Braiding Statistics of Loop Excitations in Three Dimensions

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While it is well known that three dimensional quantum many-body systems can support nontrivial braiding statistics between particlelike and looplike excitations, or between two looplike excitations, we argue that a more fundamental quantity is the statistical phase associated with braiding one loop α around another loop β, while both are linked to a third loop γ. We study this three-loop braiding in the context of $(\mathbb{Z}_N)^K$ gauge theories which are obtained by gauging a gapped, short-range entangled lattice boson model with $(\mathbb{Z}_N)^K$ symmetry. We find that different short-range entangled bosonic states with the same $(\mathbb{Z}_N)^K$ symmetry (i.e., different symmetry-protected topological phases) can be distinguished by their three-loop braiding statistics.

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Introduction.—A powerful way to characterize the topological properties of two dimensional gapped quantum many-body systems is to examine their quasiparticle braiding statistics [1]. Thus, it is natural to wonder: what is the analogous quantity that characterizes three dimensional (3D) systems? The simplest candidate—3D quasiparticle statistics—is of limited use since 3D systems can only support bosonic and fermionic quasiparticles. On the other hand, 3D systems can support much richer braiding statistics between particlelike excitations and looplike excitations [2–4] or between two looplike excitations [5–7]. Thus, one might guess that particle-loop and loop-loop braiding statistics are the natural generalizations of quasiparticle statistics to three dimensions.

In this Letter, we argue that this guess is incorrect: particle-loop and loop-loop braiding statistics do not fully capture the topological structure of 3D many-body systems. Instead, more complete information can be obtained by considering a three-loop braiding process in which a loop α is braided around another loop β, while both are linked with a third loop γ (Fig. 1). We believe that three-loop braiding statistics is one of the basic pieces of topological data that describe 3D gapped many-body systems, and much of this Letter is devoted to understanding the general properties of this quantity. Also, as an application, we show that three-loop statistics can be used to distinguish different short-range entangled many-body states with the same (unitary) symmetry—i.e., different symmetry-protected topological (SPT) phases [8–10]. The latter result shows that the braiding statistics approach to SPT phases, outlined in Ref. [11], can be extended to three dimensions.

Discrete gauge theories.—For concreteness, we focus our analysis on a simple 3D system with looplike excitations, namely lattice $(\mathbb{Z}_N)^K$ gauge theory [12]. More specifically, we consider a 3D lattice boson model built out of K different species of bosons, where the number of bosons in each species is conserved modulo N so that the system has a $(\mathbb{Z}_N)^K$ symmetry. We suppose that the ground state of the boson model is gapped and short-range entangled—that is, it can be transformed into a product state by a local unitary transformation [13]. We then imagine coupling such a lattice boson model to a $(\mathbb{Z}_N)^K$ lattice gauge field [14].

In general, these gauge theories contain two types of excitations: pointlike "charge" excitations which carry gauge charge, and stringlike "vortex loop" excitations which carry gauge flux. The most general charge excitations can carry gauge charge $q = (q_1, \ldots, q_K)$ where each component q_m is an integer defined modulo N. The most general vortex loop can carry gauge flux $\phi = (\phi_1, \ldots, \phi_K)$ where ϕ_m is a multiple of $2\pi/N$. In fact, since we can always attach a charge to a vortex loop to obtain another vortex loop, a general vortex loop excitation carries both flux and charge.

Let us try to understand the braiding statistics of these excitations. In general, there are three types of braiding processes we can consider: processes involving two charges, processes involving a charge and a loop, and processes involving multiple loops. Clearly, the first type of process cannot give any statistical phase since the charges are excitations of the short-range entangled boson model and, therefore, must be bosons. On the other hand, the

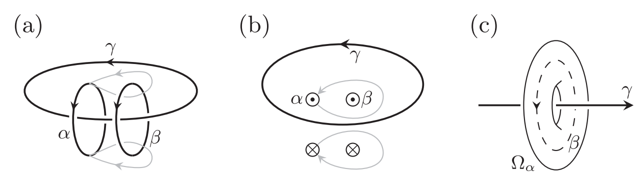

FIG. 1. (a) Three-loop braiding process. The gray curves show the paths of two points on the moving loop α. (b) A top view of the braiding process within the plane that γ lies in. (c) A torus Ω_α is swept out by α during the braiding. Loop β (dashed circle) is enclosed by Ω_α.

second kind of process, involving charges and loops, can give a nontrivial phase. More specifically, if we braid a charge $q = (q_1, \ldots, q_K)$ around a vortex loop with gauge flux $\phi = (\phi_1, \ldots, \phi_K)$, the resulting statistical phase is given by the Aharonov-Bohm formula

$$\theta = q \cdot \phi, \tag{1}$$

where "·" denotes the vector dot product.

All that remains is to examine the braiding statistics of loops. The simplest process one can consider [15] involves braiding a loop α around another loop β as shown in Fig. 2(a). To analyze this process, we use two facts about unlinked vortex loops: First, a subset of vortex loops, which we call "neutral" loops, can be shrunk to a point and annihilated by local gauge invariant operators. Second, all other vortex loops can be obtained from neutral loops by attaching an appropriate amount of charge. With these facts in mind, let us first suppose that both α, β are neutral. In this case, it follows from general principles that the statistical phase $\theta_{\alpha\beta} = 0$, since we can "smoothly" [16] deform the two-loop braiding process into another process in which α is braided around the vacuum [Fig. 2(b)]. Now, consider the general case where α, β carry charge. In this case, α, β can be thought of as neutral loops with some attached charge. It then follows from the Aharonov-Bohm formula (1) that the Berry phase associated with braiding α around β is

$$\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha, \tag{2}$$

where q_α, q_β and ϕ_α, ϕ_β denote the charge and flux carried by α, β, respectively. To see this, note that during the two-loop braiding, the charge q_α is braided around the flux ϕ_β, and the flux ϕ_α is braided around the charge q_β.

While the above calculations show that 3D gauge theories can exhibit nonvanishing braiding statistics, we can see that these statistical phases are the same for all gauge theories with gauge group $(\mathbb{Z}_N)^K$, independent of the properties of the bosonic matter. Yet, we expect that the bosonic matter should be important: if two lattice boson models realize different short-range entangled phases with the same symmetry (i.e., different SPT phases [8]), then, presumably, the corresponding gauge theories belong to distinct phases as well, by analogy with the 2D case [11,17]. Clearly, if we want to distinguish these different types of 3D gauge theories, we must consider braiding processes with more than two loops.

Three-loop braiding statistics.—For these reasons, we are naturally led to consider a braiding process involving two loops α, β which are linked with a third "base" loop γ (Fig. 1). When the loop α sweeps around β in a right-handed manner, it can acquire a statistical Berry phase which we will denote by $\theta_{\alpha\beta,c}$ where $\phi_\gamma = (2\pi/N)c$ with c being an integer vector. We use the notation $\theta_{\alpha\beta,c}$, rather than $\theta_{\alpha\beta,\gamma}$ because θ is insensitive to the charge attached to γ and depends only on its flux $\phi_\gamma = (2\pi/N)c$. Similarly, we will also consider an exchange or half-braiding process in which two identical loops α, which are linked with a base loop with flux $(2\pi/N)c$, are braided through one another and exchange places. The statistical phase associated with this exchange will be denoted by $\theta_{\alpha,c}$. Note that, throughout this Letter, we assume the loops have Abelian statistics.

These three-loop braiding processes are fundamentally different from the two-loop case because in the three-loop topology, the base loop γ may prevent us from shrinking α and β to a point and annihilating them locally. As a result, the above argument that vortex loop statistics follow the Aharonov-Bohm law (2) is no longer valid. Thus, the three-loop braiding statistics are less constrained than the two-loop case.

Constraints on $\theta_{\alpha\beta,c}$ and $\theta_{\alpha,c}$.—We now discuss the basic physical constraints on the three-loop braiding statistics. One of the simplest constraints is that $\theta_{\alpha\beta,c} = \theta_{\beta\alpha,c}$. To derive this property, we note that a process in which α winds around β can be smoothly deformed into one in which β winds around α. Therefore, since the statistical phase is invariant under smooth deformations of the braiding path, $\theta_{\alpha\beta,c}$ must be symmetric in α and β. Another obvious constraint is that $\theta_{\alpha\alpha,c} = 2\theta_{\alpha,c}$. This relation is clear since a full braiding is equivalent to performing two exchanges in series.

Even more powerful constraints on θ can be obtained by thinking about "fusion" of vortex loops. More specifically, there are two distinct ways to fuse loops together. In the first type of fusion process [Fig. 3(a)], two loops β_1, β_2 that are linked to the same loop γ can be fused to form a new loop "$\beta_1 + \beta_2$" that is also linked to γ. In the second type of fusion process [Fig. 3(b)], two loops β_1, β_2 that share the same flux $\phi_{\beta_1} = \phi_{\beta_2}$ but are linked with two different loops γ_1 and γ_2, can be fused to form a loop "$\beta_1 \oplus \beta_2$", which is linked to both γ_1 and γ_2. It is not hard to see that $\theta_{\alpha\beta,c}$ must be linear under both fusion processes [18] (Fig. 4)

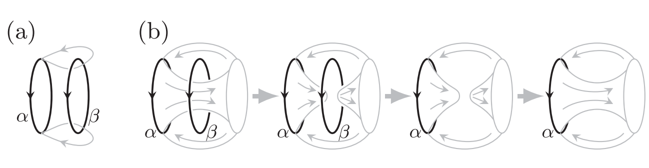

FIG. 2. (a) Braiding of two loops α, β. (b) If α, β are neutral, the two-loop process can be smoothly deformed into a process in which α is braided around the vacuum.

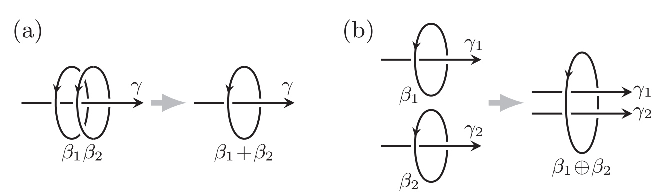

FIG. 3. Two ways to fuse loops together.

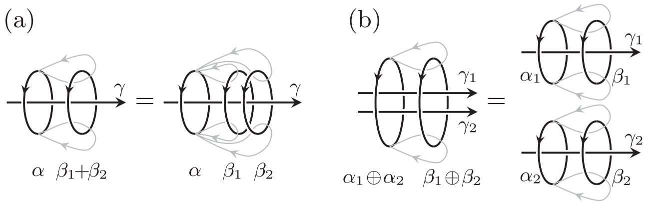

FIG. 4. Braiding processes associated with equations (3) [panel (a)] and (4) [panel (b)]. Here, $\phi_\gamma = (2\pi/N)c$, while $\phi_{\gamma_1} = (2\pi/N)c_1$ and $\phi_{\gamma_2} = (2\pi/N)c_2$.

$$\theta_{\alpha(\beta_1+\beta_2),c} = \theta_{\alpha\beta_1,c} + \theta_{\alpha\beta_2,c}, \tag{3}$$

$$\theta_{(\alpha_1\oplus\alpha_2)(\beta_1\oplus\beta_2),(c_1+c_2)} = \theta_{\alpha_1\beta_1,c_1} + \theta_{\alpha_2\beta_2,c_2}. \tag{4}$$

To derive these identities, it suffices to show that the processes defined by the left-hand sides of Figs. 4(a)–4(b) can be smoothly deformed into the processes corresponding to the right-hand sides of Figs. 4(a)–4(b). These deformations are described in the Supplemental Material [19].

One implication of the linearity of θ (3)–(4) is that we can reconstruct all the three-loop statistics from the statistics of vortex loops with unit flux. The statistics of these unit fluxes can in turn be summarized by two tensors $\Theta_{ij,k}$ and $\Theta_{i,k}$. These tensors are defined by

$$\Theta_{ij,k} \equiv N\theta_{\alpha\beta,e_k}, \qquad \Theta_{i,k} \equiv N\theta_{\alpha,e_k}, \tag{5}$$

where α, β are any two loops carrying unit flux $\phi_\alpha = (2\pi/N)e_i$ and $\phi_\beta = (2\pi/N)e_j$, respectively, and where $e_i \equiv (0, \ldots, 1, \ldots, 0)$ with a 1 in the ith entry and 0 everywhere else. To see why the tensor $\Theta_{ij,k}$ is well-defined modulo 2π, note that, if we choose another set of loops α', β' with the same flux, then the only possible topological difference between α', β' and α, β is that they may have different amounts of charge attached to them. But from the Aharonov-Bohm formula (1), we see that attaching charge to α and β can only shift the value of $\theta_{\alpha\beta,e_k}$ by multiples of $2\pi/N$ and, hence, can only shift $\Theta_{ij,k}$ by multiples of 2π. Similar reasoning applies in the case of $\Theta_{i,k}$.

Given that the $\Theta_{ij,k}$ and $\Theta_{i,k}$ effectively summarize all the three-loop statistics, all that remains is to find the physical constraints on these two quantities. We, now, argue that these constraints are as follows:

$$\Theta_{ij,k} = \Theta_{ji,k}, \qquad \Theta_{ii,k} = 2\Theta_{i,k}, \tag{6}$$

$$\Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} = 0, \tag{7}$$

$$\Theta_{ik,i} + \Theta_{i,k} = 0, \qquad \Theta_{i,i} = 0, \tag{8}$$

$$\Theta_{ij,k} = \frac{2\pi}{N}(\text{integer}), \qquad \Theta_{i,k} = \frac{2\pi}{N}(\text{integer}). \tag{9}$$

The first two constraints (6) are obvious, since they are special cases of the more general relations discussed above. The quantization conditions (9) are also easy to derive: for example, to prove the first equation in (9), consider a thought experiment in which a loop α carrying flux $\phi_\alpha = (2\pi/N)e_i$, together with N identical loops β carrying flux $\phi_\beta = (2\pi/N)e_j$, are all linked to a common base loop with flux $(2\pi/N)e_k$. Now, imagine we fuse the β loops together to form a new loop B, and then, we braid α around B. By the linearity of θ (3), the resulting statistical phase is

$$\theta_{\alpha B,e_k} = N\theta_{\alpha\beta,e_k} = \Theta_{ij,k}. \tag{10}$$

At the same time, we can see that $\phi_B = N\phi_\beta = 0$, so B is a pure charge. It then follows, from the Aharonov-Bohm formula (1), that $\theta_{\alpha B,e_k} = q_B \cdot \phi_\alpha$, which is a multiple of $2\pi/N$. Combining these two observations, we deduce that $\Theta_{ij,k}$ is a multiple of $2\pi/N$. The proof of the second equation in (9) is similar.

Equations (7)–(8) are the most interesting constraints on Θ, as these relations have no analogues in the theory of 2D braiding statistics. We call Eq. (7) the cyclic relation. A physical derivation of the cyclic relation is given in the Supplemental Material [19]. The first equation in (8) can be proved in a similar manner. On the other hand, we do not have a physical derivation of $\Theta_{i,i} = 0$, so this constraint on Θ is simply a conjecture. This conjecture is supported by two pieces of evidence: first, all the microscopic models constructed below obey this relation. Second, we can prove the weaker, but closely related relation $3\Theta_{i,i} = 0 \pmod{2\pi}$ using the second equation in (6) together with the first equation in (8).

Dimensional reduction.—We now derive a formula for the three-loop statistics that will be useful in analyzing the microscopic models discussed later. This formula is obtained by considering our system in an $L_x \times L_y \times L_z$ torus geometry—i.e., a geometry with periodic boundary conditions in all three directions. Let α, β be two loops linked with a base loop γ carrying flux $\phi_\gamma = (2\pi/N)c$ [Fig. 5(a)]. For concreteness, suppose that γ lies in the xy plane while α, β lie in the xz plane. When α sweeps around β, it gives rise to a statistical phase $\theta_{\alpha\beta,c}$ which we wish to compute. To this end, we stretch α in the z direction until it wraps all the way around the periodic z direction. We can then fuse α with itself, thereby splitting α into two noncontractible loops α' and α'' [Fig. 5(b)]. Similarly, we can stretch β in the z direction and fuse it with itself so that it splits into β' and β''. It is clear that the braiding process involving α and β can now be decomposed into two separate processes in which α' is braided around β' and α'' is braided around β''. Since these two processes are separate, we can think of them as taking place in two separate systems [Fig. 5(c)]. Furthermore, for the process involving α', β' we can stretch γ in the xy plane so that it fuses and annihilates itself. This effectively leaves a gauge flux $\phi_\gamma = (2\pi/N)c$ through one of the three holes of

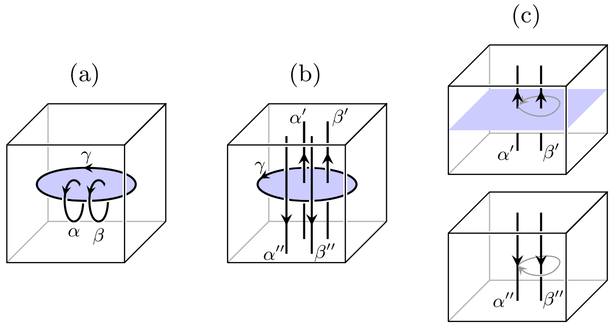

FIG. 5 (color online). Computing three-loop statistics from 2D braiding.

the 3D torus—more precisely, the hole bounded by a noncontractible cycle along the z direction (the "z hole"). Likewise, for the process involving α'', β'' we can shrink γ in the xy plane until it fuses and annihilates itself, leaving no gauge flux through the z hole. In this way, we see that $\theta_{\alpha\beta,c}$ can be expressed as

$$\theta_{\alpha\beta,c} = \theta_{\alpha'\beta',c} - \theta_{\alpha''\beta'',0}, \tag{11}$$

where the quantities on the right-hand side are statistical phases associated with braiding two vortex lines around one another in the xy plane. In the first term, this braiding takes place in the presence of a gauge flux $(2\pi/N)c$ through the z hole, while, in the second term, there is no such gauge flux. The relative sign comes from the fact that the two pairs are braided in opposite directions. The formula (11) is useful because each of the terms on the right-hand side can be thought of as braiding statistics of a 2D system if we take the thermodynamic limit $L_x, L_y \to \infty$, while keeping L_z finite but larger than the correlation length. An analogous formula can be derived for the exchange statistics $\theta_{\alpha,c}$.

Microscopic models.—To obtain examples of systems with nontrivial three-loop statistics, we consider "gauged" SPT models—that is, we take the exactly soluble lattice boson models of Ref. [8] which realize different SPT phases, and we couple them to a gauge field. These gauged SPT models can be equivalently [11] thought of as Dijkgraaf-Witten models [20]. As above, we focus on the case where the symmetry group is $G = (\mathbb{Z}_N)^K$.

As discussed in Ref. [8], the basic input for constructing a 3D gauged SPT model is a four-cocycle $\omega: G^4 \to U(1)$. If two cocycles ω_1, ω_2 differ by a four-coboundary ν, i.e., $\omega_1 = \omega_2 \cdot \nu$, then the corresponding models belong to the same SPT phase. Thus, inequivalent models are classified by elements of the cohomology group $H^4[G, U(1)]$. Here, we focus on four-cocycle ω of the form

$$\omega(a,b,c,d) = e^{(i2\pi/N^2)\sum_{ijk} M_{ijk} a_i b_j (c_k + d_k - [c_k + d_k])}, \tag{12}$$

where M_{ijk} is an integer tensor, and we parametrize the different group elements of $G = (\mathbb{Z}_N)^K$ with integer vectors $a = (a_1, \ldots, a_K)$ with $a_i = 0, \ldots, N-1$. The square bracket $[c_k + d_k]$ is defined to be $c_k + d_k$ (mod N) with values taken in the range $0, \ldots, N-1$.

Our task is to compute the three-loop statistics $\Theta_{ij,k}$ and $\Theta_{i,j}$ of the gauged SPT model with cocycle ω. The details of this calculation, which is based on the formula (11), can be found in the Supplemental Material [19]. The end result is

$$\Theta_{ij,k} = \frac{2\pi}{N}(M_{ikj} - M_{kij} + M_{jki} - M_{kji}),$$
$$\Theta_{i,j} = \frac{2\pi}{N}(M_{iji} - M_{jii}). \tag{13}$$

As a consistency check, one can easily verify that these expressions satisfy conditions (6)–(9). Conversely, it is a straightforward mathematical exercise to check that every $\Theta_{ij,k}$ and $\Theta_{i,j}$ that obeys (6)–(9) can be written in the form (13) for some M_{ijk}. Hence, every solution to (6)–(9) can be physically realized as a gauged SPT model.

Examples.—The simplest example is $G = \mathbb{Z}_N$. In this case, M has only one component M_{111}, and (13) gives trivial loop statistics, $\Theta_{ii,i} = \Theta_{i,i} = 0$, for any choice of M. This is a reasonable result since $H^4[\mathbb{Z}_N, U(1)] = 0$, so all the SPT models with $G = \mathbb{Z}_N$ are equivalent to product states [8].

The simplest nontrivial example is given by $G = (\mathbb{Z}_N)^2$. In this case, if we choose $M_{211} = p_1$, $M_{122} = p_2$, and all other components vanishing, we obtain the three-loop statistics shown in Table I. We can see that there are N^2 distinct types of statistics that can be realized by the gauged SPT models with $G = (\mathbb{Z}_N)^2$. This is also a reasonable result since $H^4[(\mathbb{Z}_N)^2, U(1)] = (\mathbb{Z}_N)^2$, so the SPT models realize N^2 distinct phases [8,21]. Evidently, each phase is associated with a different type of three-loop statistics.

TABLE I. $\Theta_{ij,k}$ for the SPT models with $(\mathbb{Z}_N)^2$ symmetry.

$\Theta_{11,1}$	$\Theta_{12,1}$	$\Theta_{22,1}$	$\Theta_{11,2}$	$\Theta_{12,2}$	$\Theta_{22,2}$
0	$(2\pi/N)p_1$	$-(4\pi/N)p_2$	$-(4\pi/N)p_1$	$(2\pi/N)p_2$	0

Discussion.—The above examples show that the gauged SPT phases with $G = \mathbb{Z}_N$ and $G = (\mathbb{Z}_N)^2$ are uniquely characterized by their three-loop statistics. More generally, we find it plausible that every 3D SPT phase with unitary symmetries is uniquely characterized by its three-loop statistics—similar to what has been proposed in the 2D case [11]. One subtlety in checking this conjecture for more general $G = (\mathbb{Z}_N)^K$ is that, when $K \geq 4$, the cocycles (12) do not exhaust all elements of $H^4[G, U(1)]$. Furthermore, the remaining elements of $H^4[G, U(1)]$ can lead to non-Abelian three-loop statistics (see Ref. [22] for examples of this phenomenon in the 2D case). Thus, a theory of non-Abelian loop statistics may be necessary to proceed further in this direction.

Is three-loop statistics measurable? In principle, three-loop statistics could be measured experimentally by performing interferometry on looplike excitations; in practice, such an experiment would be challenging. A more straightforward application is to numerical simulations, where three-loop statistics could be directly extracted from an appropriate Berry phase computation.

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Note added.—Recently, we became aware of an independent work [23] containing related results on three-loop statistics. Other recent work on this topic includes Refs. [24,25].

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[16] We say that a multiloop braiding process can be "smoothly" deformed into another process if we can interpolate between the two processes via a sequence of local changes to the worldsheets swept out by the loops. This sequence of local changes can be arbitrary except that the worldsheets of any two loops that are being braided around one another must stay far apart during every step.
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