measurable but nevertheless cannot violate a large class of Bell inequalities.

Consider the set of three dichotomic POVMs (acting on $\mathbb{C}^2$) given by the following positive operators:

$$M'_{0|x}(\eta) = \frac{1}{2}(\mathbb{1} + \eta\sigma_x), \tag{8}$$

for $x = 1, 2, 3$, where σ_1, σ_2, σ_3 are the Pauli matrices, and $0 \le \eta \le 1$. Indeed, one has that $M_{1|x}(\eta)' = \mathbb{1} - M'_{0|x}(\eta)$. This set of POVMs should be understood as noisy Pauli measurements. The set is jointly measurable if and only if $\eta \le 1/\sqrt{3}$, although any pair of POVMs is jointly measurable for $\eta \le 1/\sqrt{2}$ [26] (see also [27]). Hence, in the range $1/\sqrt{3} < \eta \le 1/\sqrt{2}$, the set $\{M'_{a|x}(\eta)\}$ forms a *hollow triangle*: it is pairwise jointly measurable but not fully jointly measurable.

We now investigate whether the above hollow triangle can lead to Bell inequality violation. The most general class of Bell inequalities to be considered here are of the form [28]

$$I = \sum_{x=1}^{3}\sum_{y=1}^{n} \gamma_{xy}\langle A_x B_y\rangle + \sum_{x=1}^{3}\alpha_x\langle A_x\rangle + \sum_{y=1}^{n}\beta_y\langle B_y\rangle \le 1, \tag{9}$$

where

$$\begin{aligned}\langle A_x B_y\rangle &= p(a = b|xy) - p(a \ne b|xy);\\ \langle A_x\rangle &= p(0|x) - p(1|x),\\ \langle B_y\rangle &= p(0|y) - p(1|y).\end{aligned} \tag{10}$$

All (tight) Bell inequalities of the above form for $n \le 5$ are known (see the Supplemental Material [29]). Using a

TABLE I. Bell inequality violation with incompatible POVMs. Specifically, we consider the sets given in Eqs. (8) and (13). For each set, we determine the smallest value of the parameter η, such that the set becomes jointly measurable (JM), and achieve Bell inequality violation. We consider tight Bell inequalities where Bob has up to $n = 5$ measurements (see Supplementary Material [29]). Note that pairwise joint measurability is equivalent to violation of the CHSH Bell inequality.

	$\{M'_{a\|x}(\eta)\}$	$\{M''_{a\|x}(\eta)\}$
Pairwise JM (CHSH violation)	$1/\sqrt{2} \approx 0.7071$	0.5858
Triplewise JM	$1/\sqrt{3} \approx 0.5774$	0.4226
Bell violation ($n = 3$): I_{3322}	0.8037	0.6635
Bell violation ($n = 4$): I^1_{3422}	0.8522	0.7913
I^2_{3422}	0.8323	0.5636
I^3_{3422}	0.8188	0.6795
Bell violation ($n = 5$): I_{3522}	0.7786	0.5636

numerical method based on semi-definite-programming [30] (see the Supplemental Material [29]), we could find the smallest value of the parameter η for which a given inequality can be violated using the set of POVMs of Eq. (8). The results are summarized in Table I. Notably, we could not find a violation in the range $1/\sqrt{3} < \eta \leq 1/\sqrt{2}$, where the set $\{M'_{a|x}(\eta)\}$ is a hollow triangle. In fact, no violation was found for $\eta \leq 0.7786$, whereas pairwise joint measurability is achieved for $\eta \leq 1/\sqrt{2} \simeq 0.7071$, thus leaving a large gap. Note also that pairwise joint measurability implies violation of the CHSH inequality here, since we have POVMs with binary outcomes [16]. We thus conjecture that there is a threshold value $\eta^* > 1/\sqrt{3}$, such that all hollow triangles with $1/\sqrt{3} < \eta \leq \eta^*$ do not violate any Bell inequality.

Moreover, we can also show that a large class of Bell inequalities of the form of Eq. (9) (for arbitrary n) cannot be violated using the hollow triangle of Eq. (8). Note that for the set of POVMs, Eq. (8), we have that $\langle A_x \rangle = \eta \mathrm{tr}(\sigma_x \rho_A)$ and $\langle A_x B_y \rangle = \eta \mathrm{tr}(\sigma_x \otimes M_{b|y} \rho_{AB})$ for $x = 1, 2, 3$. Hence, we can write the Bell polynomial as

$$I = \eta \tilde{I} + (1 - \eta) \sum_y \beta_y \langle B_y \rangle, \tag{11}$$

where $\tilde{I}$ is the Bell expression I evaluated for projective (Pauli) measurements on Alice's side. Note that $\tilde{I} \leq I_{\mathbb{C}^2}$, where $I_{\mathbb{C}^2}$ denotes the maximal value of I for qubit strategies. Hence, no Bell inequality violation is possible when

$$\sum_y |\beta_y| \leq \frac{1 - \eta I_{\mathbb{C}^2}}{1 - \eta}, \tag{12}$$

given that $\eta I_{\mathbb{C}^2} \leq 1$. Notably this includes all full correlation Bell inequalities ($\alpha_x = \beta_y = 0$), i.e., XOR games, for which it is known that the amount of violation is upper bounded for qubit strategies. More precisely, one has that $I_{\mathbb{C}^2} \leq K_3$ [31] where $K_3 \leq 1.5163$ is the Grothendieck constant of order 3. Hence, for $1/\sqrt{3} < \eta < 1/K_3 \simeq 0.6595$ we get that the hollow triangle of Eq. (8) cannot violate any full correlation Bell inequality.

From the above, one may actually wonder whether Bell inequality violation is possible at all using a set of POVMs forming a hollow triangle. We now show that this is the case. Consider the set of three dichotomic POVMs (acting on $\mathbb{C}^2$) given by the following positive operators:

$$M''_{0|x}(\eta) = \frac{\eta}{2}(\mathbb{1} + \sigma_x), \tag{13}$$

for $x = 1, 2, 3$ and $0 \leq \eta \leq 1$. Again, one has that $M''_{1|x}(\eta) = \mathbb{1} - M''_{0|x}(\eta)$. To determine the range of the parameter η for which the above set of POVMs is pairwise jointly measurable, and fully jointly measurable, we use the semi-definite-programming techniques of Ref. [16]. We find that the set $\{M''_{a|x}(\eta)\}$ is a hollow triangle for $0.4226 \leq \eta \leq 0.5858$. However, Bell nonlocality can be obtained by considering a Bell inequality with $n = 4$ measurements for Bob, for $\eta > 0.5636$. Values are summarized in Table I, while details of the construction are given in the Supplemental Material [29]. This shows that a set of partially compatible measurements, here a hollow triangle, can be used to violate a Bell inequality. Moreover, this suggests that detecting the nonlocality of a set of three incompatible POVMs is a hard problem, since a large number of measurements on Bob's side (possibly infinite) might be needed. This contrasts with the case of two POVMs, where two measurements (via CHSH) were enough [16]. Finally, note that we could find a hollow triangle with only real numbers (i.e., with all Bloch vectors in a plane of the sphere) which violates a Bell inequality with $n = 3$ measurements [32], i.e., the simplest possible case (see the Supplemental Material [29]).

Finally, an interesting open question is the following. Considering a set of arbitrarily many POVMs, it is known that any partial compatibility configuration can be realized [33]. Is it then possible to violate a Bell inequality for any possible configuration?

Discussion.—We have discussed the relation between joint measurability and quantum nonlocality. First, we showed that a set of POVMs is incompatible if and only if it can be used to demonstrate EPR steering. Hence, EPR steering provides a new operational interpretation of joint measurability. Second, we explored the link between joint measurability and Bell nonlocality. We gave evidence that these two notions are inequivalent, by showing that a hollow triangle (a set of 3 POVMs that is only pairwise compatible) can never lead to a violation of a large class of Bell inequalities. We conjecture that this hollow triangle is Bell local, that is, it cannot be used to violate any Bell inequality. Hence, such a measurement would represent the analogue, for a quantum measurement, of a local entangled state.

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Note added.—While the present work was under review, we became aware of a related work [34].

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