Integrable Model with Parafermion Zero Energy Modes

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Parafermion zero energy modes are a vital element of fault-tolerant topological quantum computation. Although it is believed that such modes form on the border between topological and normal phases, this has been demonstrated only for Z_2 (Majorana) and Z_3 parafermions. I consider an integrable model of one-dimensional fermions where such a demonstration is possible for Z_N parafermions with any N. The procedure is easily generalizable for more complicated symmetry groups.

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Non-Abelian anyons possess the most exotic statistics known to man. Their permutation transforms one ground state into another one locally indistinguishable from the first [1,2]. In conformal field theories, this property appears as nontrivial braiding of the conformal blocks [3] (see also [4]). There are possible applications of non-Abelian statistics to fault-tolerant topological quantum computation [5–8], but it is also interesting in its own right.

The simplest anyons emerge in models of Majorana fermions in which material realization has possibly been already achieved [9]. The conceptually simplest and most straightforward generalization of Majorana fermions is Z_N parafermions. The former ones have Z_2 symmetry, and the parafermions have Z_N $(N > 2)$ symmetry. In quantum computation applications, information is supposed to be stored nonlocally in parafermionic zero energy modes, and one has to learn how to manipulate them in order to process it. To this end, several schemes have been recently suggested [10,11]. In systems with many anyon zero energy modes, they will interact so that the degeneracy will be lifted, placing restrictions on the workings of the device. This makes multianyon systems an interesting subject of research, and many lattice models of interacting anyons have been considered (see, for example, [12–14], and references therein).

The most obvious problem in this context is how to obtain anyon zero energy modes. It has been argued, in direct analogy with the Majorana zero energy modes, that they emerge on a boundary between ground states with different topological properties (see, for example, [11]). The problem is, however, that for $N > 2$ in all even remotely realistic models the parafermions are interacting objects, which makes a consideration of inhomogeneous cases difficult. So far, the existence of the zero energy modes was demonstrated only for the $N = 3$ case, which can be treated by the Abelian bosonization [11]. As far as noninteracting parafermions are concerned, their Hamiltonian was found to be non-Hermitian with complex energy eigenvalues [15].

Here I suggest a solvable Hermitian fermionic model which contains inhomogeneities of the required type. In this model, anyon zero energy modes are located on mobile solitons whose number and average velocity can be varied by changing the temperature and the chemical potential. The analysis of the corresponding Bethe ansatz equations supports the idea that a boundary between topologically different states does contain parafermionic zero energy modes. Although it is not a proof that parafermion zero energy modes *always* emerge on the boundary between ground states of different topology, this is at least a demonstration that they may emerge there. The derivation is easily generalizable to parafermions from other simple Lie groups such as, for instance, $SU_k(N)$ (see, for example, [16]). I also derive an effective model describing a finite density of such modes and obtain its exact solution. The latter solutions allow one to estimate the interaction strength between the parafermions.

The field theoretical definition and properties of massless Z_N chiral parafermionic fields ψ, ψ^+ and $\bar{\psi}, \bar{\psi}^+$ can be extracted from the $SU_N(2)$ Kac-Moody algebra. The corresponding current operators can be defined in terms of free chiral fermion fields R, R^+ and L, L^+:

$$J^a = \sum_{k=1}^{N} R^+_{k\alpha} S^a_{\alpha\beta} R_{k\beta}, \qquad \bar{J}^a = \sum_{k=1}^{N} L^+_{k\alpha} S^a_{\alpha\beta} L_{k\beta}, \quad (1)$$

where S^a are spin $S = 1/2$ matrices. On the other hand, these currents can be written as [17]

$$J^+ = \frac{\sqrt{N}}{2\pi} e^{i\sqrt{8\pi/N}\varphi}\psi, \qquad J^- = \frac{\sqrt{N}}{2\pi} e^{-i\sqrt{8\pi/N}\varphi}\psi^+,$$
$$J^z = i\sqrt{N/2\pi}\partial_z\varphi, \quad (2)$$

$$\bar{J}^+ = e^{-i\sqrt{8\pi/N}\bar{\varphi}}\bar{\psi}^+, \qquad \bar{J}^- = e^{i\sqrt{8\pi/N}\bar{\varphi}}\bar{\psi},$$
$$\bar{J}^z = -i\sqrt{N/2\pi}\partial_{\bar{z}}\bar{\varphi}, \quad (3)$$

where ψ, ψ^+ are chiral parafermion fields and $\varphi, \bar{\varphi}$ are chiral components of the bosonic scalar field $\Phi = \varphi + \bar{\varphi}$ governed by the Gaussian action

$$S = \frac{1}{2}\int d^2x(\partial_\mu\Phi)^2. \quad (4)$$

From (2) and (3), one can deduce expressions for the two- and multipoint correlators of the parafermion fields which for $N > 2$ reveal their nontrivial braiding properties. For the two-point functions, we have

$$\langle\langle\psi(z)\psi^+(0)\rangle\rangle \sim z^{-2(1-1/N)},$$
$$\langle\langle\bar{\psi}(\bar{z})\bar{\psi}^+(0)\rangle\rangle \sim \bar{z}^{-2(1-1/N)}. \quad (5)$$

For the $2n$-point correlation functions, we have the identity

$$\langle\psi(1)\cdots\psi(n)\psi^+(n+1)\cdots\psi^+(2n)\rangle$$
$$= \langle J^+(1)\cdots J^+(n)J^-(n+1)\cdots J^-(2n)\rangle$$
$$\times \prod_{i<j\leq n} z_{ij}^{-2/N} \prod_{n<i<j\leq 2n} z_{ij}^{-2/N} \prod_{i,j\leq n} z_{i,j+n}^{2/N}, \quad (6)$$

which shows that for $N > 2$ multipoint correlators of parafermions do not satisfy Wick's theorem.

In a direct analog to the Majorana fermions, one can introduce a mass term for the Z_N parafermions. The corresponding action is

$$S = Z_N[\psi,\bar{\psi}] - \lambda\int d^2x[\psi\bar{\psi} + \psi^+\bar{\psi}^+], \quad (7)$$

where the Z_N term describes the critical part of the parafermion action. The mass term changes the long distance asymptotics of the parafermion correlation functions but not their braiding properties. For $N > 2$, this is an interacting theory, though its properties can be studied since it is integrable [18].

In the $N = 2$ case, it is easy to study a situation where λ is coordinate dependent. It is well known that when $\lambda(x)$ changes sign (a kink) the Schrödinger equation has a zero energy solution where the eigenfunction is localized at the kink (zero energy Majorana bound state). An important question is whether such bound states exist for $N > 2$. Here I suggest an indirect approach demonstrating existence of the parafermion zero energy modes.

Let us consider the fermionic model with the Hamiltonian density

$$\mathcal{H}_f = i(-R^+_{k\alpha}\partial_x R_{k\alpha} + L^+_{k\alpha}\partial_x L_{k\alpha})$$
$$+ g_\parallel J^z\bar{J}^z + \frac{g_\perp}{2}(J^+\bar{J}^- + J^-\bar{J}^+). \quad (8)$$

This fermionic model was solved by the Bethe ansatz for $g_\perp = g_\perp$ in Ref. [19] and for the general case in Ref. [20]. The subsequent discussion will rely on this solution whose main logic I will discuss in some detail.

I have started with the fermionic model, because fermions constitute elementary particles and therefore fermionic models present a more natural starting point for our consideration. Parafermions exist as collective excitations of many-body fermionic theories and are in that sense secondary objects.

To build a bridge from model (8) to models of parafermions, I will use conformal embedding. Conformal embedding defines "fractionalization rules" for breaking up free fermion Hamiltonians into sums of commuting Hamiltonians of different critical models [21]. The required embedding is

$$\mathrm{U}(2N) = \mathrm{U}(1) \oplus \mathrm{SU}_2(N) \oplus \mathrm{SU}_N(2), \quad (9)$$

which means that the free fermionic Hamiltonian with $\mathrm{U}(2N)$ symmetry can be written as a sum of three commuting Hamiltonians—one Gaussian model and two Wess-Zumino-Novikov-Witten (WZNW) models, the $\mathrm{SU}_N(2)$ and the $\mathrm{SU}_2(N)$ one:

$$\int dxi(-R^+_{k\alpha}\partial_x R_{k\alpha} + L^+_{k\alpha}\partial_x L_{k\alpha})$$
$$= H_{\mathrm{Gauss}} + W[\mathrm{SU}_N(2)] + W[\mathrm{SU}_2(N)]. \quad (10)$$

The current-current interaction commutes with Hamiltonians H_{Gauss} and $W[\mathrm{SU}_2(N)]$. Hence the corresponding sectors of the original fermionic model (8) remain gapless. The interacting sector is described by the $\mathrm{SU}_N(2)$ WZNW model perturbed by the anisotropic current-current interaction. The corresponding Hamiltonian density is

$$\mathcal{H} = \frac{2\pi}{N+2}(:J^aJ^a: + :\bar{J}^a\bar{J}^a:)$$
$$+ g_\parallel J^z\bar{J}^z + \frac{g_\perp}{2}(J^+\bar{J}^- + J^-\bar{J}^+). \quad (11)$$

This is exactly the theory which we can rely to the parafermions. At $g_\parallel > 0$ the theory is massive and has solitons and antisolitons. As is evident from the exact solution, each (anti)soliton carries a parafermion zero energy mode which supplies it with the non-Abelian statistics. The corresponding S matrix in the soliton sector is a tensor product of the XXZ S matrix (the scattering matrix of the sine Gordon model) and the restricted solid-on-solid (RSOS) one [22]. At sufficiently large $g_\parallel$, it also has soliton-antisoliton bound states.

It turns out that we can use one more conformal embedding. Namely, the Lagrangian density for Hamiltonian (11) can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^2 + Z_N[\psi,\bar{\psi}] - \lambda(e^{i\beta\Phi}\psi\bar{\psi} + \mathrm{H.c.}), \quad (12)$$

where $\lambda \sim Ng_\perp$ and β is related to $g_\parallel$ so that at small couplings we have $\beta^2 = (1 + Ng_z/4\pi)^{-1}$.

The last term in (12) is similar to the last term in (7), where the role of static function $\lambda(x)$ is played by dynamic field $\exp[i\beta\Phi]$. Since this field changes sign on soliton configuration, one can use model (12) as a substitute for the model of parafermions (7) with a coordinate-dependent mass gap provided one meets certain requirements. First, the solitons must be slow to be considered quasistatic and on average be far from each other. More accurate criteria for these will be extracted from the exact solution. Second, quantum fluctuations of the bosonic exponent should be small so that it will mimic a static $\lambda(x)$ in (7). This requires small β.

The above requirements are met in the following setup. Let us apply a magnetic field (it is coupled to the bosonic sector) whose strength is slightly below the soliton mass threshold. The field breaks the symmetry between the solitons and the antisolitons. The magnitude of the field is slightly below the soliton mass so that

$$T \ll M - H \ll M. \tag{13}$$

At that temperature, we have a rarified gas of thermally excited slow solitons and no antisolitons. The thermal velocity of these solitons is

$$\sqrt{\langle v^2 \rangle} = \sqrt{2T/M} \ll 1, \tag{14}$$

and the density is

$$n \sim e^{-(M-H)/T} \ll 1, \tag{15}$$

so they can exist undisturbed between collisions for the exponentially long time $\tau \sim \exp[(M-H)/T]$.

By looking at the thermodynamic Bethe ansatz (TBA) equations, we can establish whether solitons carry parafermionic zero energy modes. The corresponding TBA describing the soliton sector of the theory in the limit (13) can be extracted, for example, from Ref. [23]. They are a part of a more general system of equations which may contain also massive soliton-antisoliton bound states (see [24]) which are irrelevant for the present discussion. The free energy F of model (12) written in the limit (13)

$$F/L = -TM \int \frac{d\theta}{2\pi} \cosh\theta \ln(1 + e^{\epsilon_N(\theta)/T}), \tag{16}$$

where L is the system size, is expressed in terms of function $\epsilon_N(\theta)$ which is determined by the following system of nonlinear integral equations:

$$\begin{aligned}\epsilon_j &= Ts\circ\ln(1+e^{\epsilon_{j-1}/T})(1+e^{\epsilon_{j+1}/T}) \\ &\quad + Ts\circ\ln(1+e^{\epsilon_N/T})\delta_{j,N-1}, \quad j=1,\ldots,N-1, \end{aligned}\tag{17}$$

$$\begin{aligned}&\epsilon_N - K\circ T\ln(1+e^{\epsilon_N/T}) \\ &= -M\cosh\theta + H + Ts\circ\ln(1+e^{\epsilon_{N-1}/T}) + O(e^{-H/T}).\end{aligned}\tag{18}$$

where kernel K is

$$K(\omega) = \frac{\sinh[\pi(\xi-1)\omega/2]}{2\cosh(\pi\omega/2)\sinh(\pi\xi\omega/2)}, \quad \xi = \frac{1}{8\pi/N\beta^2 - 1},$$

and

$$s\circ f(x) = \int_{-\infty}^{\infty} \frac{dy f(y)}{\pi\cosh(x-y)}.$$

I am interested in limit (13). Then in the first approximation one can replace quasienergies ϵ_j ($j = 1, \ldots, N-1$) by their constant asymptotic values for which the corresponding integral equations (17) become algebraic. The solution is

$$1 + e^{\epsilon_j/T} = \left\{\frac{\sin[\frac{\pi(j+1)}{N+2}]}{\sin(\frac{\pi}{N+2})}\right\}^2. \tag{19}$$

Substituting this into (16), we obtain the following expression for the free energy:

$$\begin{aligned}F/L = &-TQ\int\frac{dp}{2\pi}e^{-(M-H)/T - p^2/2MT} \\ &+ O(\exp[-2(M-H)/T]),\end{aligned}\tag{20}$$

$$Q = 2\cos\left(\frac{\pi}{N+2}\right). \tag{21}$$

This expression describes the free energy of an ideal gas of particles of mass M with a chemical potential H. The prefactor Q indicates that the state of $\mathcal{N}$ particles with given energy is degenerate, so that in the thermodynamic limit the degeneracy is equal to $Q^{\mathcal{N}}$. This degeneracy obviously comes from the parafermionic zero energy modes bound to the solitons. The fact that Q is not an integer is a direct indication that the operators describing zero energy modes attached to different kinks do not commute with each other. For $N = 2$, we reproduce the known result $D(2)_{\mathcal{N}} = 2^{[\mathcal{N}/2]}$ for the dimensionality of the Clifford algebra representation of $\mathcal{N}$ gamma matrices. For $N = 3$, the obtained dimensionality is the large $\mathcal{N}$ asymptotic of Fibonacci numbers:

$$\begin{aligned}\phi &= 2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}, \\ D(3)_{\mathcal{N}} &= [\phi^{\mathcal{N}} - (-\phi)^{-\mathcal{N}}]/\sqrt{5}.\end{aligned}\tag{22}$$

Expression (20) is the first term in the expansion of the free energy in the soliton density and, as I have said, describes the ideal gas of anyons. One can move further and extract from (17) the equations for interacting anyon gas. The interactions lift the ground state degeneracy.

At lowest temperatures, we invert the matrix kernel in (17) to get the equations in the form where the kernel acts on the term which vanish in the $T = 0$ limit:

$$T\ln(1+e^{\epsilon_j/T}) - T\mathcal{A}_{jk}\ln(1+e^{-\epsilon_k/T}) = \mathcal{A}_{j,N-1}\circ s\circ T\ln(1+e^{\epsilon_N/T}), \quad (23)$$

$$j,k = 1,\ldots,N-1, \quad (24)$$

where

$$\mathcal{A}_{jk}(\omega) = 2\coth(\pi\omega/2) \times \frac{\sinh\{\pi[N-\max(j,k)]\omega/2\}\sinh\{\pi\min(j,k)\omega/2\}}{\sinh(N\pi\omega/2)}.$$

At temperatures $T \ll M$, the distribution function in the right-hand side (RHS) of (23) is very sharp and can be replaced by a delta function:

$$\mathcal{A}_{j,N-1}\circ s\circ T\ln(1+e^{\epsilon_N(\theta)/T}) \approx n_{\mathrm{sol}}(T)\mathcal{A}_{j,N-1}\circ s(\theta), \quad (25)$$

where n_{sol} is the number of solitons. Then Eqs. (23) with such a RHS look like the TBA for the ferromagnetic XXZ model with n_{sol} sites and anisotropy $\gamma = \pi/N$ with an additional restriction forbidding solutions with rapidities shifted by $i\pi/2$. Such restricted equations describe the critical RSOS models [25,26] with conformal charge $c = 2(N-1)/(N+2)$. The RHS of Eqs. (25) is $\sim n_{\mathrm{sol}}$, which means that the bandwidth of the excitations of the interacting anyon gas is proportional to the average distance between the solitons $\lambda \sim n_{\mathrm{sol}}^{-1}$. This contradicts a naive expectation that this bandwidth is proportional to the overlap of the zero energy mode wave functions which would be exponentially small in $M\lambda$. Instead, in the model with mobile solitons we have the bandwidth which is related to the time between their collisions.

Conclusions.—In this Letter, I used fermionic integrable model (8) to describe a state where the mass term of Z_N parafermions alternates its sign. In this approach, parafermions exist as collective excitations of the fermionic theory. The analysis of the Bethe ansatz equations shows that the parafermions create zero energy bound states attached to the domain walls (solitons) of a bosonic field. This bosonic field is also a collective degree of freedom related to smooth fluctuations of the spin density, and its solitons exist as dynamic excitations. By fine-tuning the temperature and magnetic field, one can create a situation when the solitons are quasistatic which imitates the desired situation of alternating parafermion mass. When the density of the domain walls is finite, the modes interact with each other which lifts the ground state degeneracy. The characteristic bandwidth of the excitations of this interacting anyon gas is proportional to the inverse collision time of solitons, i.e., to the domain wall density.

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