Universal Covariance Formula for Linear Statistics on Random Matrices

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We derive an analytical formula for the covariance $\mathrm{cov}(A, B)$ of two smooth linear statistics $A = \sum_i a(\lambda_i)$ and $B = \sum_i b(\lambda_i)$ to leading order for $N \to \infty$, where $\{\lambda_i\}$ are the N real eigenvalues of a general one-cut random-matrix model with Dyson index β. The formula, carrying the universal $1/\beta$ prefactor, depends on the random-matrix ensemble only through the edge points $[\lambda_-, \lambda_+]$ of the limiting spectral density. For $A = B$, we recover in some special cases the classical variance formulas by Beenakker and by Dyson and Mehta, clarifying the respective ranges of applicability. Some choices of $a(x)$ and $b(x)$ lead to a striking *decorrelation* of the corresponding linear statistics. We provide two applications—the joint statistics of conductance and shot noise in ideal chaotic cavities, and some new fluctuation relations for traces of powers of random matrices.

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Introduction.—The discovery of the phenomenon of universal conductance fluctuations (UCF) in disordered metallic samples, pioneered by Altshuler [1] and Lee and Stone [2], has had a profound impact on our current understanding of the mechanisms of quantum transport at low temperatures and voltage. There are two aspects of this universality: (i) the variance of the conductance is of order $(e^2/h)^2$, independent of sample size or disorder strength, and (ii) this variance decreases by precisely a factor of 2 if time-reversal symmetry is broken by a magnetic field. Both features, observed in several experiments and numerical simulations (see Ref. [3] for a review), naturally emerge from a random-matrix theoretical formulation of the electronic transport problem [4,5]. The phenomenon of UCF is just, however, one of the very many incarnations of a more general and intriguing property of sums of strongly correlated random variables.

Consider first, for instance, a set of N independent and identically distributed $\mathcal{O}(1)$ random variables $\{X_i\}$. The random variable $A = \sum_i a(X_i)$, for any function $a(x)$ (hereafter, all summations run from 1 to N), is called a linear statistics of the sample $\{X_i\}$. For large N, both the average $\langle A \rangle$ and the variance $\mathrm{var}(A)$ typically grow linearly with N. But, what happens if the N variables are instead strongly correlated? A prominent example is given by the N real eigenvalues $\{\lambda_i\}$ of a random matrix. In this case, a completely different behavior emerges: if $a(x)$ is twice differentiable, while the average is still of order $\mathcal{O}(N)$, the variance attains a *finite* value for $N \to \infty$. Moreover, quite generally, $\mathrm{var}(A) \propto 1/\beta$, where β (the Dyson index) is related to the symmetries of the ensemble, and on the scale $\mathcal{O}(1)$ of typical fluctuations around the average, the distribution of A is Gaussian [6–15]. Recalling that the conductance in chaotic cavities can indeed be written as a linear statistics of a random matrix (see below), the phenomenon of UCF is readily understood. The issue of fluctuations of generic linear statistics has, however, a longer history in the physics and mathematics literature [6–15], due to its relevance for a variety of applications beyond UCF, ranging from quantum transport in metallic conductors [16] and entanglement of trapped fermion chains [17] to the statistics of extrema of disordered landscapes [18]—to mention just a few.

For a smooth $a(x)$, there exist two celebrated formulas in the physics literature by Dyson and Mehta (DM) [19] and Beenakker (B) [20,21] for $\mathrm{var}(A)$, the latter precisely derived in the context of the quantum transport problem introduced earlier (see also Ref. [22] for a generalized B formula). They are deemed *universal*—not dependent on the microscopic detail of the random-matrix ensemble under consideration—and correctly predict an $\mathcal{O}(1)$ value for $N \to \infty$ and a universal β^{-1} prefactor.

What happens now if *two* linear statistics $A(\boldsymbol{\lambda}) = \sum_i a(\lambda_i)$ and $B(\boldsymbol{\lambda}) = \sum_i b(\lambda_i)$ are simultaneously considered? Motivated by applications to the quantum transport problem [23] and multivariate data analysis [24], we set for ourselves the task to find a universal formula for the *covariance* $\mathrm{cov}(A, B)$ that would reduce to DM or B for $A \equiv B$. But, before proceeding, it felt natural to first check under which precise conditions should we expect to recover one formula or the other.

Much to our surprise, we have failed to find a sufficiently transparent (at least to our eyes) account that encompasses all possible cases in an accessible and systematic way. The goal of this Letter is thus to produce a so-far unavailable universal formula for $\mathrm{cov}(A, B)$ of large dimensional random matrices. As a by-product of our result, we generalize DM and B formulas for $A = B$. We introduce a "conformal map" method which encloses all possible cases (old and new) into a neat and unified framework. We further employ our formula to probe a quite interesting phenomenon of *decorrelation*; namely, for some choices of $a(x)$ and $b(x)$, we get $\mathrm{cov}(A, B) = \mathcal{O}(N^{-1})$. Examples are given for (i) conductance and shot noise in ideal chaotic cavities supporting a large number of electronic channels and (ii) fluctuation relations for traces of powers of random matrices.

Setting and results.—We consider an ensemble of $N \times N$ random matrices $\mathcal{H}$, whose joint probability density of the N eigenvalues $\lambda_i \in \Lambda$ (a generic interval of the real line) can be cast in the Gibbs-Boltzmann form

$$\mathcal{P}_\beta(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}} e^{-\beta[-\sum_{i<j}\ln|\lambda_i-\lambda_j| + N\sum_i V(\lambda_i)]} \equiv \frac{e^{-\beta E(\boldsymbol{\lambda})}}{\mathcal{Z}}. \quad (1)$$

Here, the normalization constant $\mathcal{Z} = \int_{\Lambda^N} d\boldsymbol{\lambda} e^{-\beta E(\boldsymbol{\lambda})}$ is the partition function of a Coulomb gas, namely, a 1D system of N particles in equilibrium at inverse temperature $\beta > 0$ (the Dyson index), whose energy $E(\boldsymbol{\lambda})$ contains a logarithmic repulsive interaction and a confining single-particle potential $V(x)$. We first define the spectral density $\rho_N(\lambda) = N^{-1}\sum_i \delta(\lambda - \lambda_i)$ (a random measure on the real line) and its average for finite ($\langle\rho_N(\lambda)\rangle$) and large N $[\rho(\lambda) = \lim_{N\to\infty}\langle\rho_N(\lambda)\rangle]$, where henceforth the angled brackets stand for averaging with respect to Eq. (1). The potential $V(x)$ is assumed to be such that $\rho(\lambda)$ is supported on a *single* interval σ of the real line (possibly unbounded).

The form of the joint probability density (1) includes classical invariant ensembles [25] such as Wigner-Gauss $\mathcal{G}$, Wishart-Laguerre $\mathcal{W}$, Jacobi $\mathcal{J}$, and Cauchy $\mathcal{C}$. In Table I, the corresponding potentials are listed. We stress, however, that the general setting in Eq. (1) applies equally well, e.g., to noninvariant ensembles such as the Dumitriu-Edelman [26] tridiagonal β ensembles, for nonquantized $\beta > 0$.

Consider now two linear statistics $A(\boldsymbol{\lambda}) = \sum_i a(\lambda_i)$ and $B(\boldsymbol{\lambda}) = \sum_i b(\lambda_i)$. Their covariance is given by the N-fold integral

$$\mathrm{cov}(A, B) = \int_{\Lambda^N} d\boldsymbol{\lambda}\mathcal{P}_\beta(\boldsymbol{\lambda})[A(\boldsymbol{\lambda}) - \langle A\rangle][B(\boldsymbol{\lambda}) - \langle B\rangle]. \quad (2)$$

For smooth $a(x)$ and $b(x)$, we show that this covariance (2) has the universal form for $N \to \infty$

$$\mathrm{cov}(A, B) = \frac{1}{\beta\pi^2}\int_0^\infty dk\varphi(k)\mathrm{Re}[\tilde{a}(k)\tilde{b}^\star(k)], \quad (3)$$

with an error term of order $\mathcal{O}(N^{-1})$, which will always be neglected henceforth. Here, Re stands for the real part and the asterisk for complex conjugation. Assume that at least one of the end points of σ is finite, as in many practical cases. Then, $\varphi(k) = k\tanh(\pi k)$ is a *universal* kernel and we have introduced a deformed Fourier transform $\tilde{f}(k) = \int_{-\infty}^{+\infty} dx e^{ikx} f(T(e^x))$, where $T(\cdot)$ is a conformal map defined by the edges of the support of $\rho(\lambda)$

$$T(x) = \begin{cases} \frac{x\lambda_- + \lambda_+}{x+1} & \text{for } \sigma = [\lambda_-, \lambda_+] \\ \lambda_- + 1/x & \text{for } \sigma = [\lambda_-, \infty) \\ \lambda_+ - x & \text{for } \sigma = (-\infty, \lambda_+] \end{cases}. \quad (4)$$

The role of $T(\cdot)$ is to map the positive half line $[0, +\infty)$ to the support σ of $\rho(\lambda)$. Since no such conformal mapping exists if $\sigma = (-\infty, +\infty)$, this (unfrequent) case (e.g., the Cauchy ensemble $\mathcal{C}$) must be treated differently. In this case, $\varphi(k) = k$ and $\tilde{f}(k) = \int_{-\infty}^{+\infty} dx e^{ikx} f(x)$ is the standard Fourier transform. Equation (3) may be used whenever the integral converges.

Let us now offer a few remarks. First, formula (3) is evidently symmetric upon the exchange $A \leftrightarrow B$, as $\mathrm{cov}(A, B) = \mathrm{cov}(B, A)$. Second, the only dependence on the Dyson index β is through the prefactor β^{-1}, as already anticipated. Third, the details of the confining potential $V(x)$ only appear in the formula (3) through the edges $\lambda_\pm$ of σ, the support of the limiting spectral density $\rho(\lambda)$ and not through the range of variability of the eigenvalues Λ. This is a consequence of universality of the (smoothed) two-point kernel [27–30]. Fourth, if $\sigma = [\lambda_-, \lambda_+]$, the covariance admits the following alternative expression in real space:

$$\mathrm{cov}(A, B) = \frac{1}{\beta\pi^2} P \iint_{\lambda_-}^{\lambda_+} d\lambda d\lambda' \phi(\lambda, \lambda') \frac{a(\lambda')}{\lambda' - \lambda}\frac{db(\lambda)}{d\lambda}, \quad (5)$$

where $\phi(\lambda, \lambda') = \sqrt{((\lambda_+ - \lambda)(\lambda - \lambda_-))/((\lambda_+ - \lambda')(\lambda' - \lambda_-))}$ and P stands for Cauchy's principal value. Formula (5), which may be more convenient than Eq. (3) in certain cases, reduces for $a(x) = b(x)$ to the generalized B formula for the variance [as given in Ref. [22], Eq. (17)]. On the other hand,

TABLE I. Summary of various classical ensembles of type (1). For $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{J}$, we provide the edges of the limiting support $\sigma = [\lambda_-, \lambda_+]$.

	$V(x)$	Λ	σ
$\mathcal{G}$	$x^2/4$	$(-\infty, \infty)$	± 2
$\mathcal{W}$	$(x/2) - \ln x^{\alpha/2}$	$[0, \infty)$	$\left(1 \pm \sqrt{1+\alpha}\right)^2$
$\mathcal{J}$	$\ln x^{\alpha_1/2}(1-x)^{\alpha_2/2}$	[0,1]	$\left(\left(\sqrt{1+\alpha_2} \pm \sqrt{(\alpha_1+1)(\alpha_1+\alpha_2+1)}\right)/(\alpha_1+\alpha_2+2)\right)^2$
$\mathcal{C}$	$\ln\sqrt{1+x^2}$	$(-\infty, \infty)$	$(-\infty, \infty)$

Eq. (3) recovers for $a(x) = b(x)$ the DM formula [19] (see Eq. (1.1) in Ref. [21]) if $\sigma = (-\infty, +\infty)$ and the B formula [20] [see Eq. (6) below] if $\sigma = [0, 1]$. Equations (3) and (4) constitute then a neat and unified summary of all possible occurrences, including the case of semi-infinite supports (relevant for some cases [31,32]). Fifth, the representation (3) in Fourier space makes apparent that the covariance *vanishes* to leading order, e.g., if $\tilde{a}(k)$ is purely imaginary and $\tilde{b}(k)$ is real, which happens, e.g., for an even potential $V(x) = V(-x)$ and $a(x)$ and $b(x)$ having different parity [33]. This simple observation immediately predicts that the moments $\mathrm{Tr}\mathcal{G}^n$ of a Gaussian matrix (or any random matrix with an even potential) are asymptotically pairwise uncorrelated $\mathrm{cov}(\mathrm{Tr}\mathcal{G}^n, \mathrm{Tr}\mathcal{G}^m) = \mathcal{O}(N^{-1})$ if n is even and m odd. We provide now two examples of applications of the covariance formula, before turning to its derivation.

Examples.—As a first example, we focus on quantum transport in mesoscopic cavities, within the random-scattering-matrix framework (see Ref. [16] for a review). In this setting, the dimensionless conductance G and shot noise P of the cavity correspond to the choices $a(x) = x$ and $a(x) = x(1-x)$, respectively, in the Landauer-Büttiker theory [34–36]. The parameter $\alpha = N_1/N_2 - 1 \geq 0$, kept fixed in the large-$N_{1,2}$ limit, accounts for the asymmetry in the number of open electronic channels. For a symmetric cavity, $\alpha = 0$. Furthermore, it is well known that the transport eigenvalues $\{\lambda_i\}$ are distributed according to a Jacobi ($\mathcal{J}$) ensemble [37,38] with $V_{\mathcal{J}}(x) = (\alpha/2)\ln x$, implying an average density $\rho(\lambda)$ supported on $[\lambda_-, \lambda_+] = [\alpha^2/(\alpha+2)^2, 1]$ (compare with Table I).

It was precisely in this quantum transport setting that B was first derived [20,21]. It reads

$$\mathrm{var}(A) = \frac{1}{\beta\pi^2}\int_0^\infty dk|F(k)|^2 k\tanh(\pi k), \tag{6}$$

where $F(k) = \int_{-\infty}^{\infty} dx e^{ikx} a(1/(1+e^x))$. It is immediate to verify that Eq. (6) is recovered from our Eq. (3) upon setting $a(x) = b(x)$ and (crucially) $\alpha = 0$, implying $[\lambda_-, \lambda_+] = [0, 1]$. If $\alpha \neq 0$ (asymmetric cavities), Eq. (6) is not applicable and the variance of conductance and shot noise *do* depend explicitly on α [39], in agreement with Refs. [40,41]. In addition, from Eq. (3), one gets the covariance of conductance and shot noise to leading order in the channel numbers

$$\mathrm{cov}(G, P) = -\frac{2}{\beta}\frac{\alpha^2(\alpha+1)^2}{(\alpha+2)^6}. \tag{7}$$

We have checked that this result is in agreement with the asymptotics of an exact finite-N expression in Ref. [41] valid for all β (see also Ref. [42] for $\beta = 1, 2, 4$, Ref. [43] for $\beta = 2$ and $\alpha = 0$, Ref. [44] for $\beta = 1, 2$, and Ref. [45] for a different large-N method). The simple form (7) shows that for $N_{1,2} \gg 1$, conductance and shot noise are *anticorrelated*

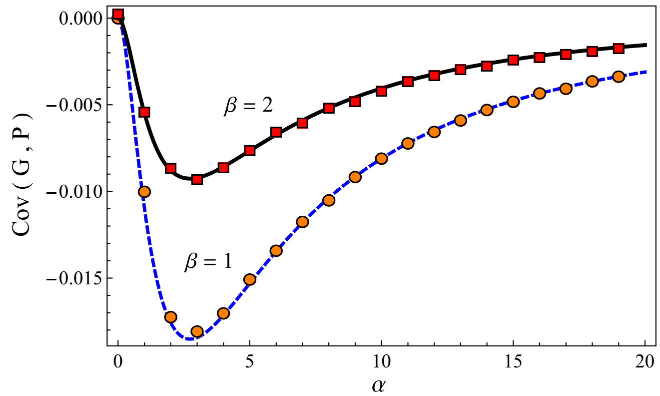

FIG. 1 (color online). Covariance of the dimensionless conductance G and shot noise P [Eq. (7)] as a function of α. The dashed blue ($\beta = 1$) and the solid black ($\beta = 2$) lines are the analytical results (7), in very good agreement with the numerical diagonalization of 10^4 Jacobi matrices of size $N = 30$ (points).

for any value of α to leading order in $N_{1,2}$. Moreover, for a symmetric ($\alpha = 0$) or highly asymmetric ($\alpha \to \infty$) cavity, the two observables are uncorrelated (for $\alpha = 0$ and $\beta = 2$, this was noticed in Ref. [41]). Given that their joint (typical) distribution is Gaussian [23], they are also independent to leading order in N for $\alpha = 0$ or $\alpha \to \infty$. As shown in Fig. 1, at $\alpha^\star = (1+\sqrt{3}) = 2.73205\ldots$ (independent of β), the anticorrelation between G and P is maximal and equal to $\mathrm{cov}(G, P)|_{\alpha=\alpha^\star} = -1/54\beta$. Since simultaneous measurements of conductance and shot noise are possible [46], a verification of this "$1+\sqrt{3}$" effect might be within reach of current experimental capabilities.

As a second example, we address the following question: what is the behavior of $\mathrm{cov}(\mathrm{Tr}\mathcal{H}^n, \mathrm{Tr}\mathcal{H}^m)$ as a function of n and m for a unitarily invariant ensemble of matrices $\mathcal{H}$? If $\sigma = [0, \lambda_+]$, we obtain from Eq. (3) for sufficiently large n and m [33]

$$\mathrm{cov}(\mathrm{Tr}\mathcal{H}^n, \mathrm{Tr}\mathcal{H}^m) \sim \frac{\lambda_+^{n+m}}{\beta\pi}\frac{\sqrt{nm}}{n+m}, \tag{8}$$

in perfect agreement with numerical simulations on $\mathcal{W}$ matrices (see Fig. 2). Setting $n = m$, we deduce the remarkable universal formula $\lim_{n\to\infty}[2\pi\beta\mathrm{var}(\mathrm{Tr}\mathcal{H}^n)]^{1/n} = \lambda_+^2$, in agreement with numerical simulations (see the inset in Fig. 2) and earlier results (see Ref. [47], Eq. (149), and Ref. [48]). We now sketch the key steps of derivation of the general formula (3) for $\sigma \neq \mathbb{R}$, treading in the same footsteps as Ref. [21]; mathematical details will be published elsewhere [33].

Derivation.—The starting point is Eq. (2) together with Eq. (1). The crucial observation is that a change of variable $\lambda_i = T(x_i)$ induced by the conformal map $T(x) = (ax+b)/(cx+d)$ with $ad - cb \neq 0$ transforms the original system into a new Coulomb gas of type (1) at the same temperature β^{-1}, with a modified potential $\tilde{V}(x)$ [49]. In these new variables, Eq. (2) becomes

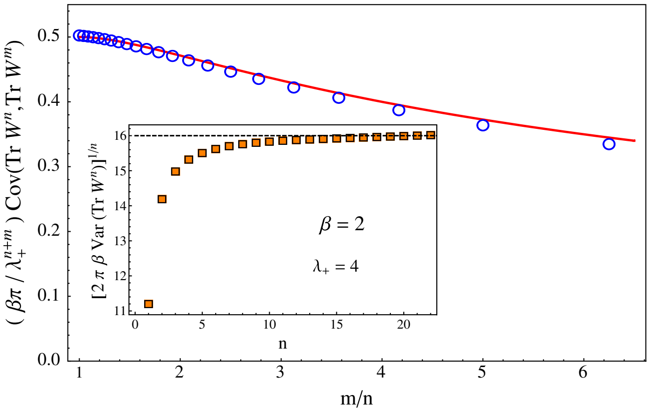

FIG. 2 (color online). The covariance $\mathrm{cov}(\mathrm{Tr}\mathcal{W}^n, \mathrm{Tr}\mathcal{W}^m)$ as a function of m/n, where $\mathcal{W}$ is a complex Wishart matrix ($\beta = 2$). For the simulation, we sampled 10^4 $\mathcal{W}$ matrices of size $N = 800$. Main: In the simulation (circles), n is fixed to 50 and m varies. The solid curve is Eq. (8). Inset: The numerical simulations (squares) show the convergence of the rescaled variance to the limit value $\lambda_+^2 = 16$.

$$\mathrm{cov}(A,B) = \int_{\tilde{\Lambda}^N} d\boldsymbol{x}\frac{1}{\tilde{\mathcal{Z}}}e^{-\beta\tilde{E}(\boldsymbol{x})}A(T(\boldsymbol{x}))B(T(\boldsymbol{x})) - \langle A\rangle\langle B\rangle, \tag{9}$$

with $\tilde{E}(\boldsymbol{x}) = -\sum_{i<j}\ln|x_i - x_j| + N\sum_i \tilde{V}(x_i) + \mathcal{O}(N)$. Introducing the spectral density of the new system $\tilde{\rho}_N(x) = N^{-1}\sum_i \delta(x - x_i)$, Eq. (9) can be reduced to the double integral

$$\mathrm{cov}(A,B) = -N^2 \iint_{\tilde{\sigma}} dx dx' \tilde{\mathcal{K}}_N(x,x') a(T(x)) b(T(x')), \tag{10}$$

where $\tilde{\mathcal{K}}_N(x,x') = -\langle\tilde{\rho}_N(x)\tilde{\rho}_N(x')\rangle + \langle\tilde{\rho}_N(x)\rangle\langle\tilde{\rho}_N(x')\rangle$ is the two-point (connected) correlation function [50]. We now denote $\tilde{\rho}(x) = \lim_{N\to\infty}\langle\tilde{\rho}_N(x)\rangle$ and $\tilde{\mathcal{K}}(x,x') = \lim_{N\to\infty}N^2\tilde{\mathcal{K}}_N(x,x')$. For a suitable choice of parameters a, b, c, and d, the corresponding density $\tilde{\rho}$ is supported on $\tilde{\sigma} = (0, +\infty)$. In summary, the maps (4) are precisely constructed to achieve these goals: (i) the 2D Coulomb interaction (logarithmic) is preserved, and (ii) the support σ is mapped into $\tilde{\sigma} = (0, +\infty)$ (this is possible whenever σ has at most one point at infinity). If $\tilde{\rho}$ is supported on the positive half line, then the kernel reads [20,21]

$$\tilde{\mathcal{K}}(x,x') = \frac{1}{\pi^2}\frac{d}{dx}\frac{d}{dx'}\ln\left|\frac{\sqrt{x} - \sqrt{x'}}{\sqrt{x} + \sqrt{x'}}\right|, \tag{11}$$

valid for $x, x' > 0$. It is derived using the following two ingredients: (i) the electrostatic integral equation for the density $\int_0^\infty dx'\tilde{\rho}(x')\ln|x - x'| = \tilde{V}(x)$, which follows from a minimization argument of the energy of the Coulomb gas (1) [25], and (ii) the functional relation $\tilde{\mathcal{K}}(x,x') = (1/\beta)(\delta\tilde{\rho}(\lambda)/\delta\tilde{V}(\lambda'))$ [20,21], which descends from the definition $\langle\tilde{\rho}_N(x)\rangle = \int_{\tilde{\Lambda}^N} d\boldsymbol{x}(1/\tilde{\mathcal{Z}})e^{-\beta\tilde{E}(\boldsymbol{x})}\tilde{\rho}_N(x)$ and the limit $N\to\infty$. Note that the universal $1/\beta$ behavior of Eq. (3) is ultimately tracked back to this functional relation. As first noticed in Ref. [20], the change of variables $x = e^y$ and $x' = e^{y'}$ makes the kernel (11) translationally invariant, and, using standard results in Fourier space, the formula (3) is readily established. The main usefulness of the conformal map method (for $\sigma \neq \mathbb{R}$) is evident: the asymptotic kernel $\tilde{\mathcal{K}}(x,x')$ of the new gas (11) becomes *universal* [independent of details of the potential $\tilde{V}(x)$ and even of the edge points $\lambda_\pm$ of the original density $\rho(\lambda)$], yielding the fixed kernel $\varphi(k)$ in Eq. (3). Every surviving trace of the original ensemble is condensed in $\lambda_\pm$, which have now been moved inside the argument of the linear statistics.

Conclusions.—In summary, we derived a universal formula (3) for the covariance $\mathrm{cov}(A, B)$ of two smooth linear statistics for one-cut random-matrix models. Remarkable features of Eq. (3) are (i) its dependence only on the edge points of σ and *not* on Λ and (ii) the possibility that $\mathrm{cov}(A, B)$ vanishes to leading order for $N\to\infty$. Hence, some linear statistics of the same ensemble may be *uncorrelated* to leading order in N, despite being functions of *strongly correlated* eigenvalues (see Refs. [51–53] for other occurrences of this phenomenon). A joint Gaussian behavior—as already detected in a few cases [23,24]—would then also imply *independence*.

We provided two applications to mesoscopic systems and multivariate analysis, leaving further examples for a forthcoming work [33] (see also Ref. [54] for another application of our formula). In the future, it will be interesting to search for the extension of formula (3) to multi-cut matrix models [55–57] as well as to the case of non-Hermitian random matrices [58]. A thorough investigation of nonsmooth linear statistics (see, e.g., Ref. [12]) is also very much called for in the context of number variance and index problems [12,18,59,60]. It should also be possible to study the covariance of linear statistics for the biorthogonal case [61], and, in general, establishing a central limit theorem for *joint* fluctuations of linear statistics is a task whose accomplishment may turn out to be relevant for several applications.

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