How the Result of a Single Coin Toss Can Turn Out to be 100 Heads

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We show that the phenomenon of anomalous weak values is not limited to quantum theory. In particular, we show that the same features occur in a simple model of a coin subject to a form of classical backaction with pre- and postselection. This provides evidence that weak values are not inherently quantum but rather a purely statistical feature of pre- and postselection with disturbance.

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In many quantum mechanical experiments, we observe a dissonance between what actually happens and what ought to happen given naïve classical intuition. For example, we would say that a particle cannot pass through a potential barrier—it is *not allowed* classically. In a quantum mechanical experiment, the "particle" can "tunnel" through a potential barrier—and a paradox is born. Most researchers spent the 20th century ignoring such paradoxes (that is, "shutting up and calculating" [1]), while a smaller group tried to understand these paradoxes [2–5] and put them to work [6].

Experimentalists can probe the quantum world through measuring the expectation value of an observable A. After many experimental trials, the expected value is

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle, \tag{1}$$

where $|\psi\rangle$ is the quantum state of the system under consideration. The measurement of such an expected value allows us to demonstrate, for example, that Bell's inequalities [5] are violated. Thus, the measurement of the expected value can have foundational significance.

In Eq. (1), the potential values one can observe are limited to the eigenvalue range of A. It was surprising, then, that Aharonov, Albert, and Vaidman [7] claimed the opposite. In 1988, they proposed the *weak value* of an observable. The weak value of A is defined as [7,8]

$$a_w = \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}, \tag{2}$$

where $|\psi\rangle$ and $|\phi\rangle$ are called *pre-* and *postselected* states. Notice that, when $\langle\phi|\psi\rangle$ is close to zero, a_w can lie far outside the range of eigenvalues of A, hence the title of Ref. [7]: "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100." When this is the case, the weak value is termed *anomalous*.

Weak values are said to have both foundational and practical significance. On one hand, they are claimed to solve quantum paradoxes [9], while, on the other, they are claimed to amplify small signals to enhance quantum metrology [10] (but compare to Refs. [11–16]). One research program in the weak value community is to examine a paradoxical quantum effect or experiment and then calculate the weak value for that situation. Often, the calculated weak value is anomalous. From this, we are supposed to conclude that the paradox is resolved (see, for example, [17] for a recent review). So it would further seem, then, that anomalous weak values, if not *the* source of quantum mysteries, provide deep insight into finding it. Indeed, since their inception, weak values have inspired deep thinking and debate about the interpretation and foundational significance of weak values [18–23].

Where a classical explanation exists, no quantum explanation is required. This is a guiding principle for quantum foundations research. In this Letter, we provide a simple classical model which shows that anomalous weak values are not limited to quantum theory. In particular, we show that the same phenomenon manifests in even the simplest classical system: a coin. This shows that the effect is an artifact of toying with classical statistics and disturbance rather than a physically observable phenomenon.

Let us begin by defining the weak value as it was formally introduced before casting it into a more general picture. We have a system with observable $A = \sum_a a|a\rangle\langle a|$ and meter system with conjugate variables Q and P so that $[Q,P] = i$. The system and meter start in states $|\psi\rangle$ and $|\Phi\rangle = (2\pi\sigma^2)^{-1/4}\int dq' \exp(-q'^2/4\sigma^2)|q'\rangle$, and we define $\Phi(q) = \langle q|\Phi\rangle$. They interact via the Hamiltonian $H = A\otimes P$ and then are measured in the bases $\{|\phi_k\rangle\}$ and $\{|q\rangle\}$, where $q\in(-\infty,\infty)$. We are interested in the joint probability distribution of this measurement: $\Pr(q,\phi|\psi,\Phi) = |\langle\phi|\langle q|e^{-ixH/\hbar}|\psi\rangle|\Phi\rangle|^2$, where x is the product of the coupling constant and interaction time. In this case, it can be shown (as in Chap. 16 of Ref. [9]), in the limit $\sigma\to\infty$ [24], that

$$\langle\phi|\langle q|e^{-ixA\otimes P/\hbar}|\psi\rangle|\Phi\rangle = \langle\phi|\psi\rangle\Phi(q - xa_w), \tag{3}$$

where a_w is the weak value given in (2) and, assuming a_w is real [25], is the average shift of the meter position given the states $|\psi\rangle$ and $|\phi\rangle$. Consider the following example.

We take the system observable $A = Z$, the Pauli Z operator, and pre- and postselected states

$$|\psi\rangle = \cos\theta/2|+1\rangle + \sin\theta/2|-1\rangle, \quad (4)$$

$$|\phi\rangle = \cos\theta/2|+1\rangle - \sin\theta/2|-1\rangle, \quad (5)$$

where $|+1\rangle$ and $|-1\rangle$ are the $+1$ and -1 eigenstates of Z, respectively. A short calculation reveals

$$a_w = \frac{1}{\cos\theta}; \quad (6)$$

thus, when $\theta \approx 1.5608$, we have $a_w = 100$. This is patently nonclassical, as the states required to observe a value $a_w > 1$, say, are in different bases. Next, we will show how to obtain an anomalous weak value from a system-meter picture and statistical conditioning.

A large class of measurements that we can perform on a quantum system can be described by a set of Kraus operators and their corresponding positive operator valued measure. Below, we will need to measure a coarse graining over a set of Kraus operators: a quantum operation. We expand the unitary to first order in x: $U(x) = \exp(-ixA \otimes P/\hbar) \approx \mathbb{I} \otimes \mathbb{I} - ixA \otimes P/\hbar$. To this order in perturbation theory, the Kraus operators for a position measurement on the meter are $M_q = \langle q|U(x)|\Phi\rangle = \mathbb{I}\langle q|\Phi\rangle - ixA\langle q|P|\Phi\rangle/\hbar$. By using $P = -i\hbar\partial/\partial q$ and $\partial_y \exp(-y^2/4\sigma^2) = (-y/2\sigma^2)\exp(-y^2/4\sigma^2)$, the Kraus operator becomes

$$M_q = \left[\mathbb{I} - q\frac{x}{2\sigma^2}A\right]\Phi(q), \quad (7)$$

where σ^2 is the initial variance of the Gaussian meter state and x is the coupling constant. Now we consider coarse-grained measurements so that $q \le 0$ is identified as the "+1" outcome of A and $q > 0$ is identified as the result "−1"; then the corresponding quantum operations are

$$\mathcal{E}_+\rho = \int_{-\infty}^{0} dq M_q \rho M_q^\dagger \quad \text{and} \quad \mathcal{E}_-\rho = \int_{0}^{\infty} dq M_q \rho M_q^\dagger. \quad (8)$$

Such quantum operations have conditional states $\rho_\pm = \mathcal{E}_\pm|\psi\rangle\langle\psi|/\mathrm{Tr}[\mathcal{E}_\pm|\psi\rangle\langle\psi|]$, which are generally mixed states. Performing the integral gives the operation

$$\mathcal{E}_\pm\rho = \frac{1}{2}\left[\rho \pm \frac{x}{\sqrt{2\pi\sigma^2}}(A\rho + \rho A)\right]. \quad (9)$$

Collecting the constants, we define $\lambda \equiv 2x/\sqrt{2\pi\sigma^2}$. Also we find it convenient to introduce a classical random variable $s \in \{\pm 1\}$ for the sign of the outcome. With these conventions, the operation becomes

$$\mathcal{E}_s\rho = \frac{1}{2}\left[\rho + s\frac{\lambda}{2}(A\rho + \rho A)\right]. \quad (10)$$

Note that as $\lambda \to 0$ the measurement approaches the trivial one, conveying no information and leaving the postmeasurement unaffected.

The trace of Eq. (10) describes the outcome statistics of weak measurement of the operator A in the state ρ. This can be seen from the probability of observing the outcome s:

$$\Pr(s|\psi) = \mathrm{Tr}[\mathcal{E}_s|\psi\rangle\langle\psi|] = \frac{1}{2}(1 + s\lambda\langle\psi|A|\psi\rangle), \quad (11)$$

which is correlated with the expectation value of the operator A.

Following Ref. [26], we now calculate the conditional expectation of the random variable s given the pre- and postselected states $|\psi\rangle$ and $|\phi\rangle$, respectively:

$$\mathbb{E}_{s|\phi,\psi}[s] = \sum_{s=\pm1} s\frac{\Pr(s,\phi|\psi)}{\Pr(\phi|\psi)} = \sum_{s=\pm1} s\frac{\langle\phi|\mathcal{E}_s(|\psi\rangle\langle\psi|)|\phi\rangle}{|\langle\phi|\psi\rangle|^2}, \quad (12)$$

where $\mathbb{E}_{x|y}[f(x)]$ denotes the conditional expectation of $f(x)$ given y and $\Pr(\phi|\psi) = \sum_s \Pr(s,\phi|\psi) = \sum_s \langle\phi|\mathcal{E}_s(|\psi\rangle\langle\psi|)|\phi\rangle$ becomes $|\langle\phi|\psi\rangle|^2$. Expanding the numerator, we obtain

$$\mathbb{E}_{s|\phi,\psi}[s] = \sum_{s=\pm1}\frac{s}{2}\frac{\langle\phi||\psi\rangle\langle\psi| + (s\lambda/2)\{|\psi\rangle\langle\psi|, A\}_+|\phi\rangle}{|\langle\phi|\psi\rangle|^2}, \quad (13)$$

where $\{A, B\}_+ = AB + BA$. This result can also be arrived at by using the Bayes rule to determine $\Pr(s|\phi,\psi)$, which is known as the "ABL rule" in quantum theory (after Aharonov, Bergmann, and Lebowitz [27]). Further expanding the numerator [25], we arrive at

$$\mathbb{E}_{s|\phi,\psi}[s] = \sum_{s=\pm1}\frac{s}{2}\left(1 + s\lambda\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}\right) \quad (14)$$

$$= \lambda\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \quad (15)$$

Thus, the conditional expectation of s results in a quantity proportional to the weak value. Since the constant of proportionality is λ, to arrive directly at the weak value we consider the conditional expectation of the random variable s/λ. Using Eq. (15), we have

$$\mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] = \frac{1}{\lambda}\mathbb{E}_{s|\phi,\psi}[s] = \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \quad (16)$$

Thus, an equivalent definition of the weak value is

$$a_w = \mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right]. \tag{17}$$

To relate this to the meter picture, note that $s/\lambda = s\sqrt{2\pi\sigma^2}/2x$. Thus, the limit of $\lambda \to 0$ is identical to $\sigma \to \infty$ [24]. It is clear from (17) that a weak value is a calculated quantity; specifically, it is the conditional expectation of the random variable s/λ.

From Eq. (11), we can see that

$$\langle\psi|A|\psi\rangle = \mathbb{E}_{s|\psi}\left[\frac{s}{\lambda}\right] = \sum_s \frac{s}{\lambda}\Pr(s|\psi). \tag{18}$$

By the classical law of total expectation, we have

$$\langle\psi|A|\psi\rangle = \mathbb{E}_{s|\psi}\left[\frac{s}{\lambda}\right] = \mathbb{E}_{\phi|\psi}\left[\mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right]\right]. \tag{19}$$

From Eq. (17), we know we can replace $\mathbb{E}_{s|\phi,\psi}[\frac{s}{\lambda}]$ with the weak value; thus,

$$\langle\psi|A|\psi\rangle = \mathbb{E}_{\phi|\psi}\left[\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}\right]. \tag{20}$$

So, the weak value arises close to the way it is often envisioned to—as a condition expectation—but, to define it properly, we need to include a renormalization by the weakness parameter λ.

Now we demonstrate that it is possible to find anomalous weak values for pre- and postselected states in the same basis provided there is classical disturbance. In particular, we take $A = Z$, $|\psi\rangle = |+1\rangle$, and later we will postselect on $|\phi\rangle = |-1\rangle$. By using the probabilities in Eq. (11), the probability of the outcome of the weak measurement is

$$\Pr(s|\psi = +1) = \frac{1}{2}(1 + s\lambda). \tag{21}$$

Since the measurement is in the same basis as the state, the state is unchanged, and the final weak value will not be anomalous. Thus, we must do something more. To simulate the disturbance, we now apply a bit-flip channel which conditionally depends on the strength and outcome of the weak measurement. This is reasonable, as one would expect, from quantum measurement theory, that the amount of disturbance should depend on the measurement. After the channel, the state becomes

$$|+1\rangle\langle+1| \mapsto (1-p)|+1\rangle\langle+1| + p|-1\rangle\langle-1|, \tag{22}$$

where p is the probability of a bit-flip error. To match the quantum case, we want p to be close to 0 when the weak value ought to be large (as the occurrence of large weak values is rare) and close to 1 when the weak value ought to be small. Such a functional form of p is as follows:

$$p = \frac{1 + s\lambda - \delta}{1 + s\lambda}, \tag{23}$$

where δ is the *disturbance parameter* which is constrained to be $0 < \delta < 1 - \lambda$ so that $0 < p < 1$. The particular form of p is not important—many choices will lead to anomalous weak values. One can even choose the p here so that it is identical to effective p from the fully quantum calculation. Here we have introduced a new parameter δ not because we must but because we can. We have chosen this form so the final expression is as simple as possible.

In explicit probabilistic notation, we have

$$\Pr(\phi = +1|s, \psi = +1) = \frac{\delta}{1 + s\lambda}(\text{no flip}), \tag{24}$$

$$\Pr(\phi = -1|s, \psi = +1) = \frac{1 + s\lambda - \delta}{1 + s\lambda}(\text{flip}). \tag{25}$$

Using the Bayes rule, we find

$$\Pr(\phi = +1, s|\psi = +1) = \frac{\delta}{2}, \tag{26}$$

$$\Pr(\phi = -1, s|\psi = +1) = \frac{1}{2}(1 + s\lambda - \delta). \tag{27}$$

Marginalizing over s, we obtain

$$\Pr(\phi = +1|\psi = +1) = \delta, \tag{28}$$

$$\Pr(\phi = -1|\psi = +1) = 1 - \delta. \tag{29}$$

We now have all the ingredients to calculate the weak value as defined in Eq. (17). An interesting case is when we preselect on $\psi = +1$ and postselect on $\phi = -1$:

$$a_w = \mathbb{E}_{s|\phi,\psi}\left[\frac{s}{\lambda}\right] \tag{30}$$

$$= \sum_{s=\pm1} \frac{s}{\lambda}\frac{\Pr(s,\phi|\psi)}{\Pr(\phi|\psi)} \tag{31}$$

$$= \sum_{s=\pm1} \frac{s}{2\lambda}\left(\frac{1 + s\lambda - \delta}{1 - \delta}\right) \tag{32}$$

$$= \frac{1}{1 - \delta}. \tag{33}$$

With $0 < \delta < 1 - \lambda$, a_w can take on arbitrary values, just as the quantum mechanical weak value in Eq. (6). This is made obvious if we make a simple change of variable $\delta = 1 - \cos\theta$. The expression in Eq. (33) also illustrates

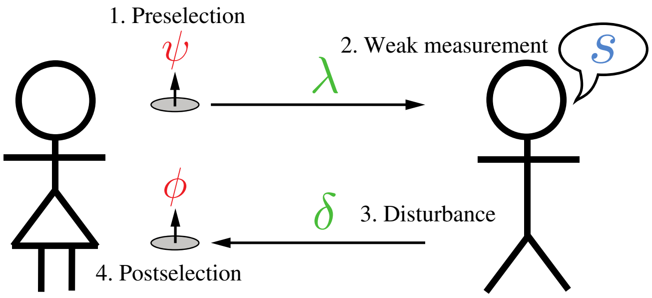

FIG. 1 (color online). An illustration of the protocol used to realize anomalous classical weak values.

the following important point: any disturbance whatsoever can result in an anomalous weak value. Thus, the effect is solely that of disturbance, postselection, and renormalization.

Since the state here remains in the Z basis at all times, this calculation is essentially classical. To make this point unequivocally clear, we now give an explicitly classical protocol to realize anomalous weak values. Our example revolves around a coin where the outcome "heads" is associated with the sign "+1" while "tails" is associated with the sign "−1." This allows us to compare the analysis above for a quantum coin case (a qubit) and a classical coin. As before, we abstract the sign into a random variable s.

An efficient strong measurement of a coin after a flip will result in an observer measuring and reporting outcome s with probability $\Pr(\text{report } s|\text{prepare } s) = 1$. A classical weak measurement of the sign of a coin $s \in \{\pm 1\}$ means that the observer did not properly ascertain if the coin was heads or tails. Such a measurement might arise from an observer not having the time to properly examine the coin or if there was oil on their glasses. We model this by introducing a probability $\Pr(\text{report } s|\text{prepare } s) = 1 - \alpha$ and $\Pr(\text{report}\neg s|\text{prepare } s) = \alpha$. To make the connection with the weak measurement in the quantum coin case [see Eq. (11)], we take $\alpha = (1-\lambda)/2$ so that

$$\Pr(s|\psi) = \frac{1}{2}(1 + \lambda s \psi). \tag{34}$$

For a coin that starts in heads, $\psi = +1$, so $\Pr(s|\psi = +1) = \frac{1}{2}(1 + \lambda s)$, which is identical to Eq. (21). In this case, the physical meaning of λ is clear—it is strength of the correlation between the result s and the preparation ψ.

We now introduce a classical protocol directly analogous to the quantum protocol that produces anomalous weak values. There are two people: Alice and Bob. The protocol is as follows (see also Fig. 1).

(i) *Preselection.*—Alice tosses the coin, the outcome ψ is recorded, and she passes it to Bob.

(ii) *Weak measurement.*—Bob reports s with the probabilities given in Eq. (34).

(iii) *Classical disturbance.*—Bob flips the coin with the probability given in Eq. (25) and returns it to Alice.

(iv) *Postselection.*—Alice looks at the coin and records the outcome ϕ.

For concreteness we preselect on heads, that is, $\psi = +1$. Bob then makes a weak measurement of the state of the coin, which is described by Eq. (34). In order to implement classical backaction, we introduce a probabilistic disturbance parameter δ to our model. Since we are free to choose this how we like, we choose the disturbance such that it results in the same flip probability in Eq. (25).

The point here is that Bob will flip the coin (i.e., $+1 \to -1$) with probability $1-\delta$. Although δ can be thought of as a "disturbance" parameter, a more entertaining interpretation is to think of Bob as a "λ liar, δ deceiver": Bob accepts the coin and lies about the outcome with probability $1/2(1-\lambda)$ and then furthermore, to cover his tracks, flips the coin before returning it to Alice with the probability depending on what he reports.

Since the probabilities for the classical and quantum cases are identical, the weak value is identical:

$$a_w = \frac{1}{1-\delta}. \tag{35}$$

In particular, we see that the classical weak value can be arbitrarily large, provided the parameter δ is close to 1 and we preselect $\psi = +1$ and postselect on $\phi = -1$. Take the example $\delta = 0.99$. The classical weak value of s, from Eq. (35) with $\delta = 0.99$, is $a_w = 100$. Thus, the outcome of the coin toss is 100 heads.

Some remarks are in order. First, we have pointed out that our model (in fact, any model) requires measurement disturbance for anomalous weak values to manifest. Since, *in theory*, classical measurements can have infinite resolution with no disturbance, some might consider our model nonclassical. However, *in practice*, classical measurements do have disturbance and do not have infinite precision. While we have not provided a physical mechanism for the disturbance here, it is clear that many can be provided. Thus, we leave the details of such a model open. We note that, in the context of Leggett-Garg inequalities, a similar observation was made: the weak value is bounded for noninvasive measurement [28].

The second, and perhaps more significant, potential criticism is that we have given a classical model where only *real* weak values occur, whereas the quantum weak value is a complex quantity in general. It is often stated that weak values are "measurable complex quantities" which further allow one to "directly" access other complex quantities [29]. However, the method to "measure" them is to perform separate measurements of the real and imaginary parts. This illustrates that the weak value is actually a *defined* quantity rather than a measured value. Thus, we can easily introduce complex weak values in our classical model with two observable quantities and simply multiply one by the imaginary unit—not unlike descriptions of circular polarization in the classical electromagnetic theory (compare to the recent classical interpretation of a weak value experiment [30]).

In conclusion, our analysis above demonstrates a simple classical model which exhibits anomalous weak values. Recall that the way in which weak values are used in foundational analyses of quantum theory is to show that they obtain anomalous values for "paradoxical" situations. To suggest that this is meaningful or explanatory, it must be the case that such values cannot be obtained classically. Here we have shown that they can indeed. Thus, the conclusion that weak values can explain some paradoxical situation or verify its quantumness is called to question. Our results provide evidence that weak values are not inherently quantum but rather a purely statistical feature of pre- and postselection with disturbance. Finally, our work suggests an interesting question for future research, namely, classical inference (including counterfactual reasoning) in the presence of classical disturbance.

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Note added.—Recently, we were made aware of similar work on "contextual values" [31], where the authors reach quite a different conclusion from ours.

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