Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations

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We investigate the new soft graviton theorem recently proposed by Cachazo and Strominger. We use the Cachazo-He-Yuan formula to prove a universal behavior for both Yang-Mills theory and gravity scattering amplitudes at the tree level in arbitrary dimension.

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Introduction.—Strominger *et al.* proposed that a certain infinite-dimensional subgroup of the Bondi–van der Burg–Metzner–Sachs (BMS) supertranslation group is an exact symmetry of the quantum gravity $\mathcal{S}$ matrix [1]. Weinberg's soft theorem [2] is a Ward identity for this subgroup [3].

Cachazo and Strominger [4] further investigated the subleading terms $S_{\rm g}^{(1)}$ and $S_{\rm g}^{(2)}$ in the soft graviton expansion

$$M_{n+1} \to (S_{\rm g}^{(0)} + S_{\rm g}^{(1)} + S_{\rm g}^{(2)})M_n \tag{1}$$

of the $(n+1)$-graviton scattering amplitude and conjectured that $S_{\rm g}^{(1)}$ is also universal. (The form of these subleading terms was known before Ref. [5].) Taking q to be the momentum and $\epsilon_{\mu\nu}$ to be the polarization tensor of the soft particle, Weinberg's soft factor is

$$S_{\rm g}^{(0)} = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} k_a^\mu k_a^\nu}{q\cdot k_a}, \tag{2}$$

while the terms $S_{\rm g}^{(1)}$ and $S_{\rm g}^{(2)}$ are given by

$$S_{\rm g}^{(1)} = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} k_a^\mu (q_\lambda J_a^{\lambda\nu})}{q\cdot k_a}, S_{\rm g}^{(2)} = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} (q_\rho J_a^{\rho\mu})(q_\lambda J_a^{\lambda\nu})}{q\cdot k_a}. \tag{3}$$

Both subleading factors depend on the total angular momentum operator $J_a^{\mu\nu} = L_a^{\mu\nu} + S_a^{\mu\nu}$ of the ath particle. Here, $L_{\mu\nu} = k_{[\mu}(\partial/\partial k^{\nu]})$, and the spin factor $S_{\mu\nu}$ can be deduced from the condition $\epsilon_{\mu\nu}k^\mu = 0$. Thus,

$$S_{\mu\nu,{\rm g}} = \epsilon_{\sigma[\mu} \frac{\partial}{\partial \epsilon^{\nu]}{}_\sigma} + \epsilon_{\sigma[\mu} \frac{\partial}{\partial \epsilon_\sigma{}^{\nu]}}. \tag{4}$$

In Yang-Mills theory, a similar factor was described in Ref. [6] where the color-stripped amplitude has the soft limit

$$A_{n+1} \to (S_{\rm YM}^{(0)} + S_{\rm YM}^{(1)})A_n, \tag{5}$$

and the operators $S_{\rm YM}^{(0)}$ and $S_{\rm YM}^{(1)}$ are

$$S_{\rm YM}^{(0)} = \sum_{\substack{a\,\text{adjacent}\,q\\ \text{signed}}} \frac{\epsilon\cdot k_a}{q\cdot k_a}, \qquad S_{\rm YM}^{(1)} = \sum_{\substack{a\,\text{adjacent}\,q\\ \text{signed}}} \frac{\epsilon_\mu q_\nu J_a^{\mu\nu}}{q\cdot k_a}. \tag{6}$$

Here, the condition $\epsilon\cdot k = 0$ leads to the spin angular momentum operator $S_{\mu\nu,{\rm YM}} = \epsilon_{[\mu}(\partial/\partial\epsilon^{\nu]})$. Gauge invariance of the Yang-Mills factor is derived from the antisymmetry in the indices. In gravity, gauge invariance of $S^{(1)}$ follows from global conservation of angular momentum while for $S^{(2)}$ it is the antisymmetry of $J^{\mu\nu}$. Relations (1)—(3) were proved in Refs. [4,6] for tree-level amplitudes using Britto-Cachazo-Feng-Witten (BCFW) recursion relations [7] in the spinor-helicity formalism using a holomorphic limit as proposed in Ref. [8]. Apart from the BCFW recursion relations, these technologies are not available in all dimensions.

Cachazo, He, and Yuan (CHY) proposed a compact integral formula [9] for tree-level scattering amplitudes of scalar ϕ^3, (pure) Yang-Mills, and gravity theories in arbitrary dimension. The amplitudes are given by an integral over points on a sphere that satisfy a set of algebraic equations, called the scattering equations. This formula generalizes the twistor string connected prescription for $\mathcal{N}=4$ SYM theory [10] to scalar, gauge, and gravity theories in arbitrary dimension.

In this Letter we perform a next-to-leading order expansion of the CHY integral in the presence of a soft particle in Yang-Mills theory and in gravity. This expansion can be used to compute the subleading soft factor for tree-level scattering amplitudes of these theories formulated on d-dimensional space-time. We show that the subleading soft factors $S_{\rm g}^{(1)}$ and $S_{\rm YM}^{(1)}$ take the same, universal form n all dimensions. Given the momentum space form of both the subleading factors (3) and the CHY amplitude, this is not unexpected because neither explicitly references dimension. However, it is also very surprising since the original conjecture of the universal subleading soft factors was based on the BMS symmetry principle, which is only available in four dimensions [11].

It would be very interesting to see how the subleading factors get corrected at loop level (see Ref. [12] for loop corrections). Given the recent progress [13] in determining a stringy action principle for the derivation of the CHY form of scattering amplitudes, it might also be possible to determine the symmetry principle generating the subleading terms in dimensions other than four. We hope that further work will clarify these questions.

Review of the CHY formula.—The CHY formula for tree-level scattering amplitudes $M_n^{(\mathbf{s})}$ is

$$\int \frac{d^n\sigma}{\mathrm{vol}SL(2,\mathbb{C})} \prod_a{}' \delta_a \left(\frac{\mathrm{tr}(T^{a_1} \ldots T^{a_n})}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \right)^{2-\mathbf{s}} \mathrm{Pf}'(\Psi)^{\mathbf{s}}, \quad (7)$$

where the "power" $\mathbf{s}$ indicates whether the integral computes colored ϕ^3 theory ($\mathbf{s}=0$), (pure) Yang-Mills theory ($\mathbf{s}=1$), or gravity ($\mathbf{s}=2$) scattering amplitudes and $\sigma_{ij} = \sigma_i - \sigma_j$. In the following, we frequently suppress $\mathrm{vol}SL(2,\mathbb{C})$. The δ distributions

$$\delta_a = \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \right) \quad (8)$$

impose $n-3$ "scattering equations"

$$\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0; \quad (9)$$

hence, the primed product $\prod_a{}' = \sigma_{ij}\sigma_{jk}\sigma_{ki}\prod_{a\neq i,j,k}$ where i, j, k may be chosen freely. The integral is to be taken over a sphere with n punctures σ_i. Because of $SL(2,\mathbb{C})$ invariance, three of these σ_i can be set to fixed values, such that the integral is $n-3$ dimensional. It is entirely fixed by the solutions to the scattering equations. We will be interested in gravity ($\mathbf{s}=2$) and gauge theory ($\mathbf{s}=1$), as the soft limit of scalar amplitudes is identically zero already at leading order. In Yang-Mills theory, we will strip off the color factor $\mathrm{tr}(T^{a_1}\ldots T^{a_n})$ and work exclusively with color-ordered amplitudes.

The factor $\mathrm{Pf}'\Psi$ is the Pfaffian of the $2n \times 2n$-dimensional matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (10)$$

with

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases}, \qquad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ -\sum_{c\neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b \end{cases} \quad (11)$$

while B looks just like A with $k \to \epsilon$. Since the matrix Ψ is singular, its Pfaffian is identically zero $\mathrm{Pf}\,\Psi = 0$. The CHY integral, therefore, prescribes the use of the "reduced Pfaffian" (indicated by the prime)

$$\mathrm{Pf}'\Psi = 2\frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}\,\Psi_{ij}^{ij}, \quad (12)$$

where Ψ_{ij}^{ij} is Ψ with the ith and jth row and column removed.

In Ref. [9], it was shown that the integral has the correct behavior in the limit of a graviton or a photon momentum going soft to leading order. We compute the next order factor in this expansion.

To begin, expand $\prod_a \delta_a$ in the presence of a soft particle. We choose the nth site to be the soft particle, i.e., $k_n \to \varepsilon k_n$, $\varepsilon \ll 1$. There are $(n-1)-3$ scattering equations that contain only one k_n in the sum and one scattering equation which is proportional to k_n. As the variables in the δ distributions are complex and the integrand does not contain branch cuts, we may treat the δ distributions as poles. This allows us to rewrite the product $\prod_a{}'\delta_a$ as

$$\frac{1}{\varepsilon \sum_{b\neq n} \frac{k_n \cdot k_b}{\sigma_n - \sigma_b}} \prod_{a\neq n}{}' \frac{1}{\varepsilon \frac{k_a \cdot k_n}{\sigma_a - \sigma_n} + \sum_{b\neq a,n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}}. \quad (13)$$

The second factor may be exactly expanded in a sum

$$\prod_{a\neq n}{}' \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \left(\frac{k_a \cdot k_n}{\sigma_{an}} \right)^i \delta^{(i)} \left(\sum_{b\neq a,n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right). \quad (14)$$

Here, we denote $\delta^{(i)}(x) = [(-1)^i i!]/(x^{i+1})$. For the Pfaffian, we may employ the useful expansion

$$\mathrm{Pf}\,A = \sum_{\substack{q=1 \\ q\neq p}}^{2n} (-1)^q a_{pq} \mathrm{Pf}\,A_{pq}^{pq}. \quad (15)$$

The leading term in the expansion for $p = n$ is given by $q = 2n$ such that $\mathrm{Pf}'\,\Psi \to C_{nn}\mathrm{Pf}'\,\Psi_{n(2n)}^{n(2n)}$. Since we are also interested in the subleading terms, we need to look at the full expansion. Expanding along the row $p = n$, each coefficient apart from $q = 2n$ in the above expansion will be of order ε. More precisely, it is

$$\mathrm{Pf}'\,\Psi = -C_{nn}\mathrm{Pf}'\,\Psi_{n(2n)}^{n(2n)} + \varepsilon \sum_{\substack{q=1 \\ q\neq i,j,n}}^{2n-1} (-1)^q [\tilde{\Psi}_{ij}^{ij}]_{nq} \mathrm{Pf}'\Psi_{nq}^{nq} \quad (16)$$

with

$$[\tilde{\Psi}_{ij}^{ij}]_{nq} = \begin{cases} \frac{k_n \cdot k_q}{\sigma_{nq}} & q \leq n \\ \frac{k_n \cdot \epsilon_{q-n}}{\sigma_{n(q-n)}} & q > n \quad \text{and} \quad q \neq 2n. \end{cases} \quad (17)$$

The authors of Ref. [9] used the fact that $\mathrm{Pf}'\,\Psi_{n(2n)}^{n(2n)}$ is actually independent of k_n and ϵ_n and represents the correct factor for the $n-1$ particle amplitude *to leading order*. At subleading order, there are still some terms proportional to

k_n in $\mathrm{Pf}\,\Psi^{n(2n)}_{n(2n)}$. The other Pfaffian minors similarly contain k_n and ϵ_n. Therefore, we will also have to expand these other minors along the row $(2n)$; i.e., we use Eq. (15) twice

$$\mathrm{Pf}\,A = \sum_{1\le q<s\le 2n} (-1)^{q+s}(a_{pq}a_{rs} - a_{ps}a_{rq})\,\mathrm{Pf}\,(A^{pq}_{pq})^{rs}_{rs}. \quad (18)$$

This will *not* provide us with another factor of ε. We will return to the importance of this second recursion and the subleading terms in $\mathrm{Pf}\,\Psi^{n(2n)}_{n(2n)}$ in due course.

The interesting subleading soft factor in gauge theory and gravity is a single derivative operator [4,6]. It can, therefore, be extracted from the expansion of the δ distributions and the leading terms in the Pfaffian recursion alone. However, the spin part $S_{\mu\nu}$ of the angular momentum operator $J_{\mu\nu}$ will not show up in this calculation since the scattering equations are explicitly independent of the polarization vectors ϵ_a. The action of $S_{\mu\nu}$ can be seen in the Pfaffian expansion.

The numerators of the soft factors follow from the leading order of the Pfaffian expansion (16). The absence of the Pfaffian in the scalar case ($\mathbf{s} = 0$) leads to the vanishing of the amplitude in the case of a scalar soft particle. In the following section, we investigate how the subleading soft factor emerges for gluon scattering amplitudes in arbitrary dimensions. Then, a similar analysis is done for graviton scattering amplitudes in the following section.

Subleading soft factor in Yang-Mills theory.—The expansion in Eq. (14) is exact. The $i = 0$ case will produce the already known leading soft factor. For our purposes we need only the $i = 1$ term in the expansion of the δ distribution. Since this provides us with a factor ε, we will not have to worry about subleading terms in the Pfaffian recursion for now. We write

$$\varepsilon \sum_{r\neq i,j,k}\left(\frac{k_n\cdot k_r}{\sigma_{nr}}\right)\delta^{(1)}\left(\sum_{b\neq r,n}\frac{k_r\cdot k_b}{\sigma_{rb}}\right){\prod_{a\neq r,n}}'\delta_a. \quad (19)$$

Note first how the product now excludes the set $\{i,j,k,r,n\}$ of indices and second the presence of the derivative on one of the δ's. Using only the leading factor coming from the Pfaffian ($C_{nn}\mathrm{Pf}'\,\Psi^{n(2n)}_{n(2n)}$), we may extract the subleading contribution of Eq. (6) to the soft factor for color-ordered gluon scattering amplitudes from

$$\int d^n\sigma \frac{\sum_{b\neq n}\frac{\epsilon_n\cdot k_b}{\sigma_{nb}}}{\sum_{b\neq n}\frac{k_n\cdot k_b}{\sigma_{nb}}}\frac{\sigma_{n-1,1}}{\sigma_{n-1,n}\sigma_{n,1}}{\sum_r}'\frac{k_n\cdot k_r}{\sigma_{nr}}\delta^{(1)}_r{\prod_{a\neq r}}'\delta_a I_{n-1}. \quad (20)$$

I_{n-1} indicates that the rest of the integrand does not depend on σ_n. It is convenient to set $i = 1$ and $j = n-1$ in the primed product and sum for the following calculation of the residues. The integrand is regular for $\sigma_n \to \infty$ just as in the leading case. Thus, the integral over σ_n can be treated as in the leading case by deforming the contour. The only contributing poles are $\sigma_n = \sigma_1$, $\sigma_n = \sigma_{n-1}$, and also $\sigma_n = \sigma_r$ for every $r \neq 1, n-1, k$ (k arbitrary) in the sum. In each case, the first fraction reduces to $(\epsilon_n\cdot k_m)/(k_n\cdot k_m)$ for $m = 1, n-1, r$. The second fraction is only interesting in the case $\sigma_n = \sigma_r$ as it is otherwise equal to unity. In the interesting case, one rewrites

$$\frac{\sigma_{n-1,1}}{\sigma_{n-1,r}\sigma_{r,1}} = \frac{\sigma_{n-1,r}+\sigma_{r1}}{\sigma_{n-1,r}\sigma_{r,1}} \quad (21)$$

to get two terms. Putting this together, Eq. (20) reduces to a sum of four terms

$$\int d^{n-1}\sigma{\sum_{r\neq n}}'\left\{\frac{\epsilon_n\cdot k_1}{k_n\cdot k_1}\frac{k_n\cdot k_r}{\sigma_{1,r}} - \frac{\epsilon_n\cdot k_r}{k_n\cdot k_r}\frac{k_n\cdot k_r}{\sigma_{1,r}} + \frac{\epsilon_n\cdot k_r}{k_n\cdot k_r}\frac{k_n\cdot k_r}{\sigma_{n-1,r}} - \frac{\epsilon_n\cdot k_{n-1}}{k_n\cdot k_{n-1}}\frac{k_n\cdot k_r}{\sigma_{n-1,r}}\right\}\delta'_r{\prod_{a\neq r}}'\delta_a I_{n-1}. \quad (22)$$

Now finally, we inspect the four terms and notice that they can be recovered from an operator of the form

$$S^{(1)} = \frac{\epsilon_{n\mu}k_{n\nu}J_1^{\mu\nu}}{k_n\cdot k_1} - \frac{\epsilon_{n\mu}k_{n\nu}J_{n-1}^{\mu\nu}}{k_n\cdot k_{n-1}} \quad (23)$$

acting on the product of δ distributions. This operator takes the exactly the same form as the operator for four dimensions [6].

As indicated above, finding the subleading soft factor from the Pfaffian recursion relation (18) and the order ε terms in the leading piece of the Pfaffian is more complicated. However, it can be done in the same way as above: First we use the leading order ($i = 0$) in the expansion of the δ distributions. It is then necessary to write out the order ε terms from the Pfaffian. This contains the pieces from the leading order and also the pieces of order ε from Eq. (16). It is useful to use Eq. (18) with $p = n$ and $r = 2n$, which produces an expansion of the Pfaffian for $n-1$ particles. Investigating the integrand reveals that it is regular at infinity and there are no branch cuts, so a contour deformation is possible. After the integration the terms can be reassembled into the form of a sum of derivative operators D acting on the Pfaffian $D\,\mathrm{Pf}\,A = \mathrm{Pf}\,A\mathrm{tr}(A^{-1}DA)$ where A is the $n-1$ particle matrix Ψ. In fact, the terms can be recovered from the use of the operator (23) on the $n-1$ particle Pfaffian. At this point, we can also see the action of the spin part $S^{\mu\nu}$ of the angular momentum operator, since there are terms in the Pfaffian expansion that cannot be recovered from the action of the orbital angular momentum operator $L^{\mu\nu}$ alone. For a detailed calculation, please refer to the Supplemental Material [14].

Subleading soft factor in gravity.—A similar analysis can be done for graviton scattering amplitudes. We use the expansion of the δ distribution in Eq. (19). Since $\mathbf{s} = 2$, there are two Pfaffian contributions. Thus, there is a pole for every particle in the amplitude

$$\int d^n\sigma \frac{(\sum_{b\neq n}\frac{\epsilon_n\cdot k_b}{\sigma_{nb}})^2}{\sum_{b\neq n}\frac{\epsilon_n\cdot k_b}{\sigma_{nb}}}\sum\nolimits_r{}' \frac{k_n\cdot k_r}{\sigma_{nr}}\delta_r^{(1)}\prod\nolimits_{a\neq r}{}'\delta_a I_{n-1}. \quad (24)$$

Because of the coefficient $(k_n \cdot k_r)/\sigma_{nr}$ of the expansion of the δ distribution, there is no *simple* pole at $\sigma_n = \sigma_r$ for every r. Instead, there is a double pole at this point. We will return to this issue shortly. Upon the use of the residue theorem in the above prescribed manner, the simple poles yield a second sum $\sum_{b\neq r}$, such that

$$\int d^{n-1}\sigma \sum\nolimits_r{}' \sum_{b\neq r} \frac{(\epsilon_n\cdot k_b)^2}{k_n\cdot k_b}\frac{k_n\cdot k_r}{\sigma_{br}}\delta_r^{(1)}\prod\nolimits_{a\neq r}{}'\delta_a I_{n-1}. \quad (25)$$

We may also consider the case of two Pfaffian factors that depend on different polarization vectors ϵ_i and $\tilde{\epsilon}_i$. Then $(\epsilon_n\cdot k_r)^2 \to (\epsilon_n\cdot k_r)(\tilde{\epsilon}_n\cdot k_r)$ in the expressions above.

This is one part of the expression one would get from acting with an operator

$$S^{(1)} = \sum_{a=1}^{n-1}\frac{\epsilon_n\cdot k_a \epsilon_{n\mu}k_{n\lambda}J_a^{\lambda\mu}}{k_n\cdot k_a} \quad (26)$$

on the product of δ distributions. Where is the rest? Interestingly, it hides in the second-order poles. To find it, we have to make use of Cauchy's integral formula

$$f^{(1)}(z_0) = \frac{1}{2\pi i}\oint\frac{f(z)dz}{(z-z_0)^2} \quad (27)$$

on the second-order poles $\sigma_n = \sigma_r$ and act with a σ_n derivative on the residue before setting $\sigma_n = \sigma_r$. After simplifying the result of the integration, one finds

$$\int d^{n-1}\sigma \sum\nolimits_r{}' \sum_{b\neq r}\left(\frac{(\epsilon_n\cdot k_r)^2 k_n\cdot k_b}{k_n\cdot k_r\sigma_{rb}} - 2\frac{\epsilon_n\cdot k_r\epsilon_n\cdot k_b}{\sigma_{rb}}\right) \times \delta_r^{(1)}\prod\nolimits_{a\neq r}{}'\delta_a I_{n-1}. \quad (28)$$

Adding Eqs. (25) and (28) produces all the terms expected from the action of the operator of Eq. (26) acting on the group of δ distributions. As before, one can also look at the result with two Pfaffians depending on different polarization vectors ϵ and $\tilde{\epsilon}$. This amounts to setting $(\epsilon_n k_r)^2 \to (\epsilon_n k_r)(\tilde{\epsilon}_n k_r)$ and $2(\epsilon_n k_r)(\epsilon_n k_b) \to (\epsilon_n k_r)(\tilde{\epsilon}_n k_b) + \epsilon \leftrightarrow \tilde{\epsilon}$ in Eq. (28).

Note how the primed product prevents us from seeing the subleading factor in the case of the four-particle amplitude. This is in accord with the low-n examples of the action of the subleading factor as given in Ref. [4] and a nice check of the result.

We performed the analysis for the Pfaffian-squared term in the integral. The major complication of the calculation for the Pfaffian factor with respect to the Yang-Mills case is the presence of poles as well as double poles for every particle due to the interaction of the leading $C_{nn}\,\mathrm{Pf}'\,\Psi_{n-1}$ piece with the order ε terms from the second Pfaffian. However, the calculation uses the same technology as the Yang-Mills case.

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