Holographic Signatures of Cosmological Singularities

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To gain insight into the quantum nature of cosmological singularities, we study anisotropic Kasner solutions in gauge-gravity duality. The dual description of the bulk evolution towards the singularity involves $\mathcal{N}=4$ super Yang-Mills theory on the expanding branch of deformed de Sitter space and is well defined. We compute two-point correlators of Yang-Mills operators of large dimensions using spacelike geodesics anchored on the boundary. The correlators show a strong signature of the singularity around horizon scales and decay at large boundary separation at different rates in different directions. More generally, the boundary evolution exhibits a process of particle creation similar to that in inflation. This leads us to conjecture that information on the quantum nature of cosmological singularities is encoded in long-wavelength features of the boundary wave function.

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Introduction.—A long-standing goal of quantum gravity is to describe physics near singularities like the big bang or inside black holes. Gauge-gravity duality is a powerful tool to apply to this problem since it maps it into a problem in ordinary QFT on a fixed spacetime background. To model a cosmological singularity using holography, one needs to construct an asymptotically anti–de Sitter (AdS) solution to Einstein's equation that evolves into (or from) a singularity which extends all the way out to infinity. This was first done in Refs. [1,2], but the dual field theory itself became singular when the bulk singularity hit the boundary. In Refs. [3,4], the same singular bulk solutions were reinterpreted as being dual to a well-defined field theory on de Sitter (dS) spacetime. However, it is not clear how (and indeed whether) the dual field theory on dS space describes the region near the singularity. This is because the probes which are best understood, such as extremal surfaces which end on the boundary, do not probe the region near the singularity [5]. Models of this type were further explored in Ref. [6], and other models were studied in Refs. [7–9].

Attempts to probe the black hole singularity were somewhat more successful in that there are geodesics with end points on the boundary which get arbitrarily close to the singularity [10]. Unfortunately, it was shown that the two-point correlator is not dominated by these geodesics, although their effects could be seen by analytic continuation [11]. Nevertheless, the presence of the black hole horizon means that clear signatures of the singularity have remained difficult to identify in the dual.

The goal of this Letter is to introduce a new holographic model of a cosmological singularity which has the advantages that (1) the dual field theory is simply strongly coupled $\mathcal{N}=4$ super Yang-Mills theory with a large number N of colors on an anisotropic generalization of de Sitter space and is manifestly well defined for all time, and (2) there are bulk geodesics with end points on the boundary which come close to the singularity. As a bonus, one can solve for the equal time correlator analytically. We indeed find distinctive behavior which, we argue, signals the presence of the bulk singularity [14]. While singularities are ultimately described by quantum gravity, i.e., the small N regime, obtaining a field theory description of a classical (large N) singularity is an important pioneering step in recasting the problem of singularities in quantum gravity in terms of the dual field theory. In particular, the transition from large to small N is a tractable problem in the field theory but remains poorly understood in the bulk.

The solution.—Solutions to Einstein's equation in five dimensions with negative cosmological constant can be obtained by starting with AdS_5 in Poincaré coordinates and replacing the flat Minkowski metric on each radial slice with any Ricci flat metric. The Kasner metric

$$ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2 \quad (1)$$

with $\sum_i p_i = 1 = \sum_i p_i^2$ is a well-known Ricci flat metric describing a homogeneous but anisotropic cosmology. It has a singularity in the Weyl curvature at $t=0$. With this metric on each radial slice of AdS_5, we obtain [8]

$$ds^2 = \frac{1}{z^2}(-dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2 + dz^2), \quad (2)$$

where we have set the AdS radius to 1. It might appear that the dual would have to live on a Kasner spacetime. However, we can divide the metric in parentheses by H^2t^2 where H is some constant, and replace the overall conformal factor by H^2t^2/z^2. Writing $Ht = e^{H\tau}$, $x_i = H^{p_i}y_i$ this yields the boundary metric

$$ds^2 = -d\tau^2 + \sum_i e^{-2(1-p_i)H\tau} dy_i^2, \tag{3}$$

which is an anisotropic deformation of dS space in flat slicing. In addition to the obvious translational symmetries, Eq. (2) is invariant under a dilation symmetry [15]:

$$z \to \lambda z, \qquad t \to \lambda t, \qquad x_i \to \lambda^{(1-p_i)} x_i. \tag{4}$$

This leaves the conformal factor Ht/z invariant and, thus, acts as an isometry of the boundary metric (3).

Two-point correlator.—In the large N limit, the leading contribution to the two-point correlator of an operator $\mathcal{O}$ of high conformal dimension Δ in the dual strongly coupled $SU(N)$ Yang-Mills theory on Eq. (3) is given by the (regulated) length of spacelike bulk geodesics connecting the two points:

$$\langle \psi | \mathcal{O}(x) \mathcal{O}(x') | \psi \rangle = e^{-m \mathcal{L}_{\text{reg}}(x,x')}, \tag{5}$$

where $|\psi\rangle$ is the state of the Yang-Mills theory, m is the mass of the bulk field that is dual to the boundary operator $\mathcal{O}$, and $\mathcal{L}_{\text{reg}}(x,x')$ is the regularized length of the bulk geodesic. When $\mathcal{O}$ is a scalar operator, we have $\Delta = 2 + \sqrt{4+m^2}$.

The length of spacelike geodesics is infinite. As usual, we regulate this length by introducing a cutoff when the conformal factor becomes large, and subtracting the divergent contribution from pure AdS. Writing $\tilde{z} = z/Ht$, our cutoff will be $\tilde{z} = \tilde{\epsilon}$.

We can solve for the bulk geodesics using the metric (2). We consider equal-time correlators for two points separated in the x_1 direction only (hereafter, referred to as x). The dilation symmetry (4) together with the translational symmetry in x_2 and x_3 imply that the correlators depend only on the proper boundary separation $\mathcal{L}_{\text{bdy}}$ between the two points, and, of course, on the exponent p_1, which we hereafter denote as p. Without loss of generality, we take the end points at $z=0$ to be $\{t=1, x=\pm\bar{x}\}$. Using t as a parameter, the geodesic equations are

$$x''(t)t = px'(t)[-2 + t^{2p} x'(t)^2], \tag{6}$$

$$z''(t)z(t) = 1 - z'(t)^2 - t^{2p-1} x'(t)^2 [t - pz(t)z'(t)]. \tag{7}$$

The solutions of Eq. (6) are hypergeometric functions for all p. For $p = \pm 1/n$, with integer n, the hypergeometric functions simplify, which makes the analysis more tractable. We first compute the correlator in a simple nonsingular example before treating the case $p = -1/4$ that describes an anisotropic dS boundary dual to a bulk with a genuine curvature singularity.

Correlators in the Milne universe: The Milne solution is a special case of the Kasner solution (2) where one of the $p_i = 1$ and the rest are zero. This metric features a coordinate singularity at $t = 0$ and is simply flat space in alternative coordinates. If $p = 0$, the effective $(2+1)$-dimensional metric determining geodesic motion is precisely AdS_3. Hence, with our choice of boundary conditions the geodesics lie entirely in the surface $t = 1$. In terms of the usual cutoff $z = \epsilon$, their length is $\mathcal{L} = 2\ln(2\bar{x}/\epsilon) = 2\ln(\mathcal{L}_{\text{bdy}}/\epsilon)$ where $\mathcal{L}_{\text{bdy}}$ is the proper boundary separation on the Minkowski boundary. With a cutoff $\tilde{\epsilon} = \epsilon/H$ appropriate for a boundary de Sitter metric,

$$\mathcal{L} = 2\ln\left(\frac{2\bar{x}}{H}\frac{H}{\epsilon}\right) = 2\ln(\mathcal{L}_{\text{bdy}}) - 2\ln(\tilde{\epsilon}), \tag{8}$$

where $\mathcal{L}_{\text{bdy}}$ is now the proper boundary separation on the de Sitter boundary. Hence, the correlator for a large-dimension operator in a $p = 0$ direction is given by

$$\langle \mathcal{O}(\bar{x}) \mathcal{O}(-\bar{x}) \rangle_{p=0} = \mathcal{L}_{\text{bdy}}^{-2\Delta}. \tag{9}$$

Note that the result is the same as flat space and independent of H, as expected for a conformal field theory on a conformally flat spacetime.

For $p = 1$, the effective $2+1$ metric seen by a geodesic can be transformed into pure AdS_3 in Poincaré coordinates by the coordinate transformation $(t,x) \to (\eta,\chi) = (t\cosh x, t\sinh x)$. Using this, we can obtain the length of a geodesic anchored at $x = \pm\bar{x}$ and $t = 1$ from the result for $p = 0$. This yields the following equal-time correlator:

$$\langle \mathcal{O}(\bar{x}) \mathcal{O}(-\bar{x}) \rangle_{p=1} = \left[\frac{2}{H}\sinh\left(\frac{H}{2}\mathcal{L}_{\text{bdy}}\right)\right]^{-2\Delta}, \tag{10}$$

which falls off exponentially with proper distance. This is precisely the correlator in a thermal state with temperature $T = H/2\pi$.

Correlators in anisotropic de Sitter space: We now turn to our central example $p = -1/4$, which describes a genuinely singular bulk solution. We will set $H = 1$ for convenience. For $p = -1/4$, the solutions of Eqs. (6) and (7) can be written as

$$x(w) = \frac{4}{15}\sqrt{c+w}(8c^2 - 4cw + 3w^2), \tag{11}$$

$$z(w) = \frac{4}{3}\sqrt{c[w^3 - 1 + 3c(1-w^2)]}, \tag{12}$$

where $w = \sqrt{t}$, and c is an integration constant. The solutions (11) and (12) describe half of the geodesics from the boundary at $w = 1$ and $x = \bar{x}$ up to a turning point in the interior at $w = w_*$ where $x = 0$. At the turning point $dt/dx = 2wdw/dx = 0$, which implies $dx/dw \to \infty$, so $w_* = -c$.

Since $\mathcal{L}_{\text{bdy}} = 2x(1)$ is quintic in $\sqrt{c}$, there are five possibly complex geodesics [17] corresponding to each boundary separation $\mathcal{L}_{\text{bdy}}$, which we require to be real and positive. We must determine which ones contribute to the

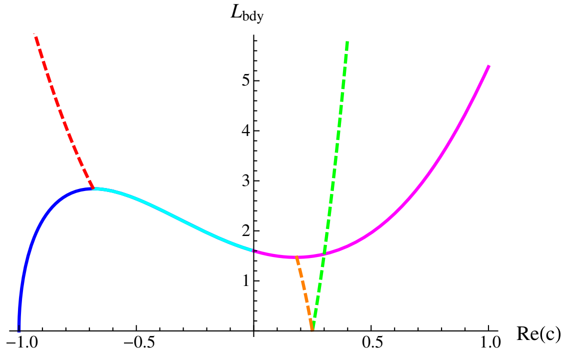

FIG. 1 (color online). The proper boundary separation $\mathcal{L}_{\text{bdy}}$ as a function of the real part of c for $p = -1/4$. The solid curve corresponds to real c, whereas the dashed curves correspond to complex conjugate pairs of c. Note that there are five possible geodesics for each $\mathcal{L}_{\text{bdy}}$.

correlator. Figure 1 shows $\mathcal{L}_{\text{bdy}}$ as a function of the real part of c. When $\text{Re}(c) > -1$, the geodesics curve towards the singularity, and $\text{Re}(c) > 0$ geodesics even propagate all the way through $t = 0$ before turning around [19]. However, we must discard the contributions from $\text{Re}(c) > 0$ geodesics because they would predict that the correlator increases as the separation between the two points grows, and they would result in an unphysical pole on a spacelike surface on the boundary. Moreover, the geodesic approximation is only justified where the spacetime is analytic [20], and our solution is certainly not analytic at $t = 0$. The net result is that the real geodesics of interest have $-1 < c < 0$ and $-c \leq w \leq 1$. As $c \to 0$, the geodesics approach the singularity.

The length of the geodesic is given by the following contour integral in the complex w plane

$$\int dw \frac{3\sqrt{1-3c+4c^3}w}{\sqrt{(c+w)}(1-w)[w^2+(1-3c)(w+1)]} \tag{13}$$

from $w = -c$ to $w = 1 - \delta$, where δ is given by the UV cutoff $\tilde{\epsilon} = z(1-\delta)$. (Since $H = t = 1$, our dS cutoff agrees with the standard cutoff in z.) The integral (13) has four singularities, at $w = 1$, $w = -c$, and two simple poles at $w_\pm = \frac{1}{2}(3c - 1 \pm \sqrt{3(3c-1)(c+1)})$. For c real and negative we may directly integrate Eq. (13) along the real axis, since the poles at $w_\pm$ do not lie on the contour of integration. When c is complex, one simply deforms the contour into the complex plane. Restricting to $\text{Re}(c) < 0$, the integral gives

$$\mathcal{L} = 2\tanh^{-1}\left[\frac{(2c-\sqrt{1-\delta})\sqrt{c+1-\delta}}{\sqrt{1+c}(2c-1)}\right], \tag{14}$$

which results in the following regulated length:

$$\mathcal{L}_{\text{reg}} = \ln\left[-\frac{64}{9}c(1+c)(2c-1)^2\right]. \tag{15}$$

The divergence of $\mathcal{L}_{\text{reg}}$ at $c = -1$ is easily seen to be the usual short-distance singularity of the correlator: $\mathcal{L}_{\text{bdy}} = 8\sqrt{1+c}$ for small $\mathcal{L}_{\text{bdy}}$, so $\mathcal{L}_{\text{reg}} = 2\ln \mathcal{L}_{\text{bdy}}$.

Now consider the divergence at $c = 0$. This occurs when the boundary separation reaches the cosmological horizon size $\mathcal{L}_{\text{hor}}$. For $\mathcal{L}_{\text{bdy}}$ slightly larger than $\mathcal{L}_{\text{hor}}$, there are bulk geodesics which come close to the singularity before returning to the boundary. As $\mathcal{L}_{\text{bdy}} \to \mathcal{L}_{\text{hor}}$, these geodesics approach a null geodesic lying entirely in the boundary which "bounces" off $\mathcal{I}^-$ (see Fig. 2).

In Ref. [10], a pole in the correlator was found corresponding to a geodesic that bounces off a black hole singularity. It was argued that this did not dominate the correlator and could only be seen by analytic continuation to a second Riemann sheet. In contrast, we believe that the pole we see at the horizon scale is physical. This is because (1) we are not in a thermal state, so there is no general argument that such a pole cannot occur, and (2) our divergence is associated with a null geodesic in the boundary and not the bulk. Physically, the pole at the horizon scale indicates that the initial state of the field theory, which describes the bulk singularity, contains particles created at each point on $\mathcal{I}^-$, moving in opposite directions. At all later times, these particles will be separated by the horizon scale.

The pole at the horizon scale in the correlator is $(\mathcal{L}_{\text{bdy}} - \mathcal{L}_{\text{hor}})^{-\Delta}$, which is weaker than the pole at short distances, which is $\mathcal{L}_{\text{bdy}}^{-2\Delta}$. This is consistent with general properties of quantum field theory.

The contributions to the equal-time correlator from the one or two geodesics with $\text{Re}(c) < 0$ are shown in Fig. 3. At small boundary separation, we obtain the requisite divergence of $\mathcal{L}_{\text{bdy}}^{-2\Delta}$ from one real geodesic. At the horizon size, a second real geodesic appears and produces the pole. At approximately twice the horizon size, the two real geodesics merge and are replaced by complex conjugate geodesics. As $\mathcal{L}_{\text{bdy}} \to \infty$, its dependence on c simplifies to $\mathcal{L}_{\text{bdy}} \propto c^{5/2}$, and we find that the asymptotic two-point

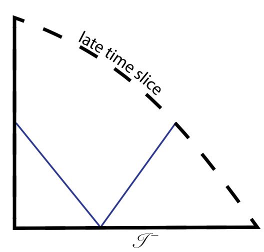

FIG. 2 (color online). A conformal diagram of the anisotropic de Sitter boundary geometry shows that two points separated by the horizon size can be connected by a null geodesic that bounces off $\mathcal{I}^-$.

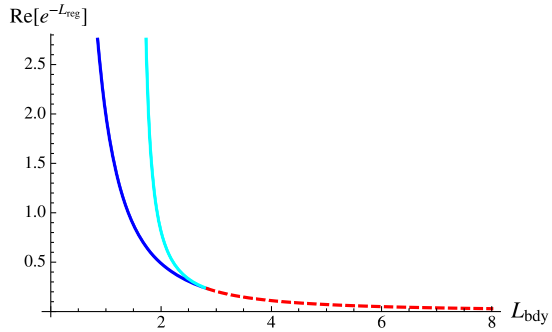

FIG. 3 (color online). $\mathrm{Re}(e^{-\mathcal{L}_{\mathrm{reg}}})$ as a function of the boundary separation for $p = -1/4$, computed from the bulk spacelike geodesics with $\mathrm{Re}(c) < 0$. The colors correspond to those in Fig. 1; e.g., the red dashed line is the contribution from each of the two complex conjugate geodesics.

correlator has a different falloff from the correlator in pure de Sitter space:

$$\langle \mathcal{O}(\bar{x})\mathcal{O}(-\bar{x})\rangle_{p=-1/4} \propto \mathcal{L}_{\mathrm{bdy}}^{-8\Delta/5}. \qquad (16)$$

Correlations in the x direction are, therefore, enhanced in the large separation limit in comparison with correlations in de Sitter space. This difference is clearly due to the anisotropy, and by extension, the bulk singularity.

The behavior for $p = -1/4$ is typical of $p < 0$. One can show [21] that when $p < 0$, geodesics always bend toward the singularity, and there always exists a family of spacelike geodesics which turn around close to the singularity. As $\mathcal{L}_{\mathrm{bdy}} \to \mathcal{L}_{\mathrm{hor}}$, these geodesics approach a null geodesic lying entirely in the boundary. For $p > 0$, the bulk geodesics bend away from the singularity, so they do not approach a null geodesic on the boundary at the horizon size. As a result, the correlator does not have a pole at the horizon scale in this case.

For general $p < 1$, the power law falloff with large boundary separation appears to satisfy

$$\langle \mathcal{O}(\bar{x})\mathcal{O}(-\bar{x})\rangle \propto \mathcal{L}_{\mathrm{bdy}}^{-2\Delta/(1-p)}. \qquad (17)$$

This holds in all cases we have checked, but we do not yet have a general derivation. A suggestive way to view this is the following: Our dilation symmetry implies that the general equal-time correlator $\langle \mathcal{O}(\bar{x},\bar{t})\mathcal{O}(-\bar{x},\bar{t})\rangle$ is only a function of one variable $\xi = \bar{t}/\bar{x}^{(1/1-p)}$. Equation (17) states that for small ξ, this function is simply $\xi^{2\Delta}$. We emphasize that this is different from the short-distance behavior, which is always given by Eq. (9).

Discussion.—We have put $\mathcal{N} = 4$ super Yang-Mills theory on an anisotropic deformation of de Sitter space and studied the two-point function of a high-dimension operator in a state dual to a cosmological singularity in the bulk. We have found two unusual features: In directions with $p < 0$, there is a pole precisely at the horizon scale, and the large-distance falloff is a power law with a power that depends on the local expansion rate. Further details and explorations of the bulk cosmological singularity using different holographic probes will be given elsewhere [21]. Since the inside region of black holes is like an anisotropic cosmology, our setup may also be useful to better understand black hole singularities.

We have focused on the singularity at $t = 0$, but our model contains another more subtle singularity at the Poincaré horizon, $z = \infty$. This can be viewed as a (null) "big crunch" singularity in the future. Alternatively, it can be removed by adding one compact dimension and starting with a six-dimensional AdS soliton. One can again replace the Minkowski slices with Kasner slices and have a big crunch in the bulk; however, now the bulk smoothly ends at finite z [22]. Our results about the pole will not be affected since they only depend on geodesics near the boundary, but the large-distance falloff will certainly be modified since one now is in a confining vacuum.

So far, we have discussed solutions with an initial "big bang" singularity. However, our results also apply to Kasner-AdS solutions with a singularity in the future. The bulk evolution from regular initial data towards the future singularity will then have a dual description in terms of $\mathcal{N} = 4$ super Yang-Mills theory on a deformed dS space expanding at different rates in different directions.

The anisotropic expansion of the boundary background breaks conformal invariance and gives rise to particle creation, just like a rolling scalar does in inflation in cosmology [23]. The relevant length scales in this process are the expansion rates in different directions. By analogy with inflation, one expects that fluctuations will be in their ground state on scales below these but exhibit particlelike excitations on larger scales. This expectation is born out by the form of the two-point correlator (17). For subhorizon boundary separations, the correlator is at all times close to that in exact dS space. By contrast, on scales larger than the horizon in a given direction it deviates significantly from the correlator in dS space reflecting the excited state on those scales.

Hence, we are led to a picture in which the holographic dual of cosmological singularities is given in terms of a boundary wave function that describes an ensemble of highly excited configurations on horizon and superhorizon scales in an anisotropic de Sitter space. It, thus, appears that signatures of the quantum nature of cosmological singularities can be found in the classical long-wavelength features predicted by the boundary theory.

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