Phase Transitions on Random Lattices: How Random is Topological Disorder?

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We study the effects of topological (connectivity) disorder on phase transitions. We identify a broad class of random lattices whose disorder fluctuations decay much faster with increasing length scale than those of generic random systems, yielding a wandering exponent of $\omega = (d-1)/(2d)$ in d dimensions. The stability of clean critical points is thus governed by the criterion $(d+1)\nu > 2$ rather than the usual Harris criterion $d\nu > 2$, making topological disorder less relevant than generic randomness. The Imry-Ma criterion is also modified, allowing first-order transitions to survive in all dimensions $d > 1$. These results explain a host of puzzling violations of the original criteria for equilibrium and nonequilibrium phase transitions on random lattices. We discuss applications, and we illustrate our theory by computer simulations of random Voronoi and other lattices.

Two of the central results on phase transitions in disordered systems are the Harris and Imry-Ma criteria. The Harris criterion [1] governs the stability of critical points against disorder. If the correlation length exponent ν of a d-dimensional clean system fulfills the inequality $d\nu > 2$, weak disorder is irrelevant and does not change the critical behavior. If $d\nu < 2$, disorder is relevant, and the character of the transition must change [2]. The Imry-Ma criterion [3] governs the stability of macroscopic phase coexistence: Disorder destroys phase coexistence by domain formation in dimensions $d \leq 2$ [4]. As a consequence, disorder rounds first-order phase transitions in $d \leq 2$. The predictions of these criteria and their generalizations to long-range correlated disorder [5,6] agree with the vast majority of explicit results on classical, quantum, and nonequilibrium systems in which the disorder stems from random coupling strengths or spatial dilution.

Puzzling results have been reported, however, on phase transitions in *topologically disordered* systems, i.e., systems on lattices with random connectivity. For example, the Ising magnet on a three-dimensional (3D) random Voronoi lattice displays the same critical behavior as the Ising model on a cubic lattice [7,8] even though Harris' inequality is violated. An analogous violation was found for the three-state Potts model on a 2D random Voronoi lattice [9]. The regular 2D eight-state Potts model features a first-order phase transition. In contrast to the prediction of the Imry-Ma criterion, the transition remains of first order on a random Voronoi lattice [10].

The nonequilibrium transition of the contact process features an even more striking discrepancy. This system violates Harris' inequality [11]. Disorder introduced via dilution or random transition rates results in an infinite-randomness critical point and strong Griffiths singularities [12,13]. In contrast, the contact process on a 2D random

Voronoi lattice shows clean critical behavior and no trace of the exotic strong-randomness physics [14].

To explain the unexpected failures of the Harris and Imry-Ma criteria, several authors suggested that, perhaps, the existing results are not in the asymptotic regime. Thus, much larger systems would be necessary to observe the true asymptotic behavior which, presumably, agrees with the Harris and Imry-Ma criteria. However, given the large systems employed in some of the cited work, this would imply enormous crossover lengths which do not appear likely because the coordination number fluctuations of the Voronoi lattice are not particularly small [15]. What, then, causes the failure of the Harris and Imry-Ma criteria on random Voronoi lattices?

In this Letter, we show that 2D random Voronoi lattices belong to a broad class of random lattices whose disorder fluctuations feature strong anticorrelations and thus decay qualitatively faster with increasing length scale than those of generic random systems. This class comprises lattices whose total coordination (total number of bonds) does not fluctuate. Such lattices are particularly prevalent in two dimensions because the Euler equation of a 2D graph imposes a topological constraint on the coordination numbers. However, higher-dimensional realizations exist as well. The suppressed disorder fluctuations lead to an important modification of the Harris criterion: The random connectivity is irrelevant at clean critical points if $(d+1)\nu > 2$. Topological disorder is thus less relevant than generic randomness. The Imry-Ma criterion is modified as well, allowing first-order transitions to survive in all dimensions $d > 1$. This explains the puzzling literature results on 2D random Voronoi lattices mentioned above. In the rest of this Letter, we sketch the derivation of these results and illustrate them by simulations.

Random lattice or cell structures occur in many areas of physics, chemistry, and biology such as amorphous solids,
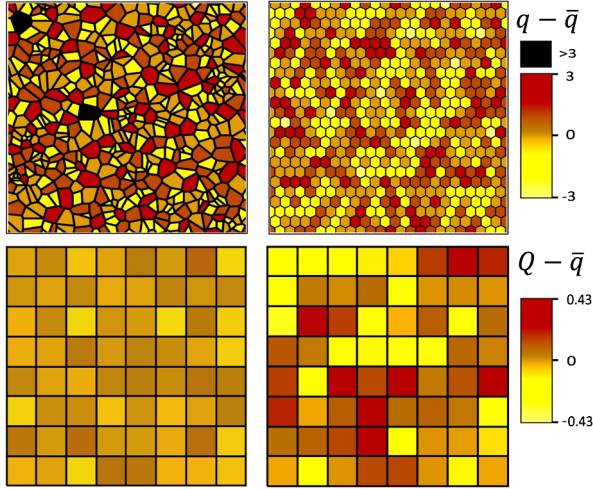
FIG. 1 (color online). Top row: coordination numbers q_i of individual sites in a random Voronoi lattice (left) and a diluted hexagonal lattice (right). Bottom row: average coordination number Q_μ of blocks with $L_b = 8$. The strong suppression of the fluctuations in the Voronoi lattice is clearly visible. [The same color (gray) scale is used left and right].

FIG. 2 (color online). Standard deviation σ_Q of the average coordination number $[Q]_\mu$ of blocks of size L_b for a random Voronoi lattice and a square lattice with 50% bond dilution (100 lattices with 5000^2 sites each). The lines are fits to $\sigma_Q \sim L_b^{-a}$, giving exponents $a = 1.001(2)$ (diluted) and $1.501(3)$ (Voronoi). Also shown is σ_Q for clusters defined via the link distance d_l (100 lattices with 2000^2 sites) giving $a = 1.52(2)$. Inset: $[Q]_\mu$ and $[Q]_\mu - \bar{q}$ of the link-distance clusters vs d_l. The line is a fit to $([Q]_\mu - \bar{q}) \sim d_l^{-b}$ yielding $b = 0.99(1)$.

foams, and biological tissue. Consider a many-particle system on such a random lattice, e.g., a classical or quantum spin system, lattice bosons, or a nonequilibrium problem such as the contact process. In all these examples, the disorder of the many-particle system stems from the random connectivity of the underlying lattice. In the following, we therefore analyze the fluctuations of the coordination number q_i (the number of nearest neighbors of site i) for different random lattices, starting with the 2D random Voronoi lattice (Fig. 1).

The Voronoi-Delaunay construction is an algorithm for building a cell network from a set of lattice sites [16]. The Voronoi cell of a site consists of all points in the plane that are closer to this site than to any other. Sites whose Voronoi cells share an edge are considered neighbors. The graph of all bonds connecting pairs of neighbors defines a triangulation of the plane called the Delaunay triangulation. Our simulations start by performing the Voronoi-Delaunay construction [17] for N points placed at independent random positions within a square of side $L = N^{1/2}$ (density fixed at unity). To study the coordination number fluctuations, we divide the system into square blocks of side L_b and calculate the block-averaged coordination number

$$Q_\mu = N_{b,\mu}^{-1} \sum_{i \in \mu} q_i \qquad (1)$$

for each block. $N_{b,\mu}$ is the number of sites in block μ, and the sum runs over all these sites. The relevant quantity is the standard deviation σ_Q of the block-averaged coordination numbers defined by

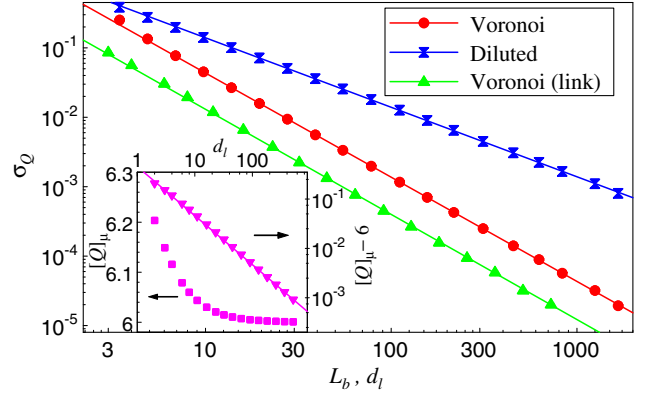$$\sigma_Q^2(L_b) = [(Q_\mu - \bar{q})^2]_\mu, \qquad (2)$$

where $[\ldots]_\mu$ denotes the average over all blocks μ, and $\bar{q}$ is the global average coordination number of the lattice.

Figure 2 compares the fluctuations in a random Voronoi lattice and a bond-diluted square lattice (both with periodic boundary conditions). In the diluted lattice, the fluctuations accurately follow $\sigma_Q \sim L_b^{-d/2} = L_b^{-1}$, as expected for uncorrelated disorder. In contrast, the fluctuations in the Voronoi lattice decay faster and follow $\sigma_Q \sim L_b^{-3/2}$. An illustration of the suppressed fluctuations in the Voronoi lattice is shown in Fig. 1.

In addition to real-space blocks, we also study clusters based on the link distance, the smallest number of bonds (links) that separate two sites. To construct such clusters, we start from a random seed site and add its neighbors, neighbors of neighbors and so on until we reach a maximum link distance d_l. This construction introduces a bias towards large q_i (as sites with more neighbors are more likely to be added to the cluster). Thus, the cluster average $[Q]_\mu$ is larger than the global average $\bar{q} = 6$; see inset of Fig. 2. The excess decays only slowly with cluster size, $([Q]_\mu - 6) \sim d_l^{-1}$. For the link-distance clusters we therefore use $\sigma_Q^2(d_l) = [(Q_\mu - [Q]_\mu)^2]_\mu$ rather than Eq. (2). The resulting data, also shown in Fig. 2, demonstrate that the fluctuations of the link-distance clusters decay with the same power, $\sigma_Q \sim d_l^{-3/2}$, as those of the real-space blocks. Had we not corrected for the size dependence of $[Q]_\mu$, we would have obtained a spurious decay exponent of (-1) [18].

How can we understand the rapidly decaying disorder fluctuations? The Euler equation of a 2D graph consisting of N sites, E edges (nearest-neighbor bonds), and F facets reads $N - E + F = \chi$. Here, χ is the Euler characteristic, a

topological invariant of the underlying surface. Periodic boundary conditions are equivalent to a torus topology, yielding $\chi = 0$ [19]. Every facet of a Delaunay triangulation is a triangle. As each triangle has three edges, and each edge is shared by two triangles, $3F = 2E$. This implies $E = 3N$; i.e., the total coordination does not fluctuate, and the average coordination number is $\bar{q} = 2E/N = 6$ for any disorder realization. (This also follows from the angle sum in any triangle being π: As each site has a total angle of 2π, 6 triangles meet at a site on average.) Now consider a block of size L_b as introduced above. The relation $3F = 2E$ holds for all triangles and edges completely inside the block. Any deviation of the block-averaged coordination number Q_μ from $\bar{q} = 6$ must thus stem from the block surface. The number of facets crossing the surface scales linearly with L_b. Assuming that each of these facets makes an independent random contribution to Q_μ leads to the estimate $\sigma_Q(L_b) \sim L_b^{1/2}/L_b^2 = L_b^{-3/2}$, in perfect agreement with the numerical data.

To substantiate these arguments, we study the coordination number correlation function

$$C(\mathbf{r}) = \frac{1}{N}\sum_{ij}(q_i - \bar{q})(q_j - \bar{q})\delta(\mathbf{r} - \mathbf{r}_{ij}), \qquad (3)$$

where $\mathbf{r}_{ij}$ is the vector from site i to j. Its integral over a block of radius r yields the bulk contribution to the fluctuations of the average coordination number

$$\sigma_{Q,\,\mathrm{bulk}}^2(r) = D(r) = \frac{2\pi}{N_r}\int_0^r dr'\, r' C(r'), \qquad (4)$$

where N_r is the number of sites in the block. The data presented in Fig. 3 show that $|C(r)|$ decays faster than exponential with distance r. Its integral $D(r)$ also decays rapidly to zero, confirming that the total coordination is not fluctuating. The topological constraint imposed by the
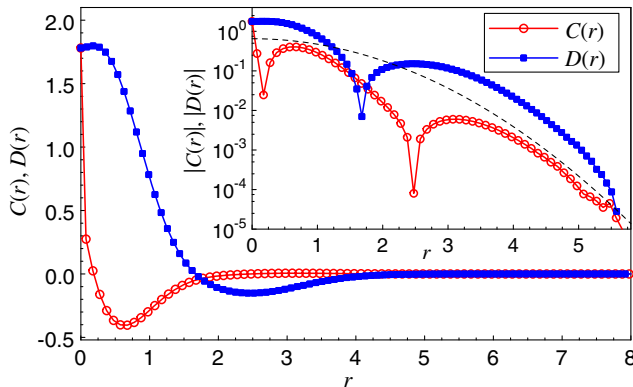

FIG. 3 (color online).   Coordination number correlation function $C(r)$ and its integral $D(r)$ vs distance r averaged over 10^7 lattices of 24^2 sites. Inset: Semi-log plot of $|C(r)|$ and $|D(r)|$. The envelope of $C(r)$ follows a Gaussian with a characteristic length $x_0 \approx 1.25$ (dashed line).

Euler equation thus leads to strong coordination number anticorrelations that are fully established within 5 or 6 typical nearest-neighbor distances.

How general are these results? Are they restricted to 2D random Voronoi lattices or do they apply to other lattices as well? The fixed total coordination is a direct consequence of the Euler equation $N - E + F = \chi$ and the triangle condition $3F = 2E$. It thus applies to any tiling of the plane with triangles. Analogously, if we tile the plane with arbitrary quadrilaterals, $4F = 2E$. This yields a fixed average coordination number of precisely $\bar{q} = 2E/N = 4$. We have thus identified a broad class of 2D lattices in which the coordination fluctuations are suppressed because the total coordination is constrained. In addition to random Voronoi lattices it includes, e.g., regular lattices with bond-exchange defects which are related to the topological models of Le Caër [20]. It also includes deterministic quasiperiodic lattices such as the Penrose and Ammann-Beenker tilings [21] (using rhombic tiles) as well as random tilings [22] whose tiles are either all triangles or all quadrilaterals.

What about higher dimensions? The Euler equation for a 3D tessellation, $N - E + F - C = \chi$, contains 1 extra degree of freedom, viz., the number C of 3D cells. The total coordination of a random tetrahedralization is therefore not fixed by a topological constraint, in agreement with the fact that the solid-angle sum in a tetrahedron is not a constant. Consequently, 3D random Voronoi lattices do not belong to our class of lattices with a constrained total coordination. However, 3D members of our class do exist. They include, e.g., lattices built exclusively from rhombohedra such as the icosahedral tiling and its random variants [23] (the solid angle sum of a rhombohedron is fixed at 4π), as well as generalizations of the bond-exchange lattices to 3D.

We now generalize to arbitrary dimension our estimate of the fluctuations of the block-averaged coordination number. As the bulk contribution is suppressed by the anticorrelations, the main contribution stems from the surface. The number of cells or facets close to the surface scales as L_b^{d-1} with block size L_b. In the generic case, i.e., in the absence of further constraints or long-range correlations, these surface cells make independent random contributions to Q_μ. This leads to

$$\sigma_Q(L_b) \sim L_b^{(d-1)/2}/L_b^d = L_b^{-(d+1)/2}. \qquad (5)$$

Casting this result in terms of the wandering exponent ω defined via $\sigma_Q \sim L_b^{-d(1-\omega)}$ [24], we obtain $\omega = (d-1)/(2d)$. This needs to be compared to uncorrelated randomness for which $\sigma_Q \sim L_b^{-d/2}$ and $\omega = 1/2$.

We have verified the prediction (5) for several lattices in addition to the 2D Voronoi lattice. The first is a random lattice produced from a triangular lattice by performing random bond exchanges. A bond exchange (left inset of Fig. 4) consists of randomly choosing a rhombus made up
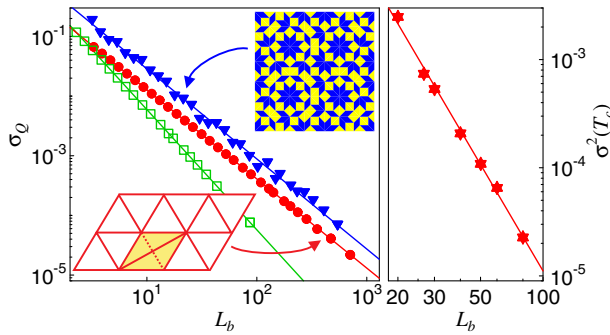
FIG. 4 (color online). Left: σ_Q vs L_b for the Ammann-Beenker tiling (8th generation, 643 008 4 sites, triangles), a triangular lattice with 50% bond-exchange defects (100 lattices with 2000^2 sites, circles), and a rhombohedral lattice with 50% bond-exchange defects (100 lattices with 300^3 sites, open squares). The lines are power-law fits giving exponents of 1.51(3), 1.498(2), and 2.01(1), respectively. Right: $\sigma^2(T_c)$ vs L_b for an Ising model on a random Voronoi lattice (100 lattices of 100^2 sites, 10^5 Monte Carlo sweeps each). The line is a fit to $\sigma(T_c) \sim L_b^{-c}$ giving $c = 1.56(7)$.

of two adjacent triangles and replacing the short diagonal (dotted) with the long one (solid). The second example is the deterministic quasiperiodic Ammann-Beenker tiling. For both lattices, the numerical data (Fig. 4) follow $\sigma_Q(L_b) \sim L_b^{-3/2}$ in agreement with (5) [25]. Finally, we have studied a 3D rhombohedral lattice with bond-exchange defects. The numerical data are in excellent agreement with the prediction $\sigma_Q(L_b) \sim L_b^{-2}$.

We now use these results to derive the analog of the Harris criterion for many-particle systems on random lattices in our class. Following Harris and Luck [1,24], we compare the fluctuations of the local distance from criticality between correlation volumes with the global distance from criticality. If the interactions between the sites are restricted to nearest neighbors and of equal strength, the disorder fluctuations are governed by (5) and decay as $\xi^{-(d+1)/2}$ with correlation length ξ. The global distance from criticality scales as $\xi^{-1/\nu}$. A clean critical point is thus stable if $\xi^{-(d+1)/2} < \xi^{-1/\nu}$ for $\xi \to \infty$. This yields the stability (Harris-Luck) criterion $(d+1)\nu > 2$. The topological disorder is thus less relevant than generic uncorrelated randomness for which the Harris criterion reads $d\nu > 2$.

The Imry-Ma criterion compares the free energy gain due to forming a domain that takes advantage of a disorder fluctuation with the energy cost of the domain wall. In our class of lattices, the gain scales as $L_b^{d\omega} = L_b^{(d-1)/2}$ while the cost of a domain wall scales as L_b^{d-1}. Forming large domains is thus unfavorable in all dimensions $d > 1$, implying that first-order transitions can survive.

The coordination number fluctuations determine the bare (in the renormalization group sense) disorder of the many-particle system. To study an example of disorder renormalizations, we calculate the local critical

temperatures T_c of the Ising model, $H = -J\sum_{\langle ij \rangle} S_i S_j$, on a random Voronoi lattice by Monte Carlo simulations. The right panel of Fig. 4 shows the variance of the block T_c (defined as the maximum of the susceptibility) as a function of block size. The data follow $\sigma(T_c) \sim L_b^{-3/2}$ in agreement with the coordination number. In general, disorder renormalizations can be expected to generate weak uncorrelated disorder even if the bare disorder is anticorrelated [26]. Our results suggest that this uncorrelated disorder, if any, is very weak (as it is invisible on length scales below $L_b \approx 100$) and thus unobservable in most experiments and simulations.

In summary, we have studied the effects of topological disorder on phase transitions. We have identified a broad class of random lattices characterized by strong disorder anticorrelations. Such lattices are ubiquitous in two dimensions because the Euler equation imposes a topological constraint on the coordination numbers. However, we have also found higher-dimensional realizations. The anticorrelations lead to modifications of the Harris and Imry-Ma criteria. This explains most of the puzzling apparent failures of the usual criteria discussed in the introduction. Note that another type of anticorrelation was recently found to protect a clean critical point in a quantum spin chain [27]. Moreover, local disorder correlations that change the degree of frustration in a spin glass can qualitatively change its phase diagram [28].

Interestingly, the 3D random Voronoi lattice does not belong to our class of lattices with constraint total coordination. Preliminary numerical results suggest that its coordination number fluctuations decay more slowly than (5) but still faster than the uncorrelated randomness result $L_b^{-d/2}$, at least for blocks with $L_b < 400$. Further work will be necessary to understand the fate of phase transitions on 3D Voronoi lattices.

So far, we have considered systems in which all pairs of neighbors interact equally strongly. If this is not so, e.g., because the interactions depend on the distance between neighboring sites, the disorder anticorrelations are destroyed. The critical behavior is thus expected to cross over to that of uncorrelated disorder. We have explicitly observed this crossover in the contact process [29].

It will be interesting to study transitions that violate even the modified stability criterion $(d+1)\nu > 2$. A prime example is the quantum phase transition of the transverse-field Ising magnet on a 2D random Voronoi lattice. Its clean critical behavior is in the $(2+1)$D Ising universality class with $\nu \approx 0.630$ and thus violates $(d+1)\nu > 2$. As the anticorrelations strongly suppress the rare region probability [29], we also expect significant modifications of the quantum Griffiths singularities.

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