Witnessing Genuine Multipartite Entanglement with Positive Maps

Marcus Huber[1,2] and Ritabrata Sengupta[3,4]
[1]ICFO-Institut de Ciencies Fotoniques, 08860 Castelldefels, Barcelona, Spain
[2]Universitat Autonoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
[3]Department of Mathematical Sciences, Indian Institute of Science Education and Research Mohali, Mohali 140306, India
[4]Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 7 S.J.S. Sansanwal Marg, New Delhi 110 016, India
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We derive a general framework that lifts any set of bipartite to multipartite entanglement witnesses and we show how positive maps can naturally be incorporated into this framework. We show that some previous approaches for multipartite entanglement detection are intimately connected to the witnesses derived from partial transposition and that such criteria can easily be outperformed in higher dimensions by nondecomposable maps. As an exemplary case we present a witness that is capable of detecting genuine multipartite entanglement in bound entangled states.

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Entanglement is a striking feature of quantum physics that lies at the very heart of many of its numerous applications [1]. Characterizing entanglement is a challenging task whose complexity scales very unfavorably with the size of the system [2–4]. In bipartite systems, a major breakthrough in entanglement detection came with the advent of simple operational criteria for detecting entanglement in mixed states [5]. One of the first of its kind was the famous Peres-Horodecki criterion, also known as positivity under partial transposition (PPT) criterion. It provides a method to tell with certainty, whether two qubit systems are entangled and also provides a criterion showing whether states cannot be distilled from multiple copies of such states into purer entanglement via local operations and classical communication (LOCC) [6]. Soon after, it was realized that one can exploit the theory of positive, yet not completely positive maps to obtain complementary and in many cases much stronger entanglement detection criteria [7]. In fact, if a state that remains positive after the application of all possible positive maps to one of its subsystems it is separable with respect to that partition [5].

When it comes to multipartite systems, the situation becomes a little more involved. The possible structure behind the infinitely many decompositions of multipartite quantum states constitutes an even harder challenge for the detection of entanglement. Famous instances exemplifying the complexity of the task are states that are entangled across every bipartite cut, yet no multipartite entanglement is necessary to describe them [2] and on the other end of the spectrum are states that are separable with respect to every bipartite cut, yet they are not completely separable [8]. Unfortunately, the paradigmatic tool for entanglement detection in bipartite systems, positive maps, is an inherently bipartite concept and applied to multipartite systems it can never reveal more than mere entanglement across bipartite cuts.

Thus, entanglement witnesses are the most commonly used tool to detect genuine multipartite entanglement in noisy multipartite quantum systems [3] and many attempts have been made to frame multipartite entanglement detection in a general framework [9–15]. In the bipartite case there is an intriguing connection between positive maps and entanglement witnesses, as the latter can be derived from the former. In this Letter, we introduce a general framework that allows us to construct witnesses for genuine multipartite entanglement directly incorporating positive maps. In fact, we even show how any nonpartial decomposability can be revealed in such a way and provide examples where our framework outperforms the best known witness constructions.

To get started let us precisely define the underlying concepts of separability, positive maps, and entanglement witnesses before we move on to our main theorem.

A state is considered to be partially separable with respect to bipartitions $b \in \mathcal{B}$ if and only if it can be written as

$$\rho_{\mathcal{B}} = \sum_{b\in\mathcal{B}} p_b \left(\sum_i q_b^i (|\phi_i\rangle\langle\phi_i|)_b \otimes (|\phi_i{}'\rangle\langle\phi_i{}'|)_{\bar{b}} \right). \quad (1)$$

This definition carries the operational meaning of which resources in terms of separability are required to create this state via LOCC. A special case are states that are biseparable; i.e., $\mathcal{B}$ is the set of all possible bipartitions $|\mathcal{B}| = 2^{n-1} - 1$. The complement of the set of biseparable states is usually referred to as *genuinely multipartite entangled* states as their creation via LOCC requires pure states that are not separable with respect to any partition. Because of the involved structure of the definition of biseparability (1) detecting genuine multipartite entanglement is a challenging task.

This is where entanglement witnesses prove useful. These are self-adjoint operators that have a positive expectation value for all states $\rho_{\mathcal{B}}$, while there is at least one state in the complement for which the expectation value is smaller than zero. The advantage of witnesses, while their detection capability is limited to a small volume of states, is of course the generically easy experimental access (especially in systems so large that a tomography is nearly impossible as, e.g., in Refs. [16,17]). If global measurements are available a single measurement is sufficient to reveal entanglement in a physical system and even for more realistic local measurements generic witnesses only require a small fraction of possible measurements [3]. Especially for revealing multipartite entanglement this is a very desirable property as a full state tomography scales very unfavorably in the number of systems involved.

Positive maps Λ (that are not completely positive), on the other hand, constitute a tool for entanglement detection that require access to the full density matrix and also the computation of eigenvalues of matrices that are exponentially large in the number of systems. There is, however, a straightforward connection that allows us to construct entanglement witnesses directly from positive maps, which we will elucidate after some preliminary definitions. For bipartite systems it is obvious that

$$(\Lambda_b \otimes \mathbb{1})_{\bar{b}}[\rho_b] = \sum_i q_i \Lambda[(|\phi_i\rangle\langle\phi_i|)_b] \otimes [(|\phi_i{}'\rangle\langle\phi_i{}'|)_{\bar{b}}] \geq 0, \tag{2}$$

such that any negative eigenvalue after application of the positive map to the subsystem immediately reveals entanglement across this bipartition into the subsystem and its complement. While this can never reveal partial separability properties in the general sense of Eq. (1), positive maps, such as those in Refs. [18–22], have proven to provide strong tools in the bipartite case [5]. There is a straightforward framework for constructing bipartite entanglement witnesses from positive maps: If the aim is to detect a given entangled target state σ and there exists a positive map Λ, such that $\Lambda \otimes \mathbb{1}[\sigma]$ has at least one negative eigenvalue with corresponding eigenvector $|n\rangle$, then

$$W(\Lambda) = (\Lambda^* \otimes \mathbb{1})[|n\rangle\langle n|], \tag{3}$$

where Λ^* is the dual of the positive map Λ, constitutes an entanglement witness that will detect the state σ to be entangled. Such a procedure is of course very helpful in experimental entanglement verification if one has a reasonable guess what the state of the system under investigation should be. Then one can apply this procedure and end up with an experimentally feasible witness operator that should be able to reveal entanglement in the system. Unfortunately this procedure only works for the bipartite case as the application of a map on a system necessarily implies a bipartition. Now we continue with the main result of our Letter, where we present a framework that enables such a construction also for partial separability and thus for genuine multipartite entanglement. We start directly with the main theorem.

Theorem: For any set of bipartite entanglement witnesses W_b across bipartitions $b|\bar{b} \in \mathcal{B}$, the following expression is always positive for mixed states ρ, which can be decomposed into pure states that are separable with respect to any of the partitions in $\mathcal{B}$:

$$\mathrm{Tr}\left[\rho\left(\sum_{b\in\mathcal{B}} \tau_b + Q\right)\right] \geq 0, \tag{4}$$

where we have used the abbreviated notation $Q = N + P$, $P = \sum_{\eta,\eta'} |\eta\rangle\langle\eta'| \max[0, \min_{b\in\mathcal{B}}[\Re e[W_b]]$, $N = \sum_{\eta,\eta'} |\eta\rangle\langle\eta'| \min[0, \max_{b\in\mathcal{B}}[\Re e[W_b]]$, and $\tau_b = [W_b - Q]_+$ (with $[A]_+$ we denote the non-negative part of the spectrum of A, i.e., we project onto the eigenspace spanned by eigenvectors belonging to positive eigenvalues).

Proof: The first observation required is

$$\mathrm{Tr}[\rho[A]_+] \geq \mathrm{Tr}[\rho A], \tag{5}$$

and thus

$$\mathrm{Tr}[\rho(\tau_b + Q)] \geq \mathrm{Tr}[\rho W_b]. \tag{6}$$

Now if we write down a state that is decomposable into states ρ_b, separable with respect to bipartition in $\mathcal{B}$ as

$$\rho_{\mathcal{B}} = \sum_b p_b \rho_b, \tag{7}$$

we find that

$$\begin{aligned} Tr&\left[\rho\left(\sum_{b\in\mathcal{B}} \tau_b + Q\right)\right] \\ &= \sum_{b\in\mathcal{B}} p_b \mathrm{Tr}[\rho_b(\tau_b + Q)] + \sum_{\{b' \neq b\}\in\mathcal{B}} p_b{}' \mathrm{Tr}[\rho_{b'}(\tau_b)] \geq 0, \end{aligned} \tag{8}$$

which completes the proof. The idea behind the theorem is straightforward: If one has a set of witnesses for detecting entanglement across different partitions that have some overlapping matrix elements collected in Q, one can separate every witness $W_b = Q + M_b$ and by making sure that M_b is positive semidefinite (as we did in our theorem by using only the positive part of the spectrum) it immediately follows that $W_{\mathrm{GME}} = Q + \sum_b M_b$ is a witness for genuine multipartite entanglement.

While this theorem lifts any set of bipartite witnesses to operators with positive expectation values on partially separable states, we need to find a negative eigenvalue of $\sum_{b\in\mathcal{B}}\tau_b + Q$ in order to guarantee that the multipartite witness is nontrivial. Here we can use the connection between positive maps and entanglement witnesses. If we use $W_b(\Lambda_b)$ as witnesses for entanglement across bipartitions we can facilitate the search for indecomposable witnesses. More importantly this opens the possibility for systematically choosing suitable witness states $|\psi_b\rangle\langle\psi_b|$ that maximize the overlap (i.e., the norm of Q) in a natural way. The problem can be formalized as follows.

Given a multipartite state ρ that is detected to be entangled across every bipartition by means of positive maps Λ_b, for each bipartition this implies that we will find witnesses of the form

$$W_b(\Lambda_b) = (\Lambda_b^* \otimes \mathbb{1}_{\bar{b}})[|\psi_b\rangle\langle\psi_b|]. \tag{9}$$

While in the bipartite case the choice of $|\psi_b\rangle\langle\psi_b|$ is obviously given via the eigenvector corresponding to the smallest (i.e., negative) eigenvalue of $\Lambda[\rho]$, this choice is not as obvious in the multipartite case. Indeed if we combine these observations with our main theorem we end up with the central witness used in all subsequent examples:

$$W_{\neg\mathcal{B}}(\{\Lambda_b, |\psi_b\rangle\}) = \sum_{b\in\mathcal{B}}[(\Lambda_b^* \otimes \mathbb{1}_{\bar{b}})[|\psi_b\rangle\langle\psi_b|] - Q]_+ + Q, \tag{10}$$

whose eigenvalues we want to minimize.

While in general this may be a hard task to achieve optimally, we can present some systematic approaches that work surprisingly well.

Starting with the most famous and widely used example of positive maps, the transposition, we illustrate the method in the three qubit case.

Example 1: If we want to detect genuine multipartite entanglement in a three qubit GHZ state $|\mathrm{GHZ}\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$, through a witness derived from the PPT criterion, the choice of $|\psi_b\rangle$ is quite straightforward. If we choose

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|011\rangle - |100\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|101\rangle - |010\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|110\rangle - |001\rangle), \tag{11}$$

we end up with a witness operator

$$W_{\neg\{1|23,2|13,3|12\}} = \frac{1}{2}\mathbb{1} - |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|, \tag{12}$$

which is well known [3] and even necessary and sufficient for detecting multipartite entanglement in GHZ-diagonal states [23]. This means that for a large class of mixed states (convex combinations of GHZ states) this simple witness construction is indeed necessary and sufficient. Indeed, Refs. [9–11] introduce a framework for multipartite entanglement detection, whose linearized version is exactly corresponds to our main theorem here. Using this framework, one can always find the corresponding $|\psi_b\rangle$ for detecting multipartite entanglement using transposition. However, as we show, this framework (and the multipartite entanglement witnesses derived from the PPT criterion) can easily be outperformed by a simpler choice of maps. For instance, Choi's map [18] and its generalizations [20,21] have the advantage that the negative off-diagonal elements in $(\Lambda_b^* \otimes \mathbb{1}_{\bar{b}})[|\psi_b\rangle\langle\psi_b|]$ generically correspond to the off-diagonal elements of $|\psi_b\rangle\langle\psi_b|$. This facilitates the search immensely as for the theorem to maximize detection strength we require the off-diagonal elements of $(\Lambda_b^* \otimes \mathbb{1}_{\bar{b}})[|\psi_b\rangle\langle\psi_b|]$ to be as similar as possible. For such maps where this is naturally the case it is sufficient for detecting a target state $|\Psi_t\rangle$ to choose $|\psi_b\rangle = |\Psi_t\rangle \forall b$ (or if one is lucky and finds an eigenvector of $|\psi_b\rangle = |\Psi_t\rangle \forall b$ in all partitions with a negative sign and high modulus, then this is, of course, the obvious choice). We will now present two simple examples (the last being similar in spirit to one example in Ref. [24]) where this advantage becomes immediately evident.

Example 2: In a similar fashion, one can construct witnesses for exemplary states that are extremal in terms of local ranks. While the GHZ is a prime example of having all local ranks equal to 2, we can also directly apply our method to higher dimensional generalizations that involve an arbitrary distribution of local ranks using the construction given in Ref. [25]. For example, for the class (4,4,3), i.e., system one and two of rank four and the third system only of rank three, we can directly write down an example as $|\psi_{(4,4,3)}\rangle = \frac{1}{2}(|001\rangle + |112\rangle + |223\rangle + |333\rangle)$. Now we can use the dual of Choi's map

$$[(a_{ij})] \mapsto \frac{1}{2}\begin{pmatrix} a_{11}+a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{22} \end{pmatrix}, \tag{13}$$

with $|\psi_b\rangle = |\psi_{(4,4,3)}\rangle$ for all partitions. Even with this rather simple choice one can detect this state to be genuinely multipartite entangled without any need for optimizing over $|\psi_b\rangle$.

Example 3: Adopting the following short hand notation for GHZ-like states:

$$|\alpha, x, y, \lambda_\alpha\rangle = \prod_{i\in\alpha}\sigma_i \otimes \mathbb{1}_{\bar{i}}\left(\sqrt{\lambda_\alpha}|x\rangle^{\otimes n} + \sqrt{\lambda_\alpha^{-1}}|y\rangle^{\otimes n}\right), \tag{14}$$

with $\sigma = |x\rangle\langle y| + |y\rangle\langle x|$. And the corresponding projector we denote as $P_\alpha(x, y, \lambda_\alpha)$, such that we can introduce the operator

$$E(\{\lambda_\alpha\}) = 3|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + \sum_{i=1,2,3}\sum_{x<y}\sum_{y=1,2} P_i(x,y,\lambda_i), \quad (15)$$

where $|\mathrm{GHZ}_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ and with this, finally, the density operator

$$\rho(\{\lambda_\alpha\}) = \frac{E(\{\lambda_\alpha\})}{\mathrm{Tr}[E(\{\lambda_\alpha\})]}. \quad (16)$$

It is immediately evident that this density matrix is invariant under partial transposition (since the off-diagonal elements of $|\mathrm{GHZ}_3\rangle$ after partial transposition in system b are the same as $|b,x,y,\lambda_b\rangle$), so the PPT criterion is not even able to reveal bipartite entanglement in this system. However, using Choi's map for $d=3$ one can easily check that this state is indeed PPT entangled (i.e., definitely bound entangled) across every bipartition for values of $0<\lambda_\alpha<1$. An immediate implication is the fact that if the state is multipartite entangled it cannot be detected by our theorem using the PPT and also not from the techniques developed in Refs. [9–11,13–15,23]. Using the very simple and straightforward choice $|\psi_1\rangle = |\psi_2\rangle = |\psi_3\rangle = |\mathrm{GHZ}_3\rangle$, $\lambda_\alpha = \lambda\forall\alpha$, and again Choi's map we can directly apply our method to check partial separability properties. We find that the witness is violated for all values of λ between 0 and $\frac{1}{3}$ and thus this bound entangled state is indeed genuinely multipartite entangled. Other examples were found for symmetric states in Refs. [26–29]) and in Ref. [30] the authors construct a different framework that allows for the construction of PPT-GME states; however, to our knowledge that is the first explicit example in a $3\otimes3\otimes3$ system and thus the smallest example of a PPT-GME state so far. The violation of this witness is even so significant that it exhibits a notable noise robustness with respect to white noise.

If we mix the state, e.g., for a choice of $\lambda=\frac{1}{9}$, with the maximally mixed state, i.e.,

$$\rho_{\mathrm{noise}}(p) = p\frac{\mathbb{1}}{27} + (1-p)\rho\left(\frac{1}{9}\right), \quad (17)$$

we find that the white noise resistance, i.e., the critical value of white noise admixture p, until which genuine multipartite entanglement can still be detected is $p_{\mathrm{crit}} = \frac{9}{179} \approx 5\%$. We have just shown that Choi's map provided an advantage for specific states, while, in general, different maps cannot be considered superior in terms of their entanglement detection strength (even with the optimal choice of Q for a given class of states). To illustrate this point let us consider the following two-parameter family of states:

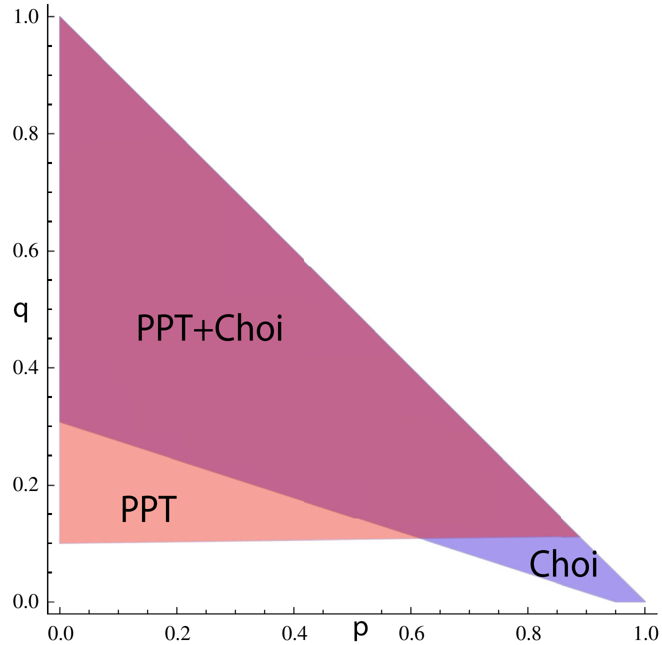

FIG. 1 (color online). Here we illustrate the detection power of two different maps for the state, Eq. (18). The regions are labeled according to which criterion detects the state for these values of p and q to be genuinely multipartite entangled (PPT refers to partial transposition and Choi to Choi's map).

$$\rho_{\mathrm{example}} = p|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + q\rho\left(\frac{1}{9}\right) + \frac{1-p-q}{27}\mathbb{1}, \quad (18)$$

and use our theorem to construct the witnesses in the same fashion as in the first two examples. The results are illustrated in Fig. 1 and showcase the detection power of the witnesses derived from our main theorem and straightforward choices of $|\psi_b\rangle$ without any optimization involved.

In conclusion we presented a framework that lifts any set of bipartite witnesses to multipartite ones. It directly connects positive maps with witnesses for partial separability. The construction is simple, operational, and experimentally friendly. We illustrated the power of the criterion by presenting the first example of a $3\otimes3\otimes3$ bound entangled state that is, at the same time, genuinely multipartite entangled. We expect that this method should find applications in all tasks that aim at characterizing multipartite entanglement. In the future it will be of interest to study the connection to semidefinite characterizations of supersets of partially separable states (as with the "PPT mixers" from Refs. [31,32]). One intriguing question that is left open is the general strength of such criteria—while in the bipartite case it is obvious that such a witness construction is capable of detecting all entangled states, the case is not so clear in the multipartite case. Can we construct multipartite entangled states that, in principle, cannot be detected by our framework?

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