3D Topological Quantum Memory with a Power-Law Energy Barrier

Kamil P. Michnicki
Department of Physics, University of Washington, Seattle, Washington 98198-1560, USA
(Received 24 June 2014; published 24 September 2014)

We discuss energy barriers and their relationship to self-correcting quantum memories. We introduce the solid code, a 3D version of Kitaev's surface code, and then combine several solid codes using a technique called welding. The resulting code is a $[[O(L^3), 1, O(L^{4/3})]]$ stabilizer code with an energy barrier of $O(L^{2/3})$, which is an exponential improvement over the previous highest energy barrier in 3D. No-go results are avoided by breaking microscopic translation invariance.

DOI: 10.1103/PhysRevLett.113.130501 PACS numbers: 03.67.Pp, 03.67.Ac, 03.67.Lx

An outstanding challenge for the development of a quantum computer is whether it is possible to build a quantum hard drive. Such a device could be used to store a quantum state and protect it from decoherence without the need to actively detect and correct errors much like a ferromagnetic hard drive does when it is powered down. The 4D toric code Hamiltonian [1,2], a spin system, is a theoretical example of such a self-correcting quantum memory. It uses a macroscopic energy barrier to prevent noise from accumulating and corrupting stored quantum information. It is a major open question whether such a system can exist in less than four dimensions. The problem is intimately related to the problem in condensed matter physics of whether topological order can exist at nonzero temperatures [3]. Most results for self-correcting quantum memories in 2D and 3D to date have been negative [4–6] or use operators of unbounded strength [7,8]. One exception has been the cubic code [9], but that too has an energy barrier of only $\log(L)$. In this Letter, we improve the best-known energy barrier for spin Hamiltonians with topological order from $O(\log L)$ to $O(L^{2/3})$.

Both the cubic code and the code presented in this Letter can be shown [10] to give a theoretical increase in storage time when the system size is increased up to a temperature-dependent maximum. The goal is to have a storage time that scales exponentially with the volume of the system. The result presented in this Letter can be viewed as a stepping stone towards this goal.

To gain intuition, consider the ferromagnetic hard disk drive. It uses the net magnetization of a ferromagnet to store bits of information. At room temperature it is stable against a global change in polarization. If the magnetization of a small domain flips, there is an energy penalty proportional to the perimeter of the domain. This energy barrier keeps these domains small, which leads to a stable classical memory that is self-correcting.

The requirements for quantum memories are more stringent than those for classical memories. Classical memories need only keep information safe while quantum memories must simultaneously hide it. Luckily, there are many local error correcting codes that achieve this, such as the 2D and 3D toric code and color codes [11–13]. These codes hide the stored superposition by making the observables depend on the topology of the system, not on any local observables.

The toric codes, color codes, cubic code, and the code presented in this Letter are all examples of stabilizer codes. We review stabilizer codes now. We use the word qubit instead of spin to emphasize that the subsystems could be any two-level system. The Pauli group is defined by

$$G = \{(i)^k P_1 \otimes \cdots \otimes P_n : k \in \{0,1,2,3\}, P_i \in \{I,X,Y,Z\}\}, \tag{1}$$

where X, Y, and Z are single-qubit Pauli operators. A stabilizer group S is a subgroup of the Pauli group such that $-I \notin S$. This implies that it is Abelian and can be simultaneously diagonalized so that there exists a subspace $\mathcal{H}_c$, called the code space, such that for all $|\psi\rangle \in \mathcal{H}_c$ and for all $h \in S$, $h|\psi\rangle = |\psi\rangle$. Given a generating set R, where $\langle R\rangle = S$, we form the Hamiltonian $H = -\sum_{h\in R} h$. The ground-state subspace and the code space are the same.

We can apply error correcting operations based on measurements of operators in the stabilizer group that bring the state back into the code space. A system is self-correcting when thermalization keeps the error corrected state very close to the original. The goal is to design a Hamiltonian that ensures that any sequence of local operations that enacts a logical operation on the error corrected state must at some point have a macroscopically large energy. The minimum such energy is called the energy barrier.

For simplicity, we choose logical operators in the Pauli group that commute with the stabilizer group. The code in this Letter has a generating set where each generator is a tensor product of either exclusively X operators or exclusively Z operators [14]. We consider logical operators of the same type.

The motivation to study the energy barrier comes from considering the lifetime of a memory subject to thermal

noise. Thermal noise is typically modeled [2,10,15,16] by a set of local jump operators. The rate to jump from state i with energy E_i to state a state j with energy E_j is denoted by r_{ij} and satisfies $r_{ij} = e^{-\beta(E_j - E_i)} r_{ji}$. Thus, the larger the energy barrier, the more noise is suppressed. The hope is that if we can find a large energy barrier in 3D, then the system will be a good quantum memory.

Main result.—There exists a local stabilizer Hamiltonian in 3D with an energy barrier of $O(L^{2/3})$ where the Hamiltonian is composed of $O(L^3)$ qubits and the qubits are of finite density. By finite density we mean that a finite number of qubits fit into a finite volume reference box. By local we mean that the terms in the Hamiltonian act on a set of qubits contained in another finite reference box.

Haah [17] proved that for local translation-invariant stabilizer codes, the highest energy barrier is $O(\log L)$. This improves a no-go theorem by Yoshida [5]. So how can the result in this Letter hold? The code in this Letter is constructed from macroscopic blocks. Each block satisfies the result by Haah. These macroscopic blocks are joined together, welded, into a macroscopic lattice. The welded code is translation invariant over a length that grows with the system size. This avoids the no-go results.

Solid codes.—We define the solid code by analogy to the surface code [1], i.e., a 3D toric code [12] with rough and smooth boundaries. These boundary conditions determine the logical operators, one of which has a constant energy barrier. This means the solid code cannot store quantum information though it is an important building block to the welded code.

We define the generators of the solid code with respect to a graph, shown in Fig. 1 (qubits are labeled by edges). The graph is mostly a cubic lattice with $d \times d \times d$ primitive cells except that horizontal edges are missing from the top and bottom boundaries of the graph. We call these boundaries rough. Terms in the Hamiltonian are labeled by vertices of the graph and faces of the primitive cells. We refer to the faces as plaquettes. The vertices are

$$V = \{\mathbf{v} = (v_1, v_2, v_3) : \ v_i \in \{1, \ldots, N\}\}. \tag{2}$$

Using the unit vectors $\boldsymbol{n_1} = (1,0,0)$, $\boldsymbol{n_2} = (0,1,0)$, and $\boldsymbol{n_3} = (0,0,1)$, the edges are

$$\begin{aligned} E = &\{\{\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{n_3}\} : \ \boldsymbol{v} \in V, v_3 \neq N\} \\ \cup &\{\{\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{n_2}\} : \ \boldsymbol{v} \in V, v_3 \neq 1, v_3 \neq N\} \\ \cup &\{\{\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{n_1}\} : \ \boldsymbol{v} \in V, v_3 \neq 1, v_3 \neq N\}. \end{aligned} \tag{3}$$

Let $\Gamma(\boldsymbol{v})$ be the set of edges that neighbor a vertex $\boldsymbol{v}$. For each $\boldsymbol{v}$ with $|\Gamma(\boldsymbol{v})| > 1$ define the term $h_{\boldsymbol{v}}^X = \prod_{e \in \Gamma(\boldsymbol{v})} X_e$. Let ∂f be the set of edges on the boundary of a plaquette f. For each plaquette f define a term $h_f^Z = \prod_{e \in \partial f} Z_e$ except that plaquettes at the top and bottom of the lattice are missing an edge. The Hamiltonian is a sum over elements of the set of vertices V and the set of faces F:

$$H = -\sum_{\boldsymbol{v} \in V : |\Gamma(\boldsymbol{v})| > 1} h_{\boldsymbol{v}}^X + -\sum_{f \in F} h_f^Z. \tag{4}$$

The X and Z terms commute and so generate a stabilizer group. The ground-state subspace of the Hamiltonian is exactly the code space of this stabilizer group.

The logical operator $\bar{X}$ of the solid code resembles an open membrane. Multiplying vertex operators generates closed membranes of qubits and horizontal pairs of open membranes. These pairs can be made to be far apart so that each membrane overlaps with a disjoint set of plaquettes. Hence, each membrane commutes with each term in the Hamiltonian, yet is not generated by them. We conclude that the logical operator $\bar{X}$ is a tensor product of single-qubit X operators on a single horizontal membrane.

The logical operator $\bar{Z}$ resembles an open string. Multiplying plaquette operators generates closed strings, strings starting and ending on the same rough boundary, and pairs of strings extending between opposite rough boundaries. These pairs of open strings can be made to be far apart so that each string overlaps with a disjoint set of vertex operators. Hence, each open string commutes with each term of the Hamiltonian, yet is not generated by them. We conclude that the logical operator $\bar{Z}$ is a tensor product of single-qubit Z operators on a single string extending between opposite rough boundaries. See Fig. 1.

The set of logical operators completely determines the ground-state degeneracy. There are two distinct nontrivial logical operators, $\bar{X}$ and $\bar{Z}$. Because they anticommute, we can only diagonalize one of them at a time. Hence, the ground-state degeneracy is 2.

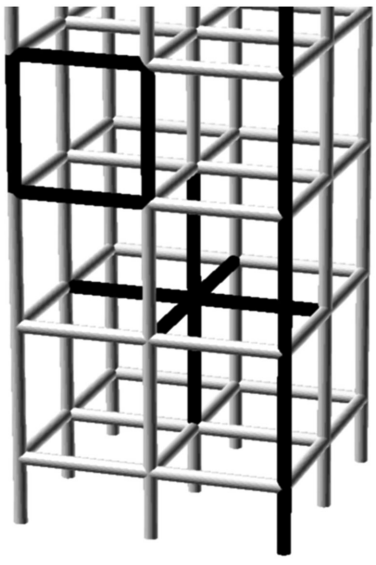

FIG. 1. A solid code with qubits represented by edges. The following operators shaded darker: a plaquette operator, a star operator, and a logical Z operator.

The logical operator $\bar{Z}$ has a constant energy barrier. Understanding why is key to doing better. Applying a Z operator on a single qubit violates either one or two terms. We call these defects. By flipping a qubit in a line starting and ending on opposite rough boundaries, we move a single defect across the lattice with no more than a constant energy penalty. One way to create a large energy barrier is to force this string of bit flips to split many times. To do this, we need qubits such that errors on them create three or more defects.

Welded solid codes.—In this section we achieve a power-law energy barrier by combining several solid codes into a 3D lattice. The final lattice is not a regular lattice. This is because each block is bent and stretched to match up with each other into a macroscopic lattice. The procedure for combining blocks of code is called welding. First, we will show how to weld three solid codes together, analyze the shape of the logical operators, and show that the energy barrier has increased. We then weld several solids into a lattice.

To gain intuition, consider a 1D Ising model of a finite length. The Hamiltonian is $\sum_{i=0}^{n-1} -Z_i Z_{i+1}$. The ground state has two degenerate eigenstates $|00\ldots0\rangle$ and $|11\ldots1\rangle$. Suppose we want to flip every qubit in a sequence that minimizes the number of defects. The best we can do is to flip the first, second, third, etc. qubits in a line until all of the qubits have been flipped. This sequence creates a single defect and moves it from one end of the string to the other. Welding is like combining three such strings so each string shares the nth qubit. When we try to move a defect past this shared qubit, it splits into two. Thus, the energy increases. For the welded solid code there is a 2D boundary between 3D blocks of qubits. Defects split when moving past them.

We now describe this boundary where defects split. We combine three solid codes along their rough boundaries. For each solid we identify qubits on the bottom rough boundaries with each other. More precisely, for all $i, j \in \{1, \ldots, N\}$, the qubit labeled by $(\{i, j, 0\}, \{i, j, 1\})$ in the first solid code is the same as the corresponding qubit in the second and third solid codes. Because the X- and Z-type terms no longer commute, we update all local Z-type stabilizers to commute with the X-type stabilizers. The rule is that whenever Z-type stabilizers agree on the shared qubits of the three solids, they are combined, welded, into a single operator. For a Z-type operator h, define $Q(h)$ to be the qubits that h acts on nontrivially. A set of Z-type operators $\{h_1, \ldots, h_n\}$ is said to be *welded* together into a Z-type operator h when $Q(h) = \cup_i Q(h_i)$. We call the resulting code a welded code. A more thorough account of the theory of welding can be found in the Supplemental Material [18] and Ref. [19].

Next, we show that the new code, the three welded solids, encodes only a single qubit by showing that all nontrivial Z-type logical operators are equivalent. The Z-type logical operators from each solid get welded together so that the new logical operator $\bar{Z}$ resembles three strings emanating from a single qubit on the shared boundary. See Fig. 2. After welding, the Z-type stabilizers have the following shapes: half-loops on rough boundaries, loops in the bulk of each solid, three welded half-loops on the shared rough boundary, and pairs of logical Z operators. There can be no other Z-type logical operator. If a Z-type operator acts with an even number of Z operators on the shared qubits, then it is in the stabilizer group, and if it acts with an odd number of Z operators on the boundary, then it is equivalent to the operator composed of three welded strings. Since there is only one nontrivial Z-type logical operator, the degeneracy is 2.

A defect caused by Z errors would split into two moving through this shared boundary, an increase in energy. This is because a single-qubit Z operator applied to this shared boundary creates three violated terms, one for each solid block. Thus, the energy barrier for the logical operator $\bar{Z}$ has increased from 1 to 2.

In order to increase the energy barrier to a power law, we generalize welding three solids on a single boundary to welding many solids into a lattice. We label the solids by their rough boundaries. A rough boundary is denoted by W_i. In the previous example we had the set of solids $E = \{\{W_1, W_4\}, \{W_2, W_4\}, \{W_3, W_4\}\}$, and they all share the rough boundary W_4. In this notation, the welded rough boundaries act as"fat" vertices and the bulk of the solid codes act as "fat" edges of a graph $G = (V = \{W_i\}, E)$. We weld the solid codes into the graph of a 3D cubic lattice to achieve a particularly high energy barrier.

We deduce the energy barrier of the logical operator $\bar{Z}$ of the welded cubic lattice by looking at the rough boundaries of each block. If there is an odd number of Z errors on the rough boundaries of a particular solid code, then there must be at least one defect in the bulk of that solid. Hence, if we weld the solid codes into a graph G, then the energy barrier

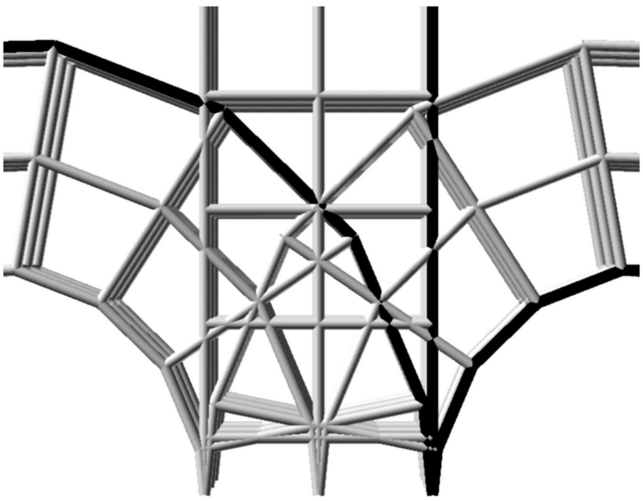

FIG. 2. Three solid codes welded together with qubits represented by edges. The bifurcating $\bar{Z}$ operator is shaded darker.

of the welded string operator is at least as big as the energy barrier for an Ising model Hamiltonian $H = \sum_{\{i,j\}\in E} -Z_i Z_j$ with precisely the same graph G. If G is a 3D cubic lattice of width R, then the energy barrier for the logical operator $\bar{Z}$ is greater than $O(R^2)$. This bound can be saturated provided we never create more than one defect within the bulk of any solid.

We deduce the energy barrier of the logical operator $\bar{X}$ in a similar way as for the logical operator $\bar{Z}$. The logical operator $\bar{X}$ is an X-type operator and hence remains unchanged. It remains a membrane in one of any solid code blocks. We show a lower bound for the energy barrier by considering the contribution to the energy barrier from the vertical plaquettes only, i.e., plaquettes in the y-z and z-x planes of each solid, leaving out the plaquettes in the x-y plane. For each vertical surface of plaquettes, defects can move up and down without creating new vertical defects, but moving between these vertical regions creates a vertical defect in each neighboring region, similarly to the case of the solid regions. These "flat" regions are connected to each other in a 2D square lattice of width $O(d)$, provided each solid is $O(d)$ qubits wide. Again, the energy barrier of this horizontal membrane is given by the energy barrier of the Ising model on a 2D square lattice. So the energy barrier is lower bounded by $O(d)$. This bound can be saturated, even with horizontal plaquettes, provided that the membrane is grown horizontally and in a single domain.

Finally, the energy barrier of solid codes welded in a cubic lattice is the minimum of the two energy barriers: $O(d)$ and $O(R^2)$. The total number of qubits is $O(d^3)$ qubits per solid with $O(R^3)$ solids, which leads to the number of qubits $N \sim O(d^3 R^3)$. The maximum energy barrier for a fixed number of qubits N is the minimum of the X and Z energy barriers. Thus, the maximum energy barrier happens when $O(d) \sim O(R^2)$, leading to an energy barrier of $\delta E \sim O(N^{2/9})$. The qubits can be placed in a box of side lengths of $O(L)$ so that the energy barrier is $O(L^{2/3})$. This demonstrates our main result.

Discussion.—We have constructed a code that has an exponentially higher energy barrier than the the logarithmic bound derived by Haah [17]. We achieved this by tuning the length over which the code is periodic to a macroscopic distance.

A lower bound on the storage time t was derived in Ref. [10]. For any stabilizer code Hamiltonian with an energy barrier δE, number of encoded qubits $k(L)$, and the error model of a Hamiltonian in the weak coupling limit, $t \sim (e^{\beta\delta E}/N)2^{-k(L)}$ when $N = O(L^3) \lesssim e^{\beta}$. This leads to a lower bound of $t \sim e^{\beta e^{(2/9)\beta}}$ for $N \lesssim e^{\beta}$ for the welded solid code. Although an upper bound is not known, it is unlikely that the memory time will go up arbitrarily with system size [20].

Further progress might be made in considering nonperiodic codes or local error correction with engineered dissipation [21–24].

I thank Aram Harrow and Steve Flammia for stimulating discussions and helpful suggestions throughout the writing process and Jeongwan Haah and Chris Laumann for feedback on the final manuscript. This work was supported by NSF Grants No. 0829937 and No. 1111382, DARPA QuEST Contract No. FA9550-09-1-0044, and IARPA via DoI NBC Contract No. D11PC20167. Part of this work was carried out while visiting at MIT.

[1] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
[2] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Open Syst. Inf. Dyn. **17**, 1 (2010).
[3] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[4] S. Bravyi and B. Terhal, New J. Phys. **11**, 043029 (2009).
[5] B. Yoshida, Ann. Phys. (Amsterdam) **326**, 2566 (2011).
[6] O. Landon-Cardinal and D. Poulin, Phys. Rev. Lett. **110**, 090502 (2013).
[7] A. Hamma, C. Castelnovo, and C. Chamon, Phys. Rev. B **79**, 245122 (2009).
[8] F. L. Pedrocchi, A. Hutter, J. R. Wootton, and D. Loss, Phys. Rev. A **88**, 062313 (2013).
[9] J. Haah, Phys. Rev. A **83**, 042330 (2011).
[10] S. Bravyi and J. Haah, Phys. Rev. Lett. **111**, 200501 (2013).
[11] A. Kitaev, Ann. Phys. (Amsterdam) **303**, 2 (2003).
[12] C. Castelnovo and C. Chamon, Phys. Rev. B **76**, 184442 (2007); **78**, 155120 (2008).
[13] H. Bombin and M. A. Martin-Delgado, Phys. Rev. Lett. **97**, 180501 (2006).
[14] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
[15] S. Chesi, D. Loss, S. Bravyi, and B. M. Terhal, New J. Phys. **12**, 025013 (2010).
[16] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).
[17] J. Haah, Commun. Math. Phys. **324**, 351 (2013).
[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.113.130501 for a detailed description of the theory of welding.
[19] K. Michnicki, arXiv:1208.3496.
[20] K. Michnicki (to be published).
[21] K. Fujii, M. Negoro, N. Imoto, and M. Kitagawa, arXiv:1401.6350.
[22] M. Herold, E. Campbell, J. Eisert, and M. Kastoryano, arXiv:1406.2338.
[23] P. Gács, J. Comput. Syst. Sci. **32**, 15 (1986).
[24] D. Gottesman, J. Mod. Opt. **47**, 333 (2000).