Sphaleron Rate in the Minimal Standard Model

Michela D'Onofrio,[1] Kari Rummukainen,[1] and Anders Tranberg[2]
[1]*Department of Physics and Helsinki Institute of Physics, PL 64 (Gustaf Hällströmin katu 2), FI-00014 University of Helsinki, Finland*
[2]*Faculty of Science and Technology, University of Stavanger, N-4036 Stavanger, Norway*

(Received 22 April 2014; published 1 October 2014)

We use large-scale lattice simulations to compute the rate of baryon number violating processes (the sphaleron rate), the Higgs field expectation value, and the critical temperature in the standard model across the electroweak phase transition temperature. While there is no true phase transition between the high-temperature symmetric phase and the low-temperature broken phase, the crossover is sharp and located at temperature $T_c = (159.5 \pm 1.5)$ GeV. The sphaleron rate in the symmetric phase ($T > T_c$) is $\Gamma/T^4 = (18 \pm 3)\alpha_W^5$, and in the broken phase in the physically interesting temperature range 130 GeV $< T < T_c$ it can be parametrized as $\log(\Gamma/T^4) = (0.83 \pm 0.01)T/\mathrm{GeV} - (147.7 \pm 1.9)$. The freeze-out temperature in the early Universe, where the Hubble rate wins over the baryon number violation rate, is $T_* = (131.7 \pm 2.3)$ GeV. These values, beyond being intrinsic properties of the standard model, are relevant for, e.g., low-scale leptogenesis scenarios.

DOI: 10.1103/PhysRevLett.113.141602 PACS numbers: 11.10.Wx, 11.15.Ha, 98.80.Cq

Introduction.—The current results from the LHC are in complete agreement with the standard model of particle physics: a Higgs boson with the mass of 125–126 GeV has been discovered [1], and no evidence of exotic physics has been observed. If the standard model is indeed the complete description of the physics at the electroweak scale, the electroweak symmetry-breaking transition in the early Universe was a smooth crossover from the symmetric phase at $T > T_c$, where the (expectation value of the) Higgs field was approximately zero, to the broken phase at $T < T_c$ GeV where it is finite, reaching the experimentally determined value $\langle|\phi|\rangle \simeq 246/\sqrt{2}$ GeV at zero temperature. The detailed physics of the transition is nonperturbative due to the infrared problems inherent in high-temperature gauge field theory. The nature of the transition was settled in 1995–1998 using lattice simulations [2–5], which indicate a first-order phase transition at Higgs boson masses $\lesssim 72$ GeV and a crossover otherwise. This scenario was also suggested by analytical computations of nonperturbative effects at the transition [6,7].

A smooth crossover means that the standard "electroweak baryogenesis" scenarios [8,9] are ineffective. These scenarios produce the matter-antimatter asymmetry of the Universe through electroweak physics only, and they require a strong first-order phase transition, with supercooling and associated out-of-equilibrium dynamics. Thus, the origin of the baryon asymmetry must rely on physics beyond the standard model.

Baryogenesis at the electroweak scale is possible in the first place through the existence of the chiral anomaly relating the baryon number of fermions to the topological Chern-Simons number $N_{\rm cs}$ of the electroweak SU(2) gauge fields

$$\Delta N_{\rm cs}(t) = \frac{1}{32\pi^2}\int_0^t dt' \int d^3x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr} F^{\mu\nu} F^{\rho\sigma}, \qquad (1)$$

where $F^{\mu\nu}$ is the SU(2) field strength [10]. A net change over time of Chern-Simons number leads to a net change in baryon number B (and lepton number L),

$$B(t) - B(0) = L(t) - L(0) = 3[N_{\rm cs}(t) - N_{\rm cs}(0)]. \qquad (2)$$

The question is then whether such a permanent change can be achieved through the dynamics around the electroweak transition, either from a symmetric initial state, such as that for electroweak baryogenesis, or when it is sourced by another mechanism such as leptogenesis [11], where an initial lepton asymmetry is converted into a baryon asymmetry.

Close to thermal equilibrium, the evolution of the Chern-Simons number is diffusive and can be described through the diffusion constant

$$\Gamma = \lim_{V,t\to\infty} \frac{\langle [N_{\rm cs}(t) - N_{\rm cs}(0)]^2 \rangle}{Vt}, \qquad (3)$$

also known as the "sphaleron rate." It enters the diffusion equation for lepton and baryon number in baryogenesis [12] and leptogenesis (see for instance, Ref. [13]).

The quantity Γ has been the focus of extensive work for many years, and a powerful framework and set of analytic and numerical tools have been developed to compute it accurately using nonperturbative lattice simulations (see Ref. [14] and references therein). Until very recently, the precise value of the Higgs mass has not been available, although extrapolation of computations at other values of this mass is possible [13]. It seems a fitting conclusion to this scientific effort to now employ all the available techniques and finally compute the sphaleron rate of the complete minimal standard model. This will also point forward to similar computations of the rate in extensions of the standard model where the electroweak transition can be strongly first order (SM + scalar singlet model, two-Higgs doublet models, supersymmetric models).

Simulation method.—We will restrict ourselves to a brief summary of the methods and techniques used and refer the reader to detailed information in the literature (Ref. [14] and references therein). The full standard model is not directly amenable to lattice simulations. However, at high temperatures the modes corresponding to scales $\geq g_W T$, including all fermionic modes, can be reliably treated with perturbative methods. The nonperturbative infrared ($k \lesssim g_W^2 T$) physics of the standard model is fully contained in an effective three-dimensional theory, which includes the Higgs field and the spatial SU(2) gauge field [15]

$$S = \int d^3x \left[\frac{1}{4} F_{ij}^a F_{ij}^a + |D_i\phi|^2 + m_3^2|\phi|^2 + \lambda_3|\phi|^4\right]. \quad (4)$$

The coefficients m_3^2, g_3^2, and λ_3 are functions of the four-dimensional continuum parameters [$\alpha_S(M_W)$, G_F, $M_{\rm Higgs}$, M_W, M_Z, $M_{\rm top}$, and the temperature T] through a set of 1- and 2-loop matching relations [15] and are shown in Fig. 1 as functions of the temperature.

We do not include the hypercharge U(1) field explicitly in the effective theory Eq. (4) because it has little effect on the infrared physics [16,17], but we take it into account in our final error analysis. Naturally, the U(1) field and the weak mixing angle do contribute to the values of the parameters of Eq. (4).

The effective action is bosonic and easily discretized on the lattice. The parameters of the lattice action are perturbatively related to the continuum action [18]; we also implement the partial $O(a)$ improvement of Ref. [19]. The effective action Eq. (4) has been very successfully used in calculations of static thermodynamic properties of the standard model, but with unphysical Higgs boson masses [2–4].

For the measurement of the sphaleron rate it is necessary to evolve the system in real time. As such, the effective theory in Eq. (4) does not describe dynamical phenomena. However, the infrared ($k \lesssim g_W^2 T$) modes have large occupation numbers and behave nearly classically. Thus, one is well motivated to apply classical equations of motion to Eq. (4) (after introducing canonical momenta). This method has been used in early studies of the sphaleron rate [20,21]. However, it has serious problems: because of the UV divergent Landau damping in the classical theory, the simulation results are lattice spacing dependent and the continuum limit does not exist [22,23]. These problems can be partially ameliorated by using more complicated effective theories that include so-called hard thermal loop effects [24], but the continuum limit is still out of reach.

A particularly attractive method was first described by Bödeker [25]: because the dynamics of the infrared modes is fully overdamped, the gauge field evolution can be described with a set of Langevin equations to leading logarithmic accuracy [$\ln^{-1}(1/g_W)$] [23,25]

$$\partial_t A_i = -\sigma_{\rm el}^{-1}\frac{\partial H}{\partial A_i} + \xi_i^a, \quad (5)$$

where $\sigma_{\rm el} \approx 0.9239T$ is the non-Abelian "color" conductivity for the standard model, and we have identified $H/T = S$ in Eq. (4). The Gaussian noise vector ξ obeys

$$\langle \xi_i^a(\mathbf{x},t)\xi_j^b(\mathbf{x}',t')\rangle = 2\sigma_{\rm el}T\delta_{ij}\delta^{ab}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \quad (6)$$

The Higgs field is parametrically much less damped than the gauge field, and it can be evolved with Langevin dynamics with a faster rate of evolution [26]. On the lattice, the Langevin dynamics can be substituted with any fully diffusive dynamics, for instance, the heat bath update with random order. The heat bath update step can be rigorously related to the Langevin time and, hence, the real evolution time [26]. The continuum limit exists, and the finite lattice spacing effects have been observed to be modest.

This method has been successfully used to measure the sphaleron rate in pure gauge theory [26] and in the standard model [14,27], but not yet using the physical Higgs mass. It has also been used to study the bubble nucleation rate in first-order electroweak phase transition [28] at unphysically small Higgs mass.

The sphaleron rate is measured using Eqs. (1)–(3). However, because topology is not well defined on a coarse lattice, we use the "calibrated cooling" method of Ref. [29], which gives a robust observable for the Chern-Simons

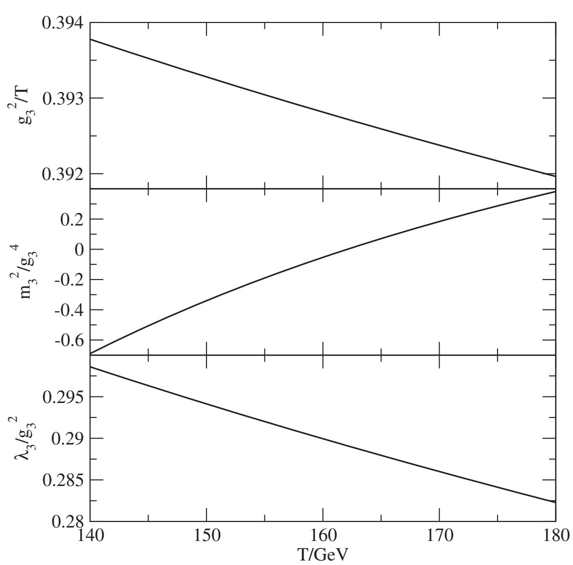

FIG. 1. The parameters of the effective theory (4) as functions of the temperature.

number. In the symmetric phase, calibrated cooling can be directly applied to the configurations generated by the heat bath evolution. Deep in the low-temperature broken phase the situation is more complicated. Although the Langevin dynamics is still correct, the potential barriers between the topological sectors become very large, because the Higgs field has to vanish in the core of the sphaleron. Hence, the rate becomes very small, and it is not practical to measure it in normal simulations. This difficulty can be overcome with a special *multicanonical* Monte Carlo computation, where the multicanonical method itself is used to calculate the height of the sphaleron barrier (∼ sphaleron energy), and special real-time runs are performed to calculate the dynamical prefactors of the tunneling process. The physical rate is then obtained by reweighting the measurements. For details of this intricate technique, we refer to Refs. [14,29]. As we will observe, in the temperature range where both methods work, these overlap smoothly.

Simulation results.—We perform the simulations using lattice spacing $a = 4/(9g_3^2)$ [i.e., $\beta_G = 4/(g_3^2 a) = 9$ in conventional lattice units] and volume $V = 32^3 a^3$. In Ref. [14] we observed that the rate measured with this lattice spacing in the symmetric phase is in practice indistinguishable from the continuum rate, and deep in the broken phase it is within a factor of 2 of our estimate for the continuum value. This accuracy is sufficient due to the exponential suppression of the rate. Indeed, other systematic effects that would not be removed by the continuum limit dominate our final error estimates, as described below. In fact, our multicanonical algorithm becomes severely inefficient deep in the broken phase at significantly smaller lattice spacings, which makes a more controlled continuum limit very costly. The simulation volume is large enough for the finite-volume effects to be negligible [14].

In Fig. 2 we show the sphaleron rate as a function of temperature. The straightforward Langevin results cover the high-temperature phase, where the rate is not too strongly suppressed by the sphaleron barrier. In fact, we were able to extend the range of the method through the crossover and into the broken phase, down to relative suppression of 10^{-3}.

Using the multicanonical simulation methods, we are able to compute the rate 4 orders of magnitude further down into the broken low-temperature phase. The results nicely interpolate with the canonical simulations in the range where both exist. In the interval $140 \lesssim T \lesssim 155$ GeV the broken phase rate is very close to a pure exponential and can be parametrized as

$$\log \frac{\Gamma_{\rm Broken}}{T^4} = (0.83 \pm 0.01)\frac{T}{\rm GeV} - (147.7 \pm 1.9). \qquad (7)$$

The error in the second constant is completely dominated by systematics. We conservatively estimate that the uncertainties of the leading logarithmic approximation and remaining lattice spacing effects [14] may affect the rate

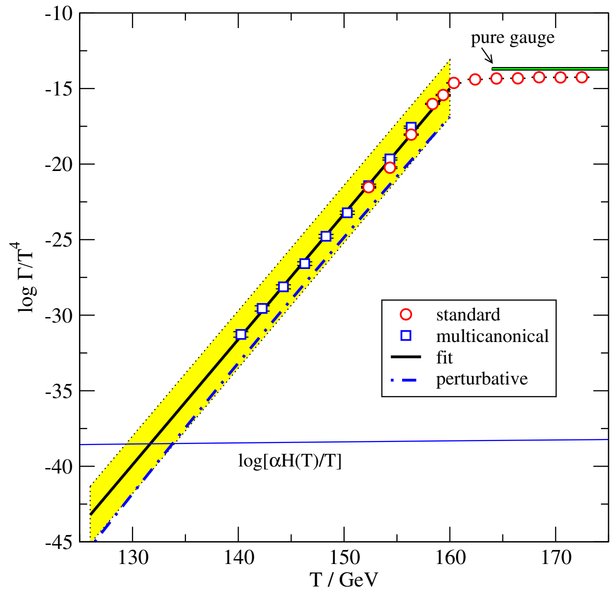

FIG. 2 (color online). The measured sphaleron rate and the fit to the broken phase rate, Eq. (7), shown with a shaded error band. The perturbative result is from Burnier *et al.* [13] with the nonperturbative correction used there removed; see main text. Pure gauge refers to the rate in hot SU(2) gauge theory [21]. The freeze-out temperature T_* is solved from the crossing of Γ and the appropriately scaled Hubble rate, shown with the almost horizontal line.

by a factor of 2. The omitted hypercharge U(1) in the effective action (with physical θ_W) can change the sphaleron energy by ≈1% [16] and shift the pseudocritical temperature by ≈1 GeV [17]. These errors have been added linearly together to obtain the error above.

In the symmetric phase the rate (divided by T^4) is approximately constant and can be presented as

$$\Gamma_{\rm Symm}/T^4 = (8.0 \pm 1.3) \times 10^{-7} \approx (18 \pm 3)\alpha_W^5, \qquad (8)$$

where, in the last form, factors of $\ln \alpha_W$ have been absorbed in the numerical constant. In pure SU(2) gauge theory the rate is $\Gamma \approx (25 \pm 2)\alpha_W^5 T^4$ [24,30]. A difference of this magnitude was also observed in Ref. [27].

In Fig. 2 we also show the perturbative result calculated by Burnier *et al.* [13]. We note that the full rate in Ref. [13] is obtained by including a large nonperturbative correction to the perturbative rate, $\log(\Gamma/T^4) = \log(\Gamma_{\rm pert}/T^4) - (3.6 \pm 0.6)$, where the correction is obtained by matching with earlier simulations in the broken phase [29]. However, these simulations were done with a Higgs mass ≈50 GeV, which is far from the physical one studied here. With the correction included, their result is a factor of ≈150 below our rate, albeit with large uncertainty. In Fig. 2 we have removed this *ad hoc* correction altogether, and the resulting purely perturbative rate agrees with our results well within the given uncertainties of both the lattice and the perturbative computation ($\delta \log \Gamma_{\rm pert}/T^4 = \pm 2$). Indeed, by applying a smaller but opposite correction $\log(\Gamma/T^4) \approx \log(\Gamma_{\rm pert}/T^4) + 1.6$, the central value agrees perfectly with our measurements. Because the perturbative result is expected to work well deep in the broken phase, the match gives us confidence to extend the range of validity of our fit (7) down to $T \approx 130$ GeV, in order to cover the physically interesting range.

The expectation value of the square of the Higgs field $v^2/T^2 = 2\langle\phi^\dagger\phi\rangle/T$ [here ϕ is the three-dimensional field introduced in Eq. (4)] measures the "turning on" of the Higgs mechanism, see Fig. 3. As mentioned above, there is no proper phase transition and $v^2(T)$ behaves smoothly as a function of the temperature. The observable $\langle\phi^\dagger\phi\rangle$ is ultraviolet divergent and is additively renormalized [2]; because of additive renormalization, $v^2(T)$ can become negative.

The crossover temperature can be defined rather accurately: using either the location of the maximum of $|dv^2/dT|$ or the obvious location where Γ changes from low-T to high-T behavior, we can estimate the crossover temperature to be $T_c = 159.5 \pm 1.5$ GeV. The error estimate is a combination of our temperature resolution and the systematics due to omitted U(1) as described above.

In Fig. 3, we also show the two-loop RG-improved perturbative result [2] for $v^2(T)$ in the broken phase. Perturbation theory reproduces T_c perfectly, and deep in the broken phase v^2 is slightly larger than the lattice measurement. In the continuum limit, we expect this difference to decrease for this observable; in Ref. [14] we extrapolated $v^2(T)$ to the continuum at a few temperature values and with Higgs mass 115 GeV. The continuum limit in the broken phase was observed to be about 6% greater than the result at $\beta_G = 9$. Thus, for $v^2(T)$ perturbation theory and lattice results match very well.

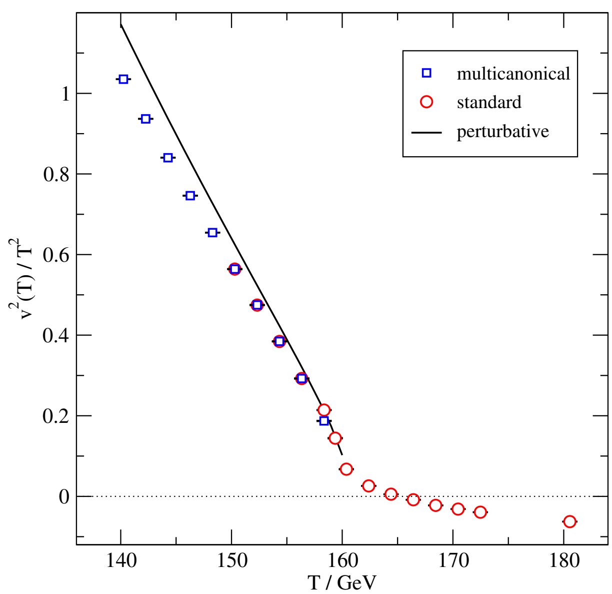

FIG. 3 (color online). The Higgs expectation value as a function of temperature, compared with the perturbative result [2].

Finally, we can use the sphaleron rate to estimate when the diffusive sphaleron rate and, hence, the baryon number becomes frozen in the early Universe. The cooling rate of the radiation-dominated Universe is given by the Hubble rate $H(T)$: $\dot{T} = -HT$. The freeze-out temperature T_* can now be solved from [13]

$$\Gamma(T_*)/T_*^3 = \alpha H(T_*), \tag{9}$$

where α is a function of the Higgs expectation value $v(T)$ but can be approximated by a constant $\alpha = 0.1015$ to better than 0.5% accuracy in the physically relevant range. Taking $H^2(T) = \pi^2 g^* T^4/(90 M_{\rm Planck}^2)$, with $g^* = 106.75$ we find $T_* = (131.7 \pm 2.3)$ GeV, as shown in Fig. 2 (we neglect g^* changing slightly as the top quark becomes massive). This temperature enters baryogenesis scenarios where the baryon number is sourced at the electroweak scale, e.g., low-scale leptogenesis scenarios (see Refs. [13,31] and references therein). For a more detailed baryon production calculation, the rates of Eqs. (7) and (8) can be entered directly into Boltzmann equations.

Conclusions.—The discovery of the Higgs particle of mass 125–126 GeV enables us to fully determine the properties of the symmetry breaking at high temperatures. Using lattice simulations of a three-dimensional effective theory, we have located the transition (crossover) point at $T_c = (159.5 \pm 1.5)$ GeV, determined the baryon number violation rate both above and well below the crossover point, and calculated the baryon freeze-out temperature in the early Universe, $T_* = (131.7 \pm 2.3)$ GeV. In addition to being intrinsic properties of the minimal standard model, these results provide input for leptogenesis calculations, in particular for models with electroweak scale leptons. It also provides a benchmark for future computations of the sphaleron rate in extensions of the standard model.

We thank Mikko Laine for discussions. This work was supported in part by a Villum Young Investigator Grant (A. T.), by the Magnus Ehrnrooth Foundation (M. D.), and by the Finnish Academy through Grants No. 1134018 and No. 1267286. The numerical work was performed using the resources at the Finnish IT Center for Science, CSC.

[1] S. Chatrchyan *et al.* (CMS Collaboration), Phys. Lett. B **716**, 30 (2012); G. Aad *et al.* (ATLAS Collaboration), Phys. Lett. B **716**, 1 (2012).

[2] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B466**, 189 (1996); K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Phys. Rev. Lett. **77**, 2887 (1996).

[3] M. Gurtler, E.-M. Ilgenfritz, and A. Schiller, Phys. Rev. D **56**, 3888 (1997).

[4] K. Rummukainen, M. Tsypin, K. Kajantie, M. Laine, and M. E. Shaposhnikov, Nucl. Phys. **B532**, 283 (1998).
[5] F. Csikor, Z. Fodor, and J. Heitger, Phys. Rev. Lett. **82**, 21 (1999).
[6] M. Reuter and C. Wetterich, Nucl. Phys. **B408**, 91 (1993).
[7] W. Buchmuller and O. Philipsen, Nucl. Phys. **B443**, 47 (1995).
[8] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, Phys. Lett. **155B**, 36 (1985).
[9] V. A. Rubakov and M. E. Shaposhnikov, Usp. Fiz. Nauk **166**, 493 (1996) [Phys. Usp. **39**, 461 (1996)].
[10] G. t'Hooft, Phys. Rev. Lett. **37**, 8 (1976).
[11] M. Fukugita and T. Yanagida, Phys. Lett. B **174**, 45 (1986); M. A. Luty, Phys. Rev. D **45**, 455 (1992).
[12] S. Y. Khlebnikov and M. E. Shaposhnikov, Nucl. Phys. **B308**, 885 (1988).
[13] Y. Burnier, M. Laine, and M. Shaposhnikov, J. Cosmol. Astropart. Phys. 02 (2006) 007.
[14] M. D'Onofrio, K. Rummukainen, and A. Tranberg, J. High Energy Phys. 08 (2012) 123.
[15] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B458**, 90 (1996).
[16] F. R. Klinkhamer and N. S. Manton, Phys. Rev. D **30**, 2212 (1984); J. Kunz, B. Kleihaus, and Y. Brihaye, Phys. Rev. D **46**, 3587 (1992).
[17] M. Laine and K. Rummukainen, Nucl. Phys. B, Proc. Suppl. **73**, 180 (1999).
[18] K. Farakos, K. Kajantie, K. Rummukainen, and M. E. Shaposhnikov, Nucl. Phys. **B425**, 67 (1994); M. Laine and A. Rajantie, Nucl. Phys. **B513**, 471 (1998).
[19] G. D. Moore, Nucl. Phys. **B523**, 569 (1998).
[20] J. Ambjorn, T. Askgaard, H. Porter, and M. E. Shaposhnikov, Nucl. Phys. **B353**, 346 (1991); O. Philipsen, Phys. Lett. B **358**, 210 (1995); J. Ambjorn and A. Krasnitz, Phys. Lett. B **362**, 97 (1995); W.-H. Tang and J. Smit, Nucl. Phys. **B482**, 265 (1996); H. P. Shanahan and A. C. Davis, Phys. Lett. B **431**, 135 (1998).
[21] G. D. Moore and K. Rummukainen, Phys. Rev. D **61**, 105008 (2000).
[22] D. Bödeker, L. McLerran, and A. Smilga, Phys. Rev. D **52**, 4675 (1995).
[23] P. B. Arnold, D. Son, and L. G. Yaffe, Phys. Rev. D **55**, 6264 (1997).
[24] G. D. Moore, C. Hu, and B. Muller, Phys. Rev. D **58**, 045001 (1998); D. Bodeker, G. D. Moore, and K. Rummukainen, Phys. Rev. D **61**, 056003 (2000).
[25] D. Bödeker, Nucl. Phys. **B559**, 502 (1999).
[26] G. D. Moore, Nucl. Phys. **B568**, 367 (2000).
[27] G. D. Moore, Phys. Rev. D **62**, 085011 (2000).
[28] G. D. Moore and K. Rummukainen, Phys. Rev. D **63**, 045002 (2001).
[29] G. D. Moore, Phys. Rev. D **59**, 014503 (1998).
[30] G. D. Moore, arXiv:hep-ph/0009161.
[31] T. Asaka and M. Shaposhnikov, Phys. Lett. B **620**, 17 (2005).