Model Independent Determination of the Gluon Condensate in Four Dimensional SU(3) Gauge Theory

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We determine the nonperturbative gluon condensate of four-dimensional SU(3) gauge theory in a model-independent way. This is achieved by carefully subtracting high-order perturbation theory results from nonperturbative lattice QCD determinations of the average plaquette. No indications of dimension-two condensates are found. The value of the gluon condensate turns out to be of a similar size as the intrinsic ambiguity inherent to its definition. We also determine the binding energy of a B meson in the heavy quark mass limit.

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The operator product expansion (OPE) [1] is a fundamental tool for theoretical analyses in quantum field theories. Its validity is only proven rigorously within perturbation theory, to arbitrary finite orders [2]. The use of the OPE in a nonperturbative framework was initiated by the ITEP group [3] (see also the discussion in Ref. [4]), which postulated that the OPE of a correlator could be approximated by the following series:

$$\mathrm{correlator}(Q) \simeq \sum_d \frac{1}{Q^d} C_d(\alpha)\langle O_d\rangle, \tag{1}$$

where the expectation values of local operators O_d are suppressed by inverse powers of a large external momentum $Q \gg \Lambda_{\mathrm{QCD}}$, according to their dimensionality d. The Wilson coefficients $C_d(\alpha)$ encode the physics at momentum scales larger than Q. These are well approximated by perturbative expansions in the strong coupling parameter α. The large-distance physics is described by the matrix elements $\langle O_d\rangle$ that usually have to be determined nonperturbatively.

Almost all QCD predictions of relevance to particle physics phenomenology are based on factorizations that are generalizations of the above generic OPE.

For correlators where $O_0 = \mathbb{1}$, the first term of the OPE expansion is a perturbative series in α. In pure gluodynamics, the first nontrivial gauge-invariant local operator has dimension four. Its expectation value is the so-called nonperturbative gluon condensate

$$\begin{aligned}\langle O_{\mathrm{G}}\rangle &= -\frac{2}{\beta_0}\left\langle \Omega\left|\frac{\beta(\alpha)}{\alpha}G^a_{\mu\nu}G^a_{\mu\nu}\right|\Omega\right\rangle \\ &= \left\langle \Omega\left|[1+\mathcal{O}(\alpha)]\frac{\alpha}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}\right|\Omega\right\rangle. \end{aligned}\tag{2}$$

This condensate plays a fundamental role in phenomenology, in particular in sum rule analyses, as for many observables it is the first nonperturbative OPE correction to the purely perturbative result. In this Letter, we will compute (and define) this object. For this purpose we use the expectation value of the plaquette calculated in Monte Carlo (MC) simulations in lattice regularization with the standard Wilson gauge action [5]

$$\langle P\rangle_{\mathrm{MC}} = \frac{1}{N^4}\sum_{x\in\Lambda_E}\langle P_x\rangle, \tag{3}$$

where Λ_E is a Euclidean spacetime lattice and

$$P_{x,\mu\nu} = 1 - \frac{1}{6}\mathrm{Tr}(U_{x,\mu\nu} + U^\dagger_{x,\mu\nu}). \tag{4}$$

For details on the notation see Ref. [6]. The corresponding OPE reads

$$\langle P\rangle_{\mathrm{MC}} = \sum_{n=0}^{\infty} p_n\alpha^{n+1} + \frac{\pi^2}{36}C_G(\alpha)a^4\langle O_G\rangle + \mathcal{O}(a^6), \tag{5}$$

where a denotes the lattice spacing.

The perturbative series is divergent due to renormalons [7] and other, subleading, instabilities. This makes any determination of $\langle O_G\rangle$ ambiguous, unless we define how to truncate or how to approximate the perturbative series. A reasonable definition that is consistent with $\langle O_G\rangle \sim \Lambda^4_{\mathrm{QCD}}$ can only be given if the asymptotic behavior of the perturbative series is under control. This has only been achieved recently [6], where the perturbative expansion of the plaquette was computed up to $\mathcal{O}(\alpha^{35})$. The observed asymptotic behavior was in full compliance with renormalon expectations, with successive contributions starting to diverge for orders around α^{27}–α^{30}, within the range of couplings α typically employed in present-day lattice simulations.

Extracting the gluon condensate from the average plaquette was pioneered in Refs. [8–11], and many attempts followed during the next decades; see, e.g., Refs. [12–21]. These suffered from insufficiently high perturbative orders and, in some cases, also finite volume effects. The failure to make contact to the asymptotic regime prevented a reliable lattice determination of $\langle O_G \rangle$. We solve this problem in this Letter.

Truncating the infinite sum at the order of the minimal contribution provides one definition of the perturbative series. Varying the truncation order will result in changes of size $\Lambda_{\rm QCD}^4 a^4$, where the dimension $d = 4$ is determined by that of the gluon condensate. We approximate the asymptotic series by the truncated sum

$$S_P(\alpha) \equiv S_{n_0}(\alpha), \quad \text{where } S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1}. \qquad (6)$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0}\alpha^{n_0+1}$ is minimal. We then obtain the gluon condensate from the relation

$$\langle O_G \rangle = \frac{36 C_G^{-1}(\alpha)}{\pi^2 a^4(\alpha)} [\langle P \rangle_{\rm MC}(\alpha) - S_P(\alpha)] + \mathcal{O}(a^2 \Lambda_{\rm QCD}^2). \qquad (7)$$

For the plaquette, the inverse Wilson coefficient

$$C_G^{-1}(\alpha) = -\frac{2\pi\beta(\alpha)}{\beta_0\alpha^2} = 1 + \frac{\beta_1}{\beta_0}\frac{\alpha}{4\pi} + \frac{\beta_2}{\beta_0}\left(\frac{\alpha}{4\pi}\right)^2 + \frac{\beta_3}{\beta_0}\left(\frac{\alpha}{4\pi}\right)^3 + \mathcal{O}(\alpha^4) \qquad (8)$$

is proportional to the β function [22,23]. For $j \le 3$, the coefficients β_j are known in the lattice scheme [see Eq. (25) of Ref. [6]]. The corrections to $C_G = 1$ are small. However, the $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^3)$ terms are of similar sizes. We will account for this uncertainty in our error budget.

Integrating the β-function results in the following dependence of the lattice spacing a on the coupling α:

$$a = \frac{1}{\Lambda_{\rm latt}} \exp\left[-\frac{1}{t} - b \ln\frac{t}{2} + s_1 bt - s_2 b^2 t^2 + \cdots\right], \qquad (9)$$

where $t = \alpha\beta_0/(2\pi)$, $b = \beta_1/(2\beta_0^2)$, $s_1 = (\beta_1^2 - \beta_0\beta_2)/(4b\beta_0^4)$, and $s_2 = (\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3)/(16b^2\beta_0^6)$. Equation (9) is not accurate in the lattice scheme for typical β values [$\beta \equiv 3/(2\pi\alpha)$] used in present-day simulations. Instead, we employ the phenomenological parametrization of [24] ($x = \beta - 6$)

$$a = r_0 \exp(-1.6804 - 1.7331x + 0.7849x^2 - 0.4428x^3), \qquad (10)$$

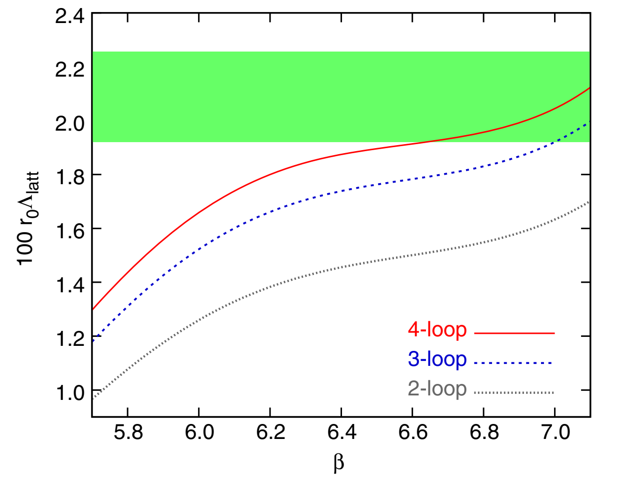

FIG. 1 (color online). Equation (10) over Eq. (9), truncated at different orders. The green band corresponds to $r_0\Lambda_{\rm latt} = 0.0209(17)$ [25].

obtained by interpolating nonperturbative lattice simulation results. Equation (10) was reported to be valid within an accuracy varying from 0.5% up to 1% in the range [24] $5.7 \le \beta \le 6.92$. We plot the ratio of the above two equations $r_0\Lambda_{\rm latt}$ in Fig. 1, where we truncate Eq. (9) at different orders. The green error band corresponds to [25] $r_0 = 0.0209(17)/\Lambda_{\rm latt} \simeq 0.5$ fm ($\Lambda_{\overline{\rm MS}} \approx 28.809\Lambda_{\rm latt}$). For large β values, this ratio should approach a constant. Up to $\beta \approx 6.7$, this appears to be the case; however, for $\beta > 6.7$ the slope of the ratio starts to increase. This may indicate violations of Eq. (10) for $\beta > 6.7$. Therefore, we will restrict ourselves to the range $\beta \in [5.8, 6.65]$, where $a(\beta)$ is given by Eq. (10). This corresponds to $(a/r_0)^4 \in [3.1 \times 10^{-5}, 5.5 \times 10^{-3}]$, covering more than 2 orders of magnitude.

Following Eq. (7), we subtract the truncated sum $S_P(\alpha)$ calculated from the coefficients p_n of Ref. [6] from the MC data on $\langle P \rangle_{\rm MC}(\alpha)$ of Ref. [26]. Multiplying this difference by $36r_0^4/(\pi^2 C_G a^4)$, where r_0/a is given by Eq. (10), gives $r_0^4\langle O_G \rangle$ plus higher-order nonperturbative terms. We show this combination in Fig. 2. The smaller error bars represent the errors of the MC data, and the outer error bars (not plotted for $N = 16$) represent the total uncertainty, including that of S_P. This part of the error is correlated between different β values. The MC data were obtained on volumes $N^4 = 16^4$ and $N^4 = 32^4$. Towards large β values, the physical volumes $N^4 a^4(\beta)$ will become small, resulting in transitions into the deconfined phase. For $\beta < 6.3$, we find no significant differences between the $N = 16$ and $N = 32$ results. In the analysis, we restrict ourselves to the more precise $N = 32$ data and, to keep finite size effects under control, to $\beta \le 6.65$. We also limit ourselves to $\beta \ge 5.8$ to avoid large $\mathcal{O}(a^2)$ corrections. At very large β values, not only does the parametrization of Eq. (10) break down but obtaining meaningful results becomes numerically challenging: the individual errors both of $\langle P \rangle_{\rm MC}(\alpha)$

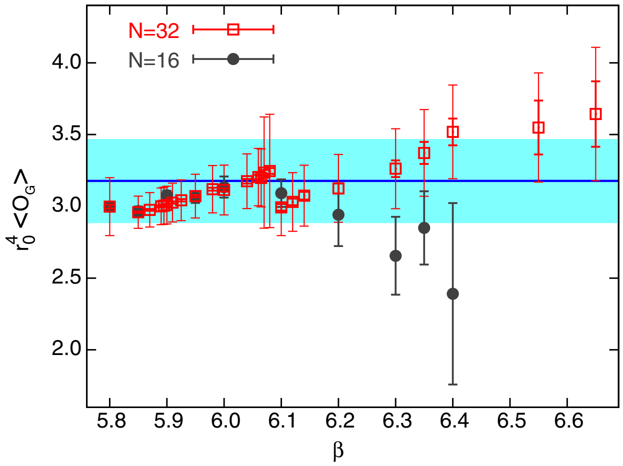

FIG. 2 (color online). Equation (7) evaluated using the $N = 16$ and $N = 32$ MC data of Ref. [26]. The $N = 32$ outer error bars include the error of $S_P(\alpha)$. The error band is our prediction for $\langle O_G\rangle$, Eq. (11).

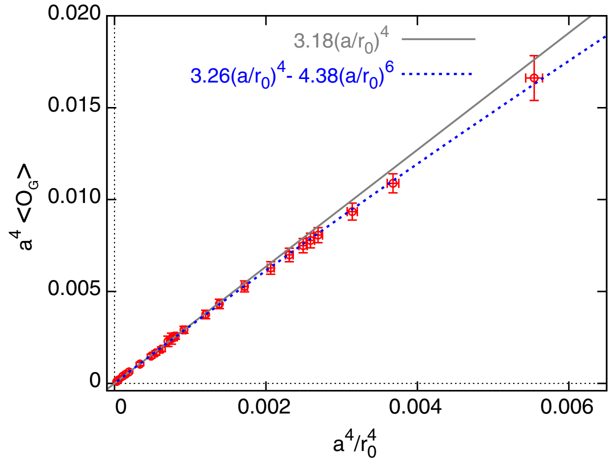

FIG. 3 (color online). Equation (7) times a^4 vs $a^4(\alpha)/r_0^4$ from Eq. (10). The linear fit with slope Eq. (11) is to $a^4 < 0.0013r_0^4$ points only.

and of $S_P(\alpha)$ somewhat decrease with increasing β. However, there are strong cancellations between these two terms, in particular at large β values, since this difference decreases with $a^{-4} \sim \Lambda_{\rm latt}^4 \exp(16\pi^2\beta/33)$ on dimensional grounds while $\langle P\rangle_{\rm MC}$ depends only logarithmically on a.

The coefficients p_n of $S_P(\alpha)$ were obtained in Ref. [6]. The p_n values carry statistical errors, and successive orders are correlated. With the use of the covariance matrix, also obtained in Ref. [6], the statistical error of $S_P(\alpha)$ can be calculated. In that reference, coefficients $p_n(N)$ were first computed on finite volumes of N^4 sites and subsequently extrapolated to their infinite volume limits p_n. This extrapolation is subject to parametric uncertainties that need to be estimated. We follow Ref. [6] and add the differences between determinations using $N \geq \nu$ points for $\nu = 9$ (the central values) and $\nu = 7$ as systematic errors to our statistical errors.

The data in Fig. 2 show an approximately constant behavior. (Note that n_0 increases from 26 to 27 at $\beta = 5.85$, from 27 to 28 at $\beta = 6.1$, and from 28 to 29 at $\beta = 6.55$. This quantization of n_0 explains the visible jump at $\beta = 6.1$.) This indicates that, after subtracting $S_P(\alpha)$ from the corresponding MC values $\langle P\rangle_{\rm MC}(\alpha)$, the remainder scales like a^4. This can be seen more explicitly in Fig. 3, where we plot this difference in lattice units against a^4. The result is consistent with a linear behavior, but a small curvature seems to be present that can be parametrized as an a^6 correction. The rightmost point ($\beta = 5.8$) corresponds to $a^{-1} \simeq 1.45$ GeV while $\beta = 6.65$ corresponds to $a^{-1} \simeq 5.3$ GeV. Note that a^2 terms are clearly ruled out.

We now determine the gluon condensate. We obtain the central value and its statistical error $\langle O_G\rangle = 3.177(36)r_0^{-1}$ from averaging the $N = 32$ data for $6.0 \leq \beta \leq 6.65$. We now estimate the systematic uncertainties. Different infinite volume extrapolations of the $p_n(N)$ data [6] result in changes of the prediction of about 6%. Another 6% error is due to including an a^6 term or not and varying the fit range. Next there is a scale error of about 2.5%, translating a^4 into units of r_0. The uncertainty of the perturbatively determined Wilson coefficient C_G is of a similar size. This is estimated as the difference between evaluating Eq. (8) to $\mathcal{O}(\alpha^2)$ and to $\mathcal{O}(\alpha^3)$. Adding all these sources of uncertainty in quadrature and using [25] $\Lambda_{\overline{\rm MS}} = 0.602(48)r_0^{-1}$ yields

$$\langle O_G\rangle = 3.18(29)r_0^{-4} = 24.2(8.0)\Lambda_{\overline{\rm MS}}^4. \qquad (11)$$

The gluon condensate of Eq. (2) is independent of the renormalization scale. However, $\langle O_G\rangle$ was obtained employing one particular prescription in terms of the observable and our choice of how to truncate the perturbative series within a given renormalization scheme. Different (reasonable) prescriptions can, in principle, give different results. One may, for instance, choose to truncate the sum at orders $n_0(\alpha) \pm \sqrt{n_0(\alpha)}$, and the result would still scale like $\Lambda_{\rm QCD}^4$. We estimated this intrinsic ambiguity of the definition of the gluon condensate in Ref. [6] as $\delta\langle O_G\rangle = 36/(\pi^2 C_G a^4)\sqrt{n_0}p_{n_0}\alpha^{n_0+1}$, i.e., as $\sqrt{n_0(\alpha)}$ times the contribution of the minimal term,

$$\delta\langle O_G\rangle = 27(11)\Lambda_{\overline{\rm MS}}^4. \qquad (12)$$

Up to $1/n_0$ corrections, this definition is scheme and scale independent and corresponds to the (ambiguous) imaginary part of the Borel integral times $\sqrt{2/\pi}$.

In QCD with sea quarks the OPE of the average plaquette or of the Adler function will receive additional contributions from the chiral condensate. For instance, $\langle O_G\rangle$ needs to be redefined, adding terms $\propto \langle\gamma_m(\alpha)m\bar{\psi}\psi\rangle$ [27]. Because of this and the problem of setting a physical

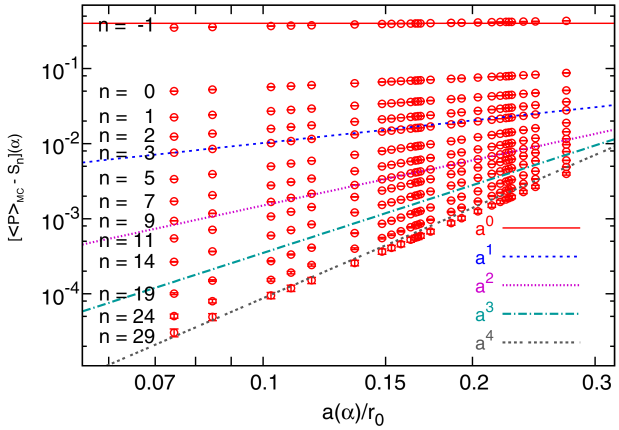

FIG. 4 (color online). Differences $\langle P\rangle_{\mathrm{MC}}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs $a(\alpha)/r_0$. The lines $\propto a^j$ are drawn to guide the eye.

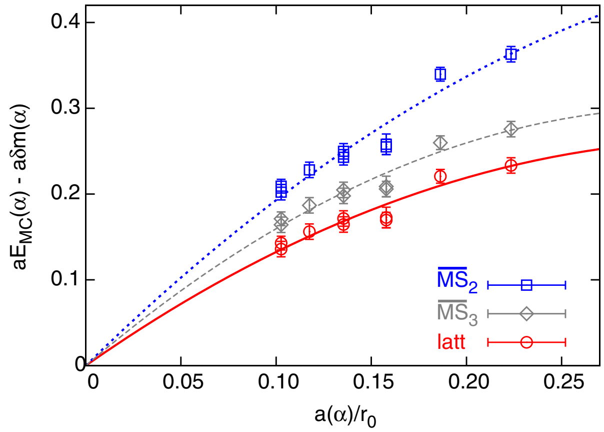

FIG. 5 (color online). $aE_{\mathrm{MC}} - a\delta m$ vs a/r_0. The expansion of $a\delta m$ was also converted into the $\overline{\mathrm{MS}}$ scheme at two ($\overline{\mathrm{MS}}_2$) and three ($\overline{\mathrm{MS}}_3$) loops. The curves are fits to $\bar{\Lambda}a + ca^2$.

scale in pure gluodynamics, it is difficult to assess the precise numerical impact of including sea quarks onto our estimates

$$\langle O_G\rangle \simeq 0.077\ \mathrm{GeV}^4, \qquad \delta\langle O_G\rangle \simeq 0.087\ \mathrm{GeV}^4, \tag{13}$$

which we obtain using $r_0 \simeq 0.5$ fm [28]. While the systematics of applying Eqs. (11)–(12) to full QCD are unknown, our main observations should still extend to this case. We remark that our prediction of the gluon condensate of Eq. (13) is significantly bigger than values obtained in one- and two-loop sum rule analyses, ranging from 0.01 GeV^4 [3,29] up to 0.02 GeV^4 [30,31]. However, these numbers were not extracted in the asymptotic regime, which for a $d = 4$ renormalon we expect to set in at orders $n \gtrsim 7$ for the $\overline{\mathrm{MS}}$ scheme. Moreover, we remark that in schemes without a hard ultraviolet cutoff, such as dimensional regularization, the extraction of $\langle O_G\rangle$ can become obscured by the possibility of ultraviolet renormalons. Independent of these considerations, all these values are smaller than the intrinsic prescription dependence of Eq. (12).

Our analysis confirms the validity of the OPE beyond perturbation theory for the case of the plaquette. Our a^4 scaling clearly disfavors suggestions about the existence of dimension-two condensates beyond the standard OPE framework [16,32–35]. In fact, we can also explain why an a^2 contribution to the plaquette was found in Ref. [16]. In the log-log plot of Fig. 4, we subtract sums S_n, truncated at different fixed orders α^{n+1}, from $\langle P\rangle_{\mathrm{MC}}$. The scaling continuously turns from $\sim a^0$ at $\mathcal{O}(\alpha^0)$ to $\sim a^4$ around $\mathcal{O}(\alpha^{30})$. Note that truncating at an α-independent fixed order is inconsistent, explaining why we never exactly obtain an a^4 slope. For $n \sim 9$, we reproduce the a^2 scaling reported in Ref. [16] for a fixed order truncation at $n = 7$. In view of Fig. 4, we conclude that the observation of this scaling power was accidental.

The methods used in this Letter can be applied to other observables. As an example, we analyze the binding energy $\bar{\Lambda} = E_{\mathrm{MC}}(\alpha) - \delta m(\alpha)$ [36–38] of heavy quark effective theory. The perturbative expansion of $a\delta m(\alpha) = \sum_n c_n\alpha^{n+1}$ was obtained in Refs. [39,40] up to $\mathcal{O}(\alpha^{20})$, and its intrinsic ambiguity $\delta\bar{\Lambda} = \sqrt{n_0}c_{n_0}\alpha^{n_0+1} = 0.748(42)\Lambda_{\overline{\mathrm{MS}}} = 0.450(44)r_0^{-1}$ was obtained in Refs. [40,41]. MC data for the ground-state energy E_{MC} of a static-light meson with the Wilson gauge action can be found in Refs. [42–44]. While for the gluon condensate we expected an a^4 scaling (see Fig. 3), for $aE_{\mathrm{MC}}(\alpha) - a\delta m(\alpha)$ we expect a scaling linear in a. Comforting enough, this is what we find, up to $a\mathcal{O}(a)$ discretization corrections; see Fig. 5. Subtracting the partial sum truncated at orders $n_0(\alpha) = 6$ from the $\beta \in [5.9, 6.4]$ data, we obtain $\bar{\Lambda} = 1.55(8)r_0^{-1}$ from such a linear plus quadratic fit, where we only give the statistical uncertainty. The errors of the perturbative coefficients are all tiny, which allows us to transform the expansion $a\delta m(\alpha)$ into $\overline{\mathrm{MS}}$-like schemes and to compute $\bar{\Lambda}$ accordingly. We define the schemes $\overline{\mathrm{MS}}_2$ and $\overline{\mathrm{MS}}_3$ by truncating $\alpha_{\overline{\mathrm{MS}}}(a^{-1}) = \alpha(1 + d_1\alpha + d_2\alpha^2 + \cdots)$ exactly at $\mathcal{O}(\alpha^3)$ and $\mathcal{O}(\alpha^4)$, respectively. The d_j are known for $j \le 3$ [40,41]. We typically find $n_0^{\overline{\mathrm{MS}}_i}(\alpha_{\overline{\mathrm{MS}}_i}) = 2, 3$ and obtain $\bar{\Lambda} \sim 2.17(8)r_0^{-1}$ and $\bar{\Lambda} \sim 1.89(8)r_0^{-1}$, respectively; see Fig. 5. We conclude that the changes due to these resummations are indeed of the size $\delta\bar{\Lambda} \sim 0.5r_0^{-1}$, adding confidence that our definition of the ambiguity is neither a gross overestimate nor an underestimate. For the plaquette, where we expect $n_0^{\overline{\mathrm{MS}}} \sim 7$, we cannot carry out a similar analysis, due to the extremely high precision that is required to resolve the differences between $S_P(\alpha)$ and $\langle P\rangle_{\mathrm{MC}}(\alpha)$, which largely cancel in Eq. (7).

In conclusion, for the first time ever, perturbative expansions at orders where the asymptotic regime is reached have been subtracted from nonperturbative MC data of the static-light meson mass and of the plaquette, thereby validating the OPE beyond perturbation theory. The scaling of the latter difference with the lattice spacing confirms the dimension $d = 4$. Dimension $d < 4$ slopes appear only when subtracting the perturbative series truncated at fixed preasymptotic orders: lower-dimensional "condensates" discussed in the literature, see, e.g., Refs. [32–35], are just approximate parametrizations of unaccounted perturbative effects, i.e., of the short-distance behavior and, thus, observable dependent (unlike the non-perturbative gluon condensate). Such simplified parametrizations introduce unquantifiable errors and, therefore, are of limited phenomenological use.

We have obtained an accurate value of the gluon condensate in SU(3) gluodynamics, Eq. (11). It is of a similar size as the intrinsic difference, Eq. (12), between (reasonable) subtraction prescriptions. This result contradicts the implicit assumption of sum rules analyses that the renormalon ambiguity is much smaller than leading non-perturbative corrections. The value of the gluon condensate obtained with sum rules can vary significantly due to this intrinsic, renormalization scheme-independent ambiguity, if determined using different prescriptions or truncating at different orders in perturbation theory. Clearly, the impact of this, e.g., on determinations of α_s from τ decays or from lattice simulations needs to be assessed carefully.

Finally, the inherent ambiguity of (reasonable) definitions of the static-light meson mass was estimated in Refs. [40,41]. Here, in a combined analysis, this estimate was confronted with MC data and confirmed.

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