Dysonian Dynamics of the Ginibre Ensemble

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We study the time evolution of Ginibre matrices whose elements undergo Brownian motion. The non-Hermitian character of the Ginibre ensemble binds the dynamics of eigenvalues to the evolution of eigenvectors in a nontrivial way, leading to a system of coupled nonlinear equations resembling those for turbulent systems. We formulate a mathematical framework allowing simultaneous description of the flow of eigenvalues and eigenvectors, and we unravel a hidden dynamics as a function of a new complex variable, which in the standard description is treated as a regulator only. We solve the evolution equations for large matrices and demonstrate that the nonanalytic behavior of the Green's functions is associated with a shock wave stemming from a Burgers-like equation describing correlations of eigenvectors. We conjecture that the hidden dynamics that we observe for the Ginibre ensemble is a general feature of non-Hermitian random matrix models and is relevant to related physical applications.

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Today, half a century after the pioneering work of Ginibre [1], random matrices with complex spectra are no longer only of academic interest. They play a role in quantum chaotic scattering [2,3], quantum information processing [4], QCD with finite chemical potential [5], in financial engineering with lagged correlations [6], and in the research on neural networks [7], to name just a few applications. Eigenvalues themselves, however, are not of sole interest in the case of non-Hermitian random matrix ensembles. The statistical properties of eigenvectors are equally significant [8], in particular, in problems concerning scattering in open chaotic cavities or random lasing [9–12]. There, the so called Petermann factor [13], a quantity describing correlations between right and left eigenvectors, modifies the quantum-limited linewidth of a laser.

On the other hand, the original Dyson's idea of a Brownian walk of real eigenvalues [14] interacting with a two-dimensional Coulombic force still leads to novel insights. Examples include the study of determinantal processes [15–17], Loewner diffusion [18], non-Hermitian deformations [19], or the fluctuations of nonintersecting interfaces in thermal equilibrium [20]. The concept of matricial stochastic evolution has been recently exploited by several authors [21–24]. In particular, it was shown that the derivatives of the logarithms of characteristic determinants of diffusing GUE (Gaussian unitary ensemble), LUE (Laguerre unitary ensemble) and CUE (Circular unitary ensemble) obey Burgers-like nonlinear equations, where the role of viscosity is played by the inverse of the matrix size. For infinite dimensions of the matrix, these equations correspond to the inviscid regime and describe evolution of the associated resolvents. Because of nonlinearity, they develop singularities (shock waves), whose positions correspond to the endpoints of the spectra. For matrices of finite size, the expansion around the shock wave solution of the initial viscid Burgers equation leads to a universal scaling of characteristic polynomials (and of the inverse characteristic polynomials as well), resulting in well-known universal oscillatory behavior of the Airy, Bessel, or Pearcey type. This approach has prompted, in particular, new perception of weak or strong coupling transition in multicolor Yang-Mills theory [25,26] and of the spontaneous breakdown of chiral symmetry in Euclidean QCD [27].

In this Letter, we unveil the intertwined evolution of eigenvalues and eigenvectors of stochastically evolving non-Hermitian matrices. To this end, we apply Dyson's idea to study diffusing matrices for the case of the Ginibre ensemble (GE). The central object of the Letter is a generalized averaged characteristic polynomial. Its logarithmic derivatives, which contain the information about both the eigenvalues and eigenvectors of the evolving matrix, fulfill a system of Burgers-like partial differential equations. We solve them to recover the spectral density, the Petermann factor encoding the correlations of eigenvectors and universal microscopic scaling at the edge of the support of the eigenvalues.

At first glance one would not expect any similarities between the GUE and the GE, even in the large N (matrix size) limit. In the case of GUE, spectra are real; end points of the spectra exhibit square root behavior and the eigenvectors decouple from the eigenvalues. In the case of GE, spectra are complex, eigenvalues form a uniform disc with a vertical cliff at the boundary and the eigenvectors are correlated [8] on the support of eigenvalues. Nonetheless, the Vandermonde determinant is present in the joint probability distribution of eigenvalues for both ensembles and this leads to a two-dimensional electrostatic Dyson's picture that underlies calculations of the spectral distribution in the large N limit. Consequently, the standard procedure for non-Hermitian ensembles relies on defining the electrostatic potential

$$V(z) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\,\ln[|z-X|^2+\epsilon^2]\rangle, \tag{1}$$

calculating the "electric field" as its gradient $G=\partial_z V$, and recovering the spectral function from the Gauss law $\rho=(1/\pi)\partial_{\bar z}G=(1/\pi)\partial_{z\bar z}V$. We use a shorthand notation defined by $|z-X|^2+\epsilon^2=(z\mathbb{1}_N-X)(\bar z\mathbb{1}_N-X^\dagger)+\epsilon^2\mathbb{1}_N$, where $\mathbb{1}_N$ is the N-dimensional identity matrix. ϵ is an infinitesimal regulator and it is crucial that the limit $N\to\infty$ is taken first. If one took the limits in an opposite order, one would obtain a trivial result. Moreover, in the case of the Ginibre ensemble, $\langle\det(z-X)\rangle=z^N$. The standard relation between zeros of the characteristic polynomials and poles of the Green's function, known from considerations of Hermitian ensembles, would therefore be lost.

The idea is to define the following object

$$\begin{aligned} D(z,w,\tau) &= \langle \det(Q-H)\rangle_\tau \\ &= \langle \det(|z-X|^2+|w|^2)\rangle_\tau, \end{aligned} \tag{2}$$

where

$$Q=\begin{pmatrix} z & -\bar w \\ w & \bar z\end{pmatrix}, \qquad H=\begin{pmatrix} X & 0 \\ 0 & X^\dagger\end{pmatrix} \tag{3}$$

and to study its evolution in the space of Q, or more precisely in the complex plane w "perpendicular" to the basic complex plane z. In other words, the regulator $i\epsilon$, which is usually treated as an infinitesimally small real variable, is promoted to a genuine complex variable w. Note that D is effectively a characteristic determinant expressed in terms of the quaternion variable Q, since $Q=q_0+i\sigma_j q_j$, where σ_j are Pauli matrices, so $z=q_0+iq_3$ and $w=q_1+iq_2$. As we shall see, the dynamics of $D(z,w,\tau)$ hidden in w captures the evolution of eigenvectors and eigenvalues of the Ginibre matrix whose elements undergo Brownian motion. It is worth mentioning that block matrices such as H and arguments Q naturally appear in non-Hermitian random matrix models, e.g., in the generalized Green's function technique [28,29], in Hermitization methods [30–32], in the derivation of the multiplication law for non-Hermitian random matrices [33], and in the weak non-Hermitian random ensembles [34].

In our notation, the meaning of the averages $\langle\ldots\rangle_\tau$ like this in (2) is $\langle F(X)\rangle_\tau=\int DXP(X,\tau|X_0,0)F(X)$, where $DX=\sum_{ab}dx_{ab}dy_{ab}$ is a flat measure over the real and imaginary parts of matrix elements, $X_{ab}=x_{ab}+iy_{ab}$, and $P(X,\tau|X_0,0)$ is the probability that the matrix will change from its initial state X_0 at $\tau=0$ to X at time τ. For a free random walk with independent increments $\langle\delta X_{ab}\rangle_\tau=0$ and $\langle\delta X_{ab}\delta\bar X_{cd}\rangle_\tau=(\delta\tau/N)\delta_{ac}\delta_{bd}$, the evolution of $P(X,\tau|X_0,0)$ is governed by the diffusion equation

$$\partial_\tau P(X,\tau|X_0,0)=\frac{1}{N}\partial_{XX^\dagger}P(X,\tau|X_0,0), \tag{4}$$

where $\partial_{XX^\dagger}$ is the standard $2N^2$-dimensional Laplacian $\partial_{XX^\dagger}=\sum_{ab}(\partial^2_{x_{ab}}+\partial^2_{y_{ab}})$. The announced dynamics of the Ginibre ensemble is hidden in equation

$$\partial_\tau D(z,w,\tau)=\frac{1}{N}\partial_{w\bar w}D(z,w,\tau), \tag{5}$$

which is central to this Letter. The derivation will be presented elsewhere, but we shortly sketch below the main steps. The determinant in (2) can be represented as a Berezin integral $\int\exp[\theta^T(Q-H)\eta]d\theta d\eta=\det(Q-H)$, where θ and η are independent vectors of Grassmann variables. Both sides of Eq. (4) can be then multiplied by this integral and integrated over DX. After some manipulations, like changing the order of integration and integrating by parts, one arrives at (5).

It is easy to see that $D(z,w,\tau)$ depends on w only through its modulus $r=|w|$. Moreover, the simplest initial condition corresponds to $X_0=0$ with $D_0(z,w)=D(z,w,0)=(|z|^2+|w|^2)^N$. The general matrix X_0 is determined by the eigenvalues Λ and a set of left-(L) and right-(R) eigenvectors $X_0R=R\Lambda$ ($L^\dagger X_0=\Lambda L^\dagger$). By applying a transformation $S=\mathrm{diag}(R,L)$, $S^{-1}=\mathrm{diag}(L^\dagger,R^\dagger)$, the off-diagonal blocks depend explicitly on the eigenvectors

$$\det(S^{-1}(Q-H)S)=\det\begin{pmatrix} z-\Lambda & -\bar w L^\dagger L \\ wR^\dagger R & \bar z-\Lambda^\dagger\end{pmatrix}. \tag{6}$$

This calculation shows that nonzero w indeed encodes full information of the underlying matrix which turns out to be valuable in what follows.

We define two convenient functions $v=v(z,r,\tau)$ and $g=g(z,r,\tau)$:

$$v\equiv\frac{1}{2N}\partial_r\ln D, \tag{7}$$

$$g\equiv\frac{1}{N}\partial_z\ln D, \tag{8}$$

which will turn out to be closely related to the eigenvector correlator and the Green's function known from the standard treatment of the Ginibre ensemble. These functions are not independent, since by construction $\partial_z v=\frac{1}{2}\partial_r g$; in particular, $g=2\int dr\partial_z v$. The diffusion equation (5) is mapped via (7), which basically is the inverse Cole-Hopf transformation [35], onto a Burgers-like equation

$$\partial_\tau v=v\partial_r v+\frac{1}{N}\left(\Delta_r-\frac{1}{4r^2}\right)v, \tag{9}$$

where $\Delta_r=\frac{1}{4}(\partial_{rr}+(1/r)\partial_r)$ is the radial part of the two-dimensional Laplacian. This equation is exact for any N.

The $1/N$ factor is a viscosity-like parameter. In the inviscid limit ($N \to \infty$), (9) reduces to

$$\partial_\tau v = v \partial_r v, \tag{10}$$

known as the Euler equation and solved by the method of characteristics. The curves along which the solution is constant are given by

$$r = \xi - v_0(\xi)\tau, \tag{11}$$

and labeled with ξ. v_0 plays the role of velocity of the front wave. We therefore have

$$v = v_0(r + \tau v). \tag{12}$$

For the initial condition $X_0 = 0$, corresponding to $v_0(r) = r/(z\bar{z} + r^2)$, we obtain a cubic algebraic equation for v. Its solution gives the (radial) dependence of v on $r = |w| \geq 0$. If one takes a cross section of the whole solution along the real axis, $\mathrm{Im}w = 0$ and $\mathrm{Re}w = \mu$, one can see that the solution consists of two symmetric branches $v(\mu) = v(-\mu)$ due to the rotational symmetry of the problem in the complex plane. In other words, the solution is represented by the pair of Cardano equations:

$$v(z\bar{z} + (\pm\mu + \tau v)^2) = \pm\mu + \tau v, \tag{13}$$

since μ, as opposed to r, may be positive or negative. The mapping between r and ξ breaks down when, at some positions $\mu = \pm r_*$, the derivative becomes singular ($d\xi/dr_* = \infty$), as visualized on the left inset at Fig. 1. The set of singular points defines the caustics (sometimes called preshocks). Physically, the singularity comes from the fact that the velocity of the flow is position dependent, which makes the solution, for a given $|z|$, nonunique after a certain time τ. Between the two symmetric caustics (which actually form a conelike surface when viewed from the whole w-complex plane) a shock is formed at $\mu = 0$ for $\tau \geq |z|^2$. Although the shock formation involves the whole (w, z) space, as depicted in Fig. 1, its dynamics is remarkably confined to the region of $r = |w| \to 0$, close to the z plane, which is the basic complex plane in our considerations. As was already mentioned, in this region r plays the role of the regulator ϵ in the formula (1). In this limit the explicit solution of (13) reads

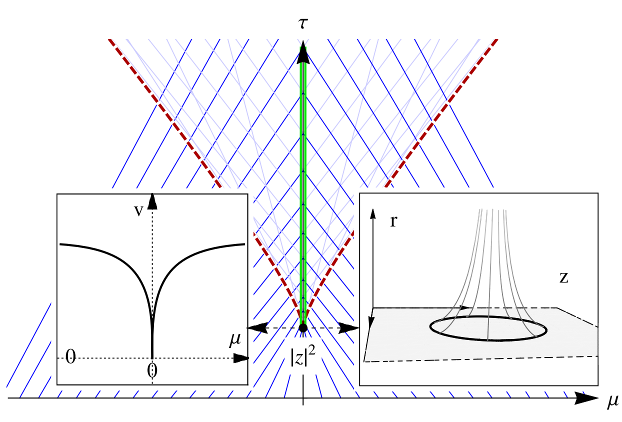

FIG. 1 (color online). The main figure shows, for a given $|z|$, the characteristics (straight lines) and caustics (dashed lines). Inside the later a shock is developed (double vertical line). Left inset shows the solution of Eq. (13) at ($\tau = |z|^2$). Right inset shows the caustics mapped to the ($r = |w|, z$) hyperplane at the same moment of time. The section $r = 0$ yields the circle $|z|^2 = \tau$, bounding the domain of eigenvalues and eigenvectors correlations for the GE.

$$v^2 = (\tau - |z|^2)/\tau^2 \quad \text{and} \quad v = 0, \quad \text{as r} \to 0. \tag{14}$$

The quantity v^2 has an explicit interpretation [36] in the large N limit, namely,

$$v^2 = \frac{\pi}{N^2}\left\langle \sum_i A_{ii}\delta^2(z - \lambda_i)\right\rangle, \tag{15}$$

where $A_{ij} = (L^\dagger L)_{ij}(R^\dagger R)_{ji}$; i.e., v^2 is a correlator between the biorthogonal sets of left and right eigenvectors introduced before, known in nuclear physics as the Bell-Steinberger matrix [12] and in the RMT context introduced in [8]. This correlator is also known from chaotic scattering theory as the Petermann factor [9]

$$K(z, \tau) = \frac{N}{\pi\rho} v^2 \tag{16}$$

(where ρ is the spectral density calculated later). Off-diagonal elements of matrix A are used to probe nonorthogonality of resonances in open quantum systems [3,37]. Figure 2 shows the time dependence of the Petermann factor for several values of $|z|$. The correlator vanishes outside the critical shock line, where, as we know from the standard approach, the Green's function is analytic, and it is nonzero inside it, where the Green's function is nonanalytic. The edge of the shock line lines up with the contour of the eigenvalue density support. To summarize, the quaternion shock wave dynamics (14) reproduces the result of [8].

Having an explicit solution for v (7), we can turn to g (8). Actually, one can show that g also fulfills a Burgers-like equation exact for any N,

$$\partial_\tau g = v \partial_r g + \frac{1}{N}\Delta_r g, \tag{17}$$

which in the inviscid limit reduces to $\partial_\tau g = v \partial_z g$ or

$$\partial_\tau g = 2v \partial_z v, \tag{18}$$

if one uses $\partial_r g = 2\partial_z v$. We see that we can calculate g by differentiating v. The initial condition $X_0 = 0$ corresponds to $g_0(r) = \bar{z}/(|z|^2 + r^2)$, in particular, $g_0(r = 0) = 1/z$. For $v = 0$ we have $\partial_\tau g = 0$ so g is constant in time, and therefore it is equal to $g = 1/z$ everywhere outside the shock line. Inside the shock line, we employ the second solution of (14), which via elementary integration leads to

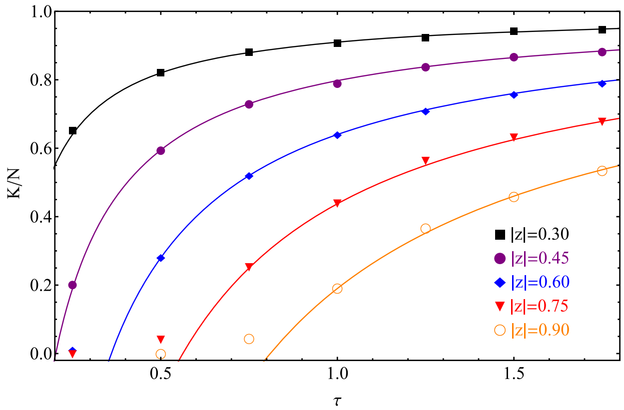

FIG. 2 (color online). The figure depicts theoretical (lines) and numerical (symbols) time dependence of the Petermann factor (rescaled by $1/N$), for different values of $|z|$. For the latter, 3×10^4, 200×200 matrices were used.

$g = \bar{z}/\tau + f(z)$. Since both solutions have to match on the line of the shock due to condition (14), the arbitrary analytic function f has to be equal to zero. Note that for $r = 0$, g coincides with the electric field $G(z, \bar{z})$ in the standard formulation mentioned earlier, so the average spectrum of the considered ensemble reads

$$\rho(z, \tau) = \frac{1}{\pi\tau}\Theta(\tau - |z|^2), \tag{19}$$

where $\Theta(x)$ is the Heaviside step function. We see that complex eigenvalues are uniformly distributed on a growing disc of radius $\sqrt{\tau}$.

Finally, we would like to comment on the solution for large but finite N, at the vicinity of the shock. Since finite size implies nonzero viscosity, the dissipative term will regularize the shock leading to the smoothening of the sharp cliff of the eigenvalue density at the edge of the disk (19). Explicit calculations show that this is indeed the case. The smoothening makes the density at the edge assume a universal shape given by the complementary error function [38]. The argument goes as follows. We use the result of [39], that the spectral density (diagonal part of the kernel) for the Ginibre ensemble is proportional to the $r \to 0$ limit of the characteristic determinant D of the type considered here. The proportionality factor is the normalization C_N and the Gaussian weight $p(z) = \exp(-(N/\tau)|z|^2)$, i.e.,

$$\rho(z, \tau) \overset{N\to\infty}{=} C_N p(z) D(z, r \to 0, \tau), \tag{20}$$

with $C_N = (2/\tau\pi)(1/(N-1)!)(N/\tau)^N$. Then, we may use the fact that the form of D is exactly known for our initial conditions, since it represents the solution for the radial diffusion [27,40,41]

$$D = \int_0^\infty q e^{-N(q^2 + r^2/\tau)} I_0\left(\frac{2Nqr}{\tau}\right)(q^2 + |z|^2)^N dq. \tag{21}$$

A careful analysis of the saddle points shows that for large N the main contribution to the integral comes from quantities which scale as $q = \theta N^{-1/4}$, $|z| - \sqrt{\tau} = \eta N^{-1/2}$, and $r = \omega N^{-3/4}$, for θ, η, and ω of order 1. We postpone the details for a future publication. Here we note, however, that this scaling is identical to the critical scaling for the cusp singularity of the Wishart or chiral random matrices. The reason for this lies in the functional form of the determinants, which happens to be identical for the two ensembles. In this way we establish additionally a somehow unexpected link between the universal scaling behavior for the Wishart and Ginibre ensembles. Taking first the large N limit and then setting $\omega = 0$, we recover from (21) a well-known result for the universal scaling at the spectral edge of the Ginibre ensemble

$$\rho(\eta) \approx \frac{1}{2\pi\tau}\mathrm{Erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right). \tag{22}$$

We conclude this Letter with several remarks. First, it is inspiring to compare the Burgers-like structures even between the simplest Hermitian model (GUE) and its non-Hermitian counterpart, i.e., the Ginibre ensemble. In the case of GUE, the characteristic determinant $D_{\mathrm{GUE}}(z)$ fulfills a complex diffusion equation $\partial_\tau D_{\mathrm{GUE}} = -(1/2N)\partial_{zz} D_{\mathrm{GUE}}$. The corresponding Burgers equation resulting from the Cole-Hopf transformation is complex too and has to be solved with complex characteristics. Singularities (shock waves) appear at discrete points (end points of the spectra) in the flow of eigenvalues [21]. On the contrary, for the GE, singularities are given by one-dimensional curves appearing in the flow of eigenvector correlations. The fact that in the Hermitian case the viscosity is negative also has far-reaching consequences. In particular, it is not smoothening the shock, like in the GE (where we observe the Erfc smearing), but it triggers violent oscillations, being the source of Airy universality. Similar universal oscillations originate from negative viscosity in other ensembles. The fact that ensembles as different as GUE, CUE, LUE, and GE have a similar underlying mathematical structure of Burgers-like equations is remarkable and deserves further studies.

Moreover, for clarity we have only considered the dynamics of the simplest non-Hermitian ensemble. Our approach works, however, for any initial condition imposed on the considered process. Additionally, the method can be used to study other non-Hermitian ensembles (e.g., non-Gaussian ones), for which the described coevolution will also be present. The resulting equations are of course much more involved in more general scenarios. Our formalism could also be exploited to expand the area of application of non-Hermitian random matrix ensembles within problems of growth [18], charged droplets in the quantum Hall effect [42], and gauge theory or geometry relations in string theory [43] beyond the subclass of complex matrices represented by normal matrices.

Finally, we would like to emphasize that a consistent description of non-Hermitian ensembles requires the knowledge of the detailed dynamics not only on the complex z plane, where eigenvalues live, but also in the "orthogonal" w plane. In several standard techniques of non-Hermitian random matrix models this second variable is treated as an auxiliary parameter, serving as a regulator only. We have shown that it governs, in the large N limit, the evolution of the standard correlator of eigenvectors which is furthermore coupled to the dynamics of the resolvent. Eigenvectors and eigenvalues evolve therefore simultaneously, and this coevolution is probably a common feature of all, also multipoint Green's functions in non-Hermitian random matrix models.

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