Area Law Scaling for the Entropy of Disordered Quasifree Fermions

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We study theoretically and numerically the entanglement entropy of the d-dimensional free fermions whose one-body Hamiltonian is the Anderson model. Using the basic facts of the exponential Anderson localization, we show first that the disorder averaged entanglement entropy $\langle S_\Lambda \rangle$ of the d dimension cube Λ of side length l admits the area law scaling $\langle S_\Lambda \rangle \sim l^{(d-1)}$, $l \gg 1$, even in the gapless case, thereby manifesting the area law in the mean for our model. For $d = 1$ and $l \gg 1$ we obtain then asymptotic bounds for the entanglement entropy of typical realizations of disorder and use them to show that the entanglement entropy is not self-averaging, i.e., has nonvanishing random fluctuations even if $l \gg 1$.

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Entanglement is a basic ingredient of the quantum description, having a great potential for applications [1]. An important quantifier of entanglement is the von Neumann entropy. In the bipartite setting, where the system is the union of a subsystem and its environment of the characteristic lengths l and L, the entropy of the reduced density matrix of the subsystem (entanglement or block entropy) may have an unusual asymptotic behavior as a function of $l, 1 \ll l \ll L$, if the whole system is in its ground state. Namely, it was shown in a number of works that the entanglement entropy is proportional to the surface area l^{d-1} of the subsystem but not to its volume l^d. The latter (extensive) length scaling is standard in quantum statistical mechanics for nonzero temperature (thermal entanglement), while the former was found first in cosmology and then in other fields and is known as the area law. Moreover, the area law is not always valid, e.g., at quantum critical points of several one-dimensional (1D) translation invariant quantum spin chains, where the entropy is proportional to $\log l, l \gg 1$. It is also believed and found for simple translation invariant models that a multidimensional analog of the above divergence is $l^{d-1} \log l$ [2,3].

More generally, the area law scaling l^{d-1} is to be valid for quantum systems with finite range interaction and a spectrum gap, while for gapless systems other scalings are possible, $l^{d-1} \log l$ in particular, which is closely related to the existence of a quantum phase transition in the corresponding system [2]. This is, however, not simple to prove, even in the translation invariant case, since the spectrum of many-body quantum systems is rather complex even for 1D exactly solvable models. On the other hand, there is a simpler model having both spectrum types and both scalings. This is the model of quasifree fermions described by the Hamiltonian quadratic in the creation and annihilation operators. For this Hamiltonian with finite range and translation invariant coefficients the large-l scaling of the entanglement entropy for any $d \geq 1$ was established first via the upper and lower bounds and certain conjectures on the subleading term in the Szegö theorem for Toeplitz determinants [3] and then rigorously [4].

Following a standard paradigm of condensed matter theory, it is natural to consider a disordered version of the model, replacing the translation invariant coefficients of the fermionic Hamiltonian with random coefficients, which are translation invariant in the mean and have fast decaying spatial correlations [5].

The analysis of quadratic fermionic forms reduces to that of a certain one-body Hamiltonian. Hence, in the case of random coefficients we obtain a problem of the theory of one-body disordered systems, which, however, proves to be quite nontrivial in general. Thus, to demonstrate the role of disorder in the asymptotic behavior of the entanglement entropy without involving too many technicalities it is natural to use a simple but nontrivial setting and to ask simpler questions, e.g., about the upper and lower bounds for the disorder averaged entanglement entropy implying its scaling (see Ref. [3] for an analogous approach in the translation invariant case), the same for the entanglement entropy of typical realizations of disorder, and/or about the self-averaging property of the entropy. Recall that a number of important characteristics of disordered system (free energy, magnetization, density of states, conductivity, etc.) possess this property, i.e., become nonrandom in the macroscopic limit [5]. This allows one to deal only with the disorder averaged characteristics, but not with their whole probability distributions.

We will show in this Letter that, for the free fermions in the random external field, (i) for any $d \geq 1$ the averaged entanglement entropy possesses the area law scaling l^{d-1}, (ii) for $d = 1$ the same in true for all typical realizations of disorder, and (iii) the entropy is not self-averaging for $d = 1$.

Model.—We consider the system of N lattice spinless fermions with the parity conserving Hamiltonian

$$\mathcal{H} = \sum_{j,k=1}^{N} A_{jk} c_j^{+} c_k, \tag{1}$$

where $c_j^{+}, c_j, j = 1, \ldots, N$ are the Fermi operators and $A = \{A_{jk}\}$ is a Hermitian $N \times N$ matrix.

By using the Bogolyubov transformation it is easy to find that if $K = \{\langle c_j^{+} c_k \rangle_G\}_{j,k=1}^{N}$, where $\langle \cdots \rangle_G$ is the "Gibbs" averaging with the "density" matrix

$$\rho = e^{-\mathcal{H}}/Z, \qquad Z = \mathrm{Tr} e^{-\mathcal{H}}/Z, \tag{2}$$

then

$$K = (1 + e^{A})^{-1}, \qquad A = -\log K(1-K)^{-1}, \tag{3}$$

$$S = -\mathrm{Tr}\rho \log_2 \rho = \mathrm{tr} h(K), \tag{4}$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x), \tag{5}$$

where Tr and tr denote the trace in the 2^N-dimensional space of N fermions and in the N-dimensional one-body configuration space, respectively.

We choose $A = (H - \mu)/T$ where $H = H_0 + V$ is the Anderson model, in which $H_0 = a\Delta$, a is the hopping parameter, Δ is the discrete Laplacian, $V = \{V_j\}_{j\in\Omega}$ is the random potential, μ is the Fermi energy, and T is the temperature. Then Eq. (3) implies

$$P = \{P_{jk}\}_{j,k\in\Omega} = K|_{T=0} = \theta(\mu - H), \tag{6}$$

where θ is the Heaviside function. Thus, P is the orthogonal projection on the ground state of the whole system; the Slater determinant on the first n eigenstates of H, where $n/|\Omega| = N(\mu)$ and $N(\mu)$ is the integrated density of states of H. Hence the entropy (4) of the whole system is zero.

Consider now a subsystem of fermions in a subcube Λ of Ω; the latter can be the whole $\mathbb{Z}^d$. We assume that Λ is centered at the origin and of side length $l = 2m + 1$. Note that the setting is not unambiguous for indistinguishable particles and we use its natural version known as the entanglement of modes [6]. Then the corresponding reduced density matrix is $\rho_\Lambda = e^{-\mathcal{H}_\Lambda}/Z_\Lambda$, where $\mathcal{H}_\Lambda$ is the entanglement Hamiltonian [2] given by Eq. (1) with $A = -\log P_\Lambda(1 - P_\Lambda)^{-1}$ and [see Eqs. (6) and (4)]

$$S_\Lambda = -\mathrm{Tr}\rho_\Lambda \log_2 \rho_\Lambda = \mathrm{tr} h(P_\Lambda), \qquad P_\Lambda = \{P_{jk}\}_{j,k\in\Lambda}. \tag{7}$$

The area law scaling for the disorder averaged entanglement entropy.—We will show now that if the spectrum of H below μ is localized, then the disorder average $\langle S_\Lambda \rangle$ scales as l^{d-1} for $l \gg 1$. To this end, we present upper and lower bounds for $\langle S_\Lambda \rangle$, which are asymptotically proportional to l^{d-1}.

We start from bounds for h of Eq. (4) [2],

$$\varphi(x) \le h(x) \le \sqrt{\varphi(x)}, \qquad \varphi(x) = 4x(1-x). \tag{8}$$

The bounds and Eq. (7) imply

$$\begin{aligned} L_\Lambda \le S_\Lambda \le U_\Lambda, &\qquad L_\Lambda = 4\mathrm{tr}\Gamma_\Lambda, \\ U_\Lambda = 2\mathrm{tr}\sqrt{\Gamma_\Lambda}, &\qquad \Gamma_\Lambda = K_\Lambda(\mathbf{1}_\Lambda - K_\Lambda). \end{aligned} \tag{9}$$

We use the equality $\sum_{k\in\mathbb{Z}^d} |P_{jk}|^2 = P_{jj}$, valid for any orthogonal projection, to write

$$L_\Lambda = 4\sum_{j\in\Lambda}\sum_{k\in\bar{\Lambda}} |P_{jk}|^2, \tag{10}$$

where $\bar{\Lambda}$ is the exterior of Λ.

Note that in the 1D translation invariant case $P_{jk} = \sin\kappa(j-k)/\pi(j-k)$, where κ is the Fermi momentum, and Eq. (13) yields $L_\Lambda \simeq 4\pi^{-2}\log l, l \gg 1$. This is a simple example of the log scaling in the translation invariant case. A more involved argument leads to the lower bound $\sim l^{d-1}\log l$ for any $d \ge 1$ [3] and to the corresponding asymptotic formula [4].

Assume that the potential is independent and identically distributed (IID) in different points. Then $\langle |P_{jk}|^2 \rangle = \Pi_{j-k}$, where $\Pi_j = \Pi_{-j}$ and is symmetric in the coordinates $(j_1, \ldots, j_d)$, and Eq. (10) implies

$$\langle L_\Lambda \rangle = 4\sum_{j\in\Lambda}\sum_{k\in\bar{\Lambda}} \Pi_{j-k}. \tag{11}$$

It is easy to find that Π_{j-k} is the integral over $\Delta \times \Delta, \Delta = (-\infty, \mu)$, of the current-current correlator $\langle (\delta(H-E_1))_{jk}(\delta(H-E_2))_{jk} \rangle$ determining the ac conductivity of free disordered fermions [7].

Note that we write here and below $a_l \simeq b_l$ if b_l is the leading term of a_l for $l \gg 1$ and $a_l \lesssim b_l$ if $a_l \le b_l$, $c_l \simeq b_l$.

We will use now a basic rigorous result of the localization of states of the d-dimensional Anderson model, according to which if the probability distribution of the IID random potential is smooth enough and either μ is close enough to the bottom of the spectrum or the hopping parameter is small enough, then $\langle |P_{jk}| \rangle \le Ce^{-\gamma|j-k|}$ for some $C < \infty$ and $\gamma > 0$ (see, e.g., Ref. [8]). This and the inequality $|P_{jk}| \le 1$ valid for any orthogonal projection imply

$$\Pi_j \le Ce^{-\gamma|j|}. \tag{12}$$

The sum over k in Eq. (11) consists of $2^d - 1$ sums such that $\binom{d}{\delta}$, $\delta = 1, \ldots, d$ of them have the coordinates $k_{\alpha_1}, \ldots, k_{\alpha_\delta}$ outside the interval $[-m, m]$ and the rest inside the interval. Since the summands of Eq. (11) are non-negative, $\langle L_\Lambda \rangle$ is bounded below by the sums with $\delta = 1$ (in fact, the leading term of $\langle L_\Lambda \rangle$ for $l \gg 1$) and then Eq. (12) yields that up to exponential small in l terms

$$\langle L_\Lambda \rangle \geq 4d \sum_{j\in\Lambda} \sum_{|k_1|>m} \Pi^{(1)}_{|j_1-k_1|} \simeq c_- l^{d-1},$$
$$c_- = 8d \sum_{t\geq 1} t\Pi_t^{(1)}, \quad \Pi_t^{(1)} = \sum_{j_2,\ldots,j_d \in \mathbb{Z}^{d-1}} \Pi_{t,j_2,\ldots,j_d}. \tag{13}$$

For the upper bound U_Λ of Eq. (9) we will use the inequality $\mathrm{Tr} f(M) \leq \sum_{j=1}^n f(M_{jj})$ valid for any $n \times n$ Hermitian M and a concave f. The inequality is a version of the Peierls variation principle [9] with the only difference being that it is usually formulated for convex f, e^{-x} in particular, thus with the opposite inequality.

We use the inequality with $M = \Gamma_\Lambda$ of Eq. (9) and $f(x) = \sqrt{x}$ to obtain [cf. Eq. (10)]

$$U_\Lambda \leq 2 \sum_{j\in\Lambda} \left(\sum_{k\in\bar{\Lambda}} |P_{jk}|^2 \right)^{1/2} \tag{14}$$

and then the Schwarz inequality $\langle \xi^{1/2} \rangle \leq \langle \xi \rangle^{1/2}$ and Eq. (12) [cf. Eq. (11)] to obtain

$$\langle U_\Lambda \rangle \leq \sum_{j\in\Lambda} \left(\sum_{k\in\bar{\Lambda}} \Pi_{k-j} \right)^{1/2} < \infty. \tag{15}$$

Since $\Pi_{jk} \geq 0$, the sum over $k = (k_1, k_2, \ldots, k_d) \in \bar{\Lambda}$ is not less than $(2^d - 1)$ times the sum over $|k_1| > l$ and $(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}$. This and the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ yield up to exponential small in l terms [cf. Eq. (13)]

$$\langle U_\Lambda \rangle \lesssim c_+ l^{d-1}, \qquad c_+ = 4(2^d - 1) \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} \Pi^{(1)}_{k+j} \right)^{1/2}. \tag{16}$$

Note that $c_\pm$ of Eqs. (13) and (16) are finite in view of Eq. (12). This and Eq. (9) prove the validity of the area law scaling $\langle S_\Lambda \rangle \sim l^{d-1}$ for the averaged entanglement entropy of free disordered fermions. For similar results on disordered oscillators see Ref. [10].

Bounds for the 1D entanglement entropy on typical realizations of disorder.—We will write L_l and U_l for L_Λ and U_Λ and $\Lambda = [-m, m]$ and $l = 2m + 1$. We have

$$L_l = 4 \sum_{|j|\leq m} \sum_{|k|>m} |P_{jk}|^2 = L_l^+ + L_l^-,$$
$$L_l^+ = 4 \sum_{|j|\leq m} \sum_{k>m} |P_{jk}|^2, \qquad L_l^- = 4 \sum_{|j|\leq m} \sum_{k<-m} |P_{jk}|^2, \tag{17}$$

thus,

$$L_l^\pm = \mathcal{L}_l^\pm + R_l^\pm, \qquad \mathcal{L}_l^\pm = 4 \sum_{j=\pm m}^{\mp\infty} \sum_{k=\pm(m+1)}^{\pm\infty} |P_{jk}|^2,$$
$$R_l^\pm = 4 \sum_{j=\mp(m+1)}^{\mp\infty} \sum_{k=\pm(m+1)}^{\pm\infty} |P_{jk}|^2. \tag{18}$$

According to Eq. (12), $\langle R_l^\pm \rangle \leq C_1 e^{-\gamma_1 l}$ where $C_1 < \infty$ and $\gamma_1 > 0$. This, the Chebyshev inequality, and the Borel-Cantelli lemma [11] imply that R_l^+ vanishes with probability 1 (i.e., for any typical realizations) as $l \to \infty$, i.e., $L_l^+ \simeq \mathcal{L}_l^+, l \gg 1$, with probability 1. Introduce the shift operator T: $(TV)_j = V_{j+1}$. Writing the Anderson Hamiltonian as $H(V)$ to make explicit its dependence on V, we find that $H_{jk}(T^a V) = H_{j+a,k+a}(V)$. This and Eq. (6) imply the same for $\{P_{jk}\}_{j,k\in\mathbb{Z}}$; thus, $\mathcal{L}_m^+(V) = \mathcal{L}_0^+(T^l V)$. The terms of the series in Eq. (18) for $\mathcal{L}_0^+$ are non-negative random functions; thus, the series is convergent with a probability 1 if the series of its averages is convergent. This is again guarantied by Eq. (12). Thus, $\mathcal{L}_0^+$ of Eq. (18) is well defined and we have $L_l^+ \simeq \mathcal{L}_0^+(T^m V), l \gg 1$, with probability 1. Likewise $L_l^- \simeq \mathcal{L}_0^-(T^{-m} V), l \gg 1$, and

$$L_l \simeq \mathcal{L}_0^+(T^m V) + \mathcal{L}_0^-(T^{-m} V), \qquad l \gg 1, \tag{19}$$

with the same probability.

A similar argument yields [cf. Eq. (19)]

$$U_l \lesssim \mathcal{U}_0^+(T^m V) + \mathcal{U}_0^-(T^{-m} V), \qquad l \gg 1,$$
$$\mathcal{U}_0^\pm(V) = 2^{3/2} \sum_{j=0}^{\mp\infty} \left(\sum_{k=\pm 1}^{\pm\infty} |P_{k,j}|^2 \right)^{1/2}. \tag{20}$$

Figure 1 presents our numerical results on the probability distributions $p_L(x)$ and $p_U(x)$ of the lower [Eq. (19)] and

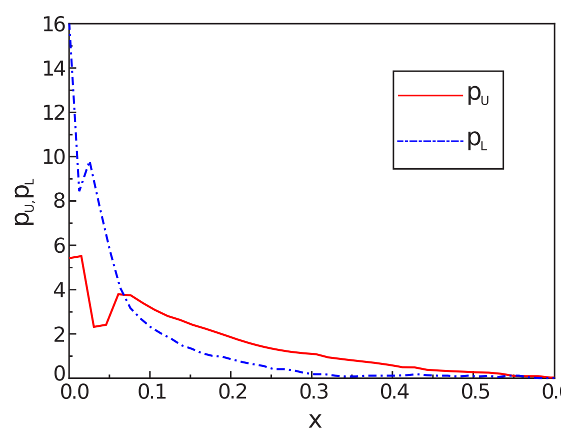

FIG. 1 (color online). The probability distributions $p_U(x)$ and $p_L(x)$ of the lower (19) and upper (20) bounds obtained from numerical data on 15 000 realizations of disorder. The hopping parameter a of the Anderson model is $1/10$, the potential is uniformly distributed over $[-1, 1]$, $\mu = -0.25$, the system size $L = 30\,000$, and the subsystem size $l = 1500$.

upper [Eq. (20)] bounds; the latter is with the optimal exponent $\log_2$ instead of $1/2$ [see item (i) in the next section]. It is important that p_L and p_U are nonzero on practically the same intervals. This implies that the entanglement entropy S_l depends nontrivially on the realizations of disorder even if $l \gg 1$; i.e., S_l is not self-averaging. Indeed, if it were self-averaging, i.e., $S_l \simeq S, l \gg 1$, for a nonrandom S, then the whole interval where the probability density p_U of the upper bound (20) is nonzero would lie on the right of S, while the whole interval where the probability density p_L of the lower bound (19) is not zero would lie on the left of S. Thus, these two probability densities would not overlap.

Besides, it follows from the analysis of numerically obtained probability distributions of U_l and L_l with growing l that they become independent of l (saturate) for $l \gg 1$. This can be explained as follows. Since the random potential is independent in different points, the first two terms of the right-hand sides of Eqs. (19) and (20) have to also be statistically independent for $l \gg 1$, and since the potential is translation and reflection symmetric in the mean, the probability distributions of these terms are independent of m and identical. Hence, for $l \gg 1$ the probability distributions p_L and p_U of Eqs. (19) and (20) are the convolutions of l-independent probability distributions of $\mathcal{L}_0^{\pm}$ and $\mathcal{U}_0^{\pm}$ and this was also checked numerically.

It is worth mentioning that our numerical results do not allow us to conclude that the probability distribution of the entanglement entropy $S_l, l \gg 1$, is concentrated on a finite interval, hence that the random function S_l is bounded by a nonrandom constant for $l \gg 1$ on the typical realizations of disorder. In fact, this seems unlikely. Rather, one has to expect that for every typical realization of disorder there exists an infinite sequence $\{l_n\}$ of values of l such that $S_{l_n} \to \infty$ as $n \to \infty$. However, these would be just rather rare peaks of randomly fluctuating entanglement entropy (7) but not its "regular" asymptotics.

Remarks.—(i) The bound $\sqrt{\varphi}$ in Eq. (8) can be replaced by a tighter one $\varphi^{\alpha}, \alpha = \log 2$. (ii) Analogous results are valid for the Rényi entropy $R_\alpha = (1-\alpha)^{-1}\mathrm{Tr}\log_2\rho_\Lambda^\alpha$, reducing to the von Neumann entropy (7) for $\alpha = 1$. (iii) The above results are based on Eq. (12) manifesting the localization for the corresponding one-body problem. It follows from Ref. [12] that an analogous bound holds for the 1D Schrödinger operator with certain incommensurate potentials. Thus, the entanglement entropy of 1D free fermions in the corresponding external fields is also bounded. (iv) We have discussed the area law for the Fermi energy lying in the localized spectrum of the Anderson model. The case where the Fermi energy is in a gap is much simpler. Here an analog of Eq. (12) can be obtained by writing Eq. (6) as the contour integral of the exponentially decaying Green's function. (v) One can ask about the asymptotics of the entanglement entropy for nonzero temperature (thermal entanglement). In this case the leading term of the entropy is proportional to l^d with a nonrandom coefficient and there are certain random or incommensurate subleading terms of various scaling (a stochastic analog of the Szegö theorem [13]).

Conclusion.—We have shown that for the free fermions in the random external field the averaged entanglement entropy of the $d \geq 1$ dimension cube of side length l is bounded from above and from below by $c_{\pm} l^{d-1}$. The result suggests the validity of the area law "in the mean" even in the gapless case for disordered free fermions. This has to be compared with the results for the translation invariant case, where the entropy scales as $l^{d-1}\log l$, and with those of a series of works (see Ref. [14] for a review) in which, by using a strong disorder version of the real space renormalization group, it was found that the averaged entropy at critical points of certain disordered spin chains scales as in the nonrandom case, although with a different prefactor of $\log l$. This could be an indication of the difference of the origin of the area law for disordered spin chains and disordered free fermions where there is no interaction and a nontrivial entanglement is due to a pure "kinematic" effect of Fermi statistics, hence simple formulas (3)–(5).

We have also obtained bounds for the $d = 1$ entanglement entropy of all typical realizations of disorder. The bounds do not imply in general that the entropy of typical realizations is bounded for $l \gg 1$ "uniformly" in realizations, i.e., by a nonrandom constant. However, we show numerically that the bounds have a nontrivial l-independent for $l \gg 1$ and overlapping probability distributions (see Fig. 1) manifesting that the entanglement entropy is not self-averaging, i.e., has nonvanishing random fluctuations for $l \gg 1$.

Our results can be viewed as an indication of an important role of disorder in the entanglement in extended systems, similarly to its role in condensed matter (Anderson localization) and phase transitions (rounding effects). This seems to be especially interesting in the dimension 1, where the Anderson localization is the case for arbitrary small disorder and all energies [5]. The results can also be used in the elaboration of the density matrix renormalization group method [2,6] for disordered systems.

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