Ultrarelativistic Electron States in a General Background Electromagnetic Field

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The feasibility of obtaining exact analytical results in the realm of QED in the presence of a background electromagnetic field is almost exclusively limited to a few tractable cases, where the Dirac equation in the corresponding background field can be solved analytically. This circumstance has restricted, in particular, the theoretical analysis of QED processes in intense laser fields to within the plane wave approximation even at those high intensities, achievable experimentally only by tightly focusing the laser energy in space. Here, within the Wentzel-Kramers-Brillouin approximation, we construct analytically single-particle electron states in the presence of a background electromagnetic field of general space-time structure in the realistic assumption that the initial energy of the electron is the largest dynamical energy scale in the problem. The relatively compact expression of these states opens, in particular, the possibility of investigating analytically strong-field QED processes in the presence of spatially focused laser beams, which is of particular relevance in view of the upcoming experimental campaigns in this field.

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The predictions of QED have been confirmed with outstanding precision in numerous experiments. The impressive agreement between the theoretical and the experimental value of the electron $(g-2)$-factor is customarily quoted as a prominent example [1]. However, the experimental scrutiny of QED becomes much less thorough when processes are involved, occurring in the presence of a strong background electromagnetic field, i.e., of the order of $F_c = m^2c^3/\hbar|e| = 1.3\times10^{16}$ V/cm $= 4.4\times10^{13}$ G (here, m and $e<0$ are the electron mass and charge, respectively) [2]. The main reason is that these values largely exceed the field strengths available in laboratories. An important exception is represented by the electric field of highly charged ions (charge number $Z\sim1/\alpha$, with $\alpha = e^2/\hbar c \approx 1/137$) at the typical QED length $\lambda_C = \hbar/mc = 3.9\times10^{-11}$ cm [3,4]. Indeed, numerous experiments on processes occurring in the presence of highly charged ions [5–8] have already successfully confirmed the predictions of QED. Correspondingly, advanced analytical methods [9], have been developed to interpret accurate experimental data beyond the exactly solvable Coulomb model of the ionic field.

Modern high-power lasers represent an alternative source of intense electromagnetic fields structurally thoroughly different from atomic fields. Although the amplitude F_0 of the strongest laser pulse ever produced is about $10^{-4}F_c$ [10], it can be boosted to an effective strength $F_0^*\sim F_c$ in the rest frame of ultrarelativistic particles colliding with the laser beam [11]. This principle has been exploited at SLAC to perform the so-far unique experimental campaign on strong-laser field QED [12], employing a laser with photon energy 1 eV and amplitude $F_0 = 2.7\times10^{10}$ V/cm, and an almost counter-propagating electron beam with energy of 45 GeV ($F_0^*\approx0.3F_c$). The relatively large pulse spot area ($\sim60\ \mu\mathrm{m}^2$) allowed for the experimental results being well reproduced theoretically within the plane wave field approximation.

Approximating the laser field as a plane wave allows one to solve exactly the Dirac equation in the resulting background electromagnetic field [13]. The corresponding electron single-particle states (Volkov states) have been extensively employed to investigate different strong-field QED processes [14–29] (see also the recent reviews [30–32]). Correspondingly, particle in cell (PIC) codes including strong-field QED effects [33–35] in the dynamics of laser-irradiated plasmas employ expressions of the QED rates calculated in the plane wave (local-constant-crossed-field) approximation. However, no analytical calculations in strong-field QED have been performed so far, which also include self-consistently the spatial focusing of the laser beam. This is especially desirable as ultrahigh intensities are attained nowadays by spatially focusing the laser energy almost down to the diffraction limit.

In the present Letter, we determine analytically the electron single-particle states in the presence of a strong background electromagnetic field of general space-time structure in the experimentally relevant case of an ultrarelativistic electron. In the realistic assumption that the initial energy of the electron is the largest dynamical energy scale in the problem, we first determine the classical worldline of the electron and then we construct the corresponding quantum states in the Wentzel-Kramers-Brillouin (WKB) or eikonal approximation [11,36]. The availability of such single-particle states and, in particular, their relatively compact expression open the possibility of investigating analytically and in a systematic way strong-field QED processes in the presence of intense background fields with complex space-time structure as, e.g., those of tightly focused laser beams.

To date, two methods have been developed to investigate QED processes in the presence of a virtually arbitrary background electromagnetic field. However, the first one, based on the quasiclassical operator technique [37,38], allows one to obtain results only at the leading order in the quasiclassical, ultrarelativistic limit and does not contain a general prescription on how to calculate neither the amplitude of a generic QED process nor high-order corrections. The second one, instead, employs the so-called "trajectory-coherent states" (TCS) [39,40], which are relativistic electron wave functions localized near the classical electron's trajectory. However, the expression of the TCS is extremely cumbersome and of limited use for practical calculations.

We first consider the classical problem of an ultra-relativistic electron moving in a background electromagnetic field, described by the four-vector potential $A^\mu(x)$ in the Lorentz gauge $\partial_\mu A^\mu = 0$ (here and below, units with $c=1$ are employed). We work in the laboratory frame where the electron initial four-momentum is $p_0^\mu = (\varepsilon_0, \boldsymbol{p}_0) = (\sqrt{m^2+\boldsymbol{p}_0^2}, \boldsymbol{p}_0)$, and we have in mind the case where the background electromagnetic field represents an intense, short, and tightly focused laser beam. Thus, we also assume that the field tensor $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = (\boldsymbol{E}(x), \boldsymbol{B}(x))$ is localized in space and time, that it has a maximum amplitude F_0, and that it is characterized by a typical angular frequency ω_0, such that the classical nonlinearity parameter $\xi_0 = |e|F_0/m\omega_0$ [see [41] for a manifestly covariant and gauge-invariant definition of the parameter ξ_0 (see also [15])] satisfies the strong inequalities: $m \ll m\xi_0 \ll \varepsilon_0$. The above assumptions well fit present and near-future experimental conditions envisaged to test strong-field QED with intense lasers. In fact, even next generation of 10 PW Ti:sapphire lasers [42] (central wavelength $\lambda_0 = 0.8\ \mu$m) are realistically expected not to exceed a peak intensity of $I_0 \sim 10^{23}\ \mathrm{W/cm^2}$, corresponding to $F_0 \sim 6\times 10^{12}\ \mathrm{V/cm} = 2\times 10^{10}\ \mathrm{G}$, $\xi_0 \sim 160$, and $m\xi_0 \sim 80$ MeV. Such a field amplitude is effectively boosted to the critical value F_c in the rest frame of an electron with an energy of $\varepsilon_0 \gtrsim 500$ MeV, which is about 6 times $m\xi_0$. In addition, electron beams with energies of about 2 GeV have been already demonstrated experimentally also with laser-plasma accelerators [43]. Our starting point is the Lorentz equation: $dp^\mu/ds = (e/m)F^{\mu\nu}p_\nu$, where $p^\mu = (\varepsilon, \boldsymbol{p}) = (\sqrt{m^2+\boldsymbol{p}^2}, \boldsymbol{p})$ is the electron four-momentum and s is its proper time. According to the analytical solution of the Lorentz equation in a plane wave [11], the condition $m\xi_0 \ll \varepsilon_0$ in the laboratory frame ensures that the electron will be only slightly deflected from its initial direction by the background field in the physically relevant situation where it is initially counterpropagating with respect to the laser field. Thus, rather than working with manifestly covariant equations, it is convenient to introduce the light cone coordinates $\phi = t - \boldsymbol{n}\cdot\boldsymbol{x}$, $\tau = (t+\boldsymbol{n}\cdot\boldsymbol{x})/2$, and $\boldsymbol{x}_\perp = \boldsymbol{x} - (\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}$, with $\boldsymbol{n} = \boldsymbol{p}_0/|\boldsymbol{p}_0|$, and the quantities $n^\mu = (1, \boldsymbol{n})$, $\tilde{n}^\mu = (1/2)(1, -\boldsymbol{n})$, and $a_j^\mu = (0, \boldsymbol{a}_j)$, with $j = 1, 2$ (in this respect, see also [44], where vacuum QED has been formulated by employing light cone coordinates in the so-called "infinite-momentum frame"). The quantities $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ introduced above are two unit vectors perpendicular to $\boldsymbol{n}$ and to each other, and such that $\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \boldsymbol{n}$. An arbitrary four-vector $v^\mu = (v^0, \boldsymbol{v})$ can be expressed as: $v^\mu = v_+ n^\mu + v_- \tilde{n}^\mu + v_1 a_1^\mu + v_2 a_2^\mu$, where $v_+ = (\tilde{n}v) = (v^0 + \boldsymbol{n}\cdot\boldsymbol{v})/2$, $v_- = (nv) = v^0 - \boldsymbol{n}\cdot\boldsymbol{v}$, and $v_j = -(a_j v) = \boldsymbol{a}_j\cdot\boldsymbol{v}$ (note that $a_j^2 = -\boldsymbol{a}_j^2 = -1$). In the original light cone notation [45], the direction $\boldsymbol{n}$ was chosen as the "third" one, i.e., $\boldsymbol{n} = (0,0,1)$. However, for the sake of convenience in the use of the final results, we prefer to keep $\boldsymbol{n}$ as an arbitrary unit vector.

The on shell condition $p^2 = m^2$ implies that $p_- = (m^2 + \boldsymbol{p}_\perp^2)/2p_+$ and, in the physical situation of interest here, we require that the condition $|\boldsymbol{p}_\perp| \sim m\xi_0 \ll p_+$ is satisfied in the laboratory frame, i.e., that the quantity $p_+ \approx \varepsilon$ is the largest dynamical energy scale in the problem. By parametrizing the electron trajectory via the "time" τ, the three independent components of the Lorentz equation can be written in the convenient form

$$\frac{dp_+}{d\tau} = eE_n + \frac{e}{2}\frac{\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp}{p_+}, \tag{1}$$

$$\frac{d\boldsymbol{p}_\perp}{d\tau} = e\boldsymbol{F}_p - eB_n\frac{\boldsymbol{n}\times\boldsymbol{p}_\perp}{p_+} + \frac{e}{4}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+^2}\boldsymbol{F}_m, \tag{2}$$

where the light cone components of the field tensor have been expressed in terms of the electromagnetic field as $F_{\tilde{n},n} = \tilde{n}_\mu F^{\mu\nu} n_\nu = \boldsymbol{n}\cdot\boldsymbol{E} = E_n$, $F_{\tilde{n},j} = \tilde{n}_\mu F^{\mu\nu} a_{j,\nu} = \boldsymbol{a}_j\cdot\boldsymbol{F}_m/2$, $F_{n,j} = n_\mu F^{\mu\nu} a_{j,\nu} = \boldsymbol{a}_j\cdot\boldsymbol{F}_p$, and $F_{1,2} = a_{1,\mu}F^{\mu\nu}a_{2,\nu} = -\boldsymbol{n}\cdot\boldsymbol{B} = -B_n$, with $\boldsymbol{F}_{p/m} = \boldsymbol{E}_\perp \pm \boldsymbol{n}\times\boldsymbol{B}_\perp$. The idea now is to solve Eqs. (1)–(2) iteratively by exploiting the appearance of different powers of the small quantity $|\boldsymbol{p}_\perp|/p_+$. We assume that the light cone components of $F^{\mu\nu}$ have all the same order of magnitude and that the relative size of each term is determined by the power of the quantity $1/p_+$. If this is not the case, in fact, a careful analysis is required, as the mentioned hierarchy could be altered. This is expected to occur more likely in the idealized case of highly symmetric fields. For example, for a constant and uniform magnetic field perpendicular to $\boldsymbol{n}$, it is $B_n \equiv 0$ and the term proportional to $1/p_+$ in Eq. (2) vanishes, unlike the one proportional to $1/p_+^2$. We also note that for a tightly focused Gaussian beam counterpropagating with respect to the electron, it is $|E_n| \sim 0.1|\boldsymbol{F}_p|$ [46,47].

Now, the field components in Eqs. (1)–(2) are calculated along the electron's trajectory. Since $p^\mu = p_+ dx^\mu/d\tau$, the following exact equations for the electron's "spatial" coordinates $\boldsymbol{r}(\tau) = (\phi(\tau), \boldsymbol{x}_\perp(\tau))$ as functions of τ can be derived:

$$\frac{d\phi}{d\tau} = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p_+^2}, \tag{3}$$

$$\frac{d^2\boldsymbol{x}_\perp}{d\tau^2} = e\frac{\boldsymbol{F}_p}{p_+} - eE_n\frac{\boldsymbol{p}_\perp}{p_+^2} - eB_n\frac{\boldsymbol{n}\times\boldsymbol{p}_\perp}{p_+^2} + \frac{e}{4}\frac{m^2+\boldsymbol{p}_\perp^2}{p_+^3}\boldsymbol{F}_m - \frac{e}{2}(\boldsymbol{F}_m\cdot\boldsymbol{p}_\perp)\frac{\boldsymbol{p}_\perp}{p_+^3}. \tag{4}$$

We set the initial conditions at a given time $\tau_0 = (t_0 + \boldsymbol{n}\cdot\boldsymbol{x}_0)/2$ as $\boldsymbol{r}(\tau_0) = \boldsymbol{r}_0 = (\phi_0, \boldsymbol{x}_{0,\perp})$, with $\phi_0 = t_0 - \boldsymbol{n}\cdot\boldsymbol{x}_0$, and as $p_+(\tau_0) = p_{0,+}$ [recall that, by definition, $\boldsymbol{p}_\perp(\tau_0) = \boldsymbol{p}_{0,\perp} = \boldsymbol{0}$ and that the on shell condition implies that $p_-(\tau_0) = p_{0,-} = m^2/2p_{0,+}$]. We also assume that $A^\mu(\tau_0, \boldsymbol{r}) = 0$, with $\boldsymbol{r} = (\phi, \boldsymbol{x}_\perp)$.

By denoting the quantities calculated up to terms proportional to $1/p_{0,+}$ via the upper index (1), we have that

$$\boldsymbol{r}^{(1)}(\tau) = \left(\phi_0, \boldsymbol{x}_{0,\perp} + \frac{e}{p_{0,+}}\int_{\tau_0}^{\tau} d\tau' \boldsymbol{G}_p(\tau', \boldsymbol{r}_0)\right), \tag{5}$$

where $\boldsymbol{G}_p(\tau, \boldsymbol{r}_0) = \int_{\tau_0}^{\tau} d\tau' \boldsymbol{F}_p(\tau', \boldsymbol{r}_0)$.

By substituting the expression of $\boldsymbol{r}^{(1)}(\tau)$ in Eqs. (1)–(2) and by integrating them, we obtain that

$$p_+^{(1)}(\tau) = p_{0,+} + \int_{\tau_0}^{\tau} d\tau' \bigg[eE_n(\tau', \boldsymbol{r}_0) + \frac{e^2}{p_{0,+}}\boldsymbol{G}_p(\tau', \boldsymbol{r}_0)\cdot\nabla_\perp\int_{\tau'}^{\tau} d\tau'' E_n(\tau'', \boldsymbol{r}_0) + \frac{e^2}{2p_{0,+}}\boldsymbol{F}_m(\tau', \boldsymbol{r}_0)\cdot\boldsymbol{G}_p(\tau', \boldsymbol{r}_0)\bigg], \tag{6}$$

$$\boldsymbol{p}_\perp^{(1)}(\tau) = \int_{\tau_0}^{\tau} d\tau' \bigg[e\boldsymbol{F}_p(\tau', \boldsymbol{r}_0) + \frac{e^2}{p_{0,+}}\boldsymbol{G}_p(\tau', \boldsymbol{r}_0)\cdot\nabla_\perp\int_{\tau'}^{\tau} d\tau'' \boldsymbol{F}_p(\tau'', \boldsymbol{r}_0) - \frac{e^2}{p_{0,+}}B_n(\tau', \boldsymbol{r}_0)(\boldsymbol{n}\times\boldsymbol{G}_p(\tau', \boldsymbol{r}_0))\bigg], \tag{7}$$

$$p_-^{(1)}(\tau) = \frac{m^2 + e^2\boldsymbol{G}_p^2(\tau, \boldsymbol{r}_0)}{2p_{0,+}}. \tag{8}$$

The condition $|\boldsymbol{p}_\perp^{(1)}(\tau)| \ll p_+^{(1)}(\tau)$ in the laboratory frame ensures that our approximated solution is accurate [see Eqs. (1)–(2)] and it is fulfilled if $|e\boldsymbol{G}_p(\tau, \boldsymbol{r}_0)| \ll p_{0,+}$. Since tightly focused laser pulses are usually localized in a space (time) region of the order of a few laser central wavelengths (periods), the above condition is equivalent in order of magnitude to the requirement $m\xi_0 \ll \varepsilon_0$ in the relevant case of an electron initially counterpropagating with respect to the laser beam. In order to highlight the qualitative novelties in the theoretical predictions brought about by the inclusion of the laser spatial focusing, in Fig. 1, we plot the momentum p_z in units of m as a function of the quantity $\omega_0 t$ for an electron (initial conditions $\boldsymbol{x}_0 = (0, 0, 1.5\ \mu\mathrm{m})$ and $\boldsymbol{p}_0 = (0, -260\ \mathrm{MeV}, 0)$ at $t_0 = 0$) initially counterpropagating with respect to a Ti:sapphire, Gaussian, $\sin^2$-pulse beam [46], linearly polarized along the z direction with duration $T = 16$ fs, spot radius $w_0 = 1.5\ \mu$m, and peak intensity $I_0 = 5.7\times 10^{22}$ W/cm^2 ($\xi_0 = 110$). The continuous (dashed) line indicates the results of the numerical integration of the Lorentz equation neglecting (including) the beam spatial focusing. The dotted line, on top of the dashed one, corresponds to the analytical result from Eq. (7) (discrepancies between the numerical and the analytical results arise at the third significant digit). The final value of p_z in the case of the Gaussian beam is 1 MeV, whereas it vanishes within the plane wave approximation (see the inset in Fig. 1), according to the Lawson-Woodward theorem (see, e.g., [48]). Note that the corresponding divergence of 4 mrad is larger than already demonstrated electron beam divergences (e.g., of 0.45 mrad at a beam energy of 245 MeV [49]). In general, by setting $\tau\to\infty$ in Eqs. (6)–(8), the relation between the final four-momentum $p^{(1),\mu}(\infty)$ and the initial one p_0^μ can be obtained. Another qualitatively new feature brought about by the laser focusing in the above-mentioned physical setup is that the quantity $p_+(\tau)$ is no more a constant of motion as in the plane wave case and, indeed, the correction to the plane wave result depends on the longitudinal electric field of the laser [see Eq. (6)].

It is interesting to observe that by keeping only the leading-order term of each component of the four-momentum obtained above (i.e., by approximating $p_+(\tau)\approx p_{0,+}$, $\boldsymbol{p}_\perp(\tau)\approx e\boldsymbol{G}_p(\tau, \boldsymbol{r}_0)$, and $p_-(\tau)\approx[m^2 + e^2\boldsymbol{G}_p^2(\tau, \boldsymbol{r}_0)]/2p_{0,+}$), the corresponding expression coincides with the exact four-momentum of an electron in a background plane wavelike field depending on τ and calculated at the initial coordinates $\boldsymbol{r}_0$. This is in agreement with the classical result that an ultrarelativistic particle "sees" an arbitrary background field in its rest frame as a plane wavelike field at leading order [11]. Note that, although it might be more convenient to express the four-momentum obtained above

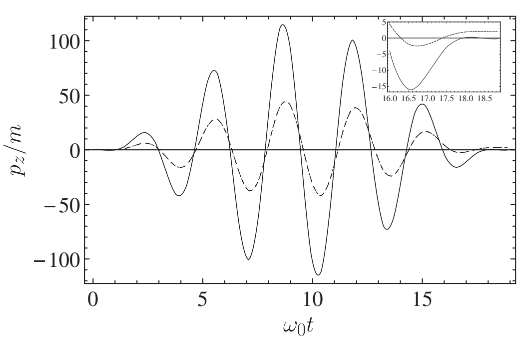

FIG. 1. Electron transverse momentum p_z in units of the electron mass as a function of $\omega_0 t$ for numerical values and details given in the text.

in terms of the four-potential $A^\mu(x)$ as in the plane wave case, we prefer to employ the physical observable electromagnetic field.

We pass now to the quantum case and we consider the Dirac equation $[\gamma^\mu(i\hbar\partial_\mu - eA_\mu) - m]\psi = 0$, where γ^μ are the Dirac matrices and $\psi(x)$ is the bispinor electron wave function [2]. Based on the general argument that the de Broglie length of an ultrarelativistic particle is very small, we apply the WKB method [36] and look for a solution of the form $\psi(x) = \exp[iS(x)/\hbar]\varphi(x)$, where $S(x)$ turns out to be the classical electron action [50–52]. For an electron with initial four-momentum p_0^μ and spin quantum number σ_0, the positive-energy wave function $\psi^{(1)}_{p_0,\sigma_0}(\tau,\boldsymbol{r})$ up to the first order in $1/p_{0,+}$ reads (see the Supplemental Material [53] for a detailed derivation):

$$\psi^{(1)}_{p_0,\sigma_0}(\tau,\boldsymbol{r}) = e^{iS^{(1)}_{p_0}(\tau,\boldsymbol{r})/\hbar}\left(1 - \frac{e}{2}\int_{\tau_0}^{\tau}\frac{d\tau'}{p_{0,+}}\{\nabla_\perp\cdot\boldsymbol{G}_p(\tau',\boldsymbol{r}) - i\Sigma\cdot[\boldsymbol{B}(\tau',\boldsymbol{r}) - \boldsymbol{n}\times\boldsymbol{E}(\tau',\boldsymbol{r})]\}\right)\frac{u_{p_0,\sigma_0}}{\sqrt{2\varepsilon_0}}, \quad (9)$$

where

$$S^{(1)}_{p_0}(\tau,\boldsymbol{r}) = -p_{0,+}\phi - \frac{m^2}{2p_{0,+}}\tau - \int_{\tau_0}^{\tau}d\tau'\left[eA_-(\tau',\boldsymbol{r}) + \frac{e^2}{2p_{0,+}}\boldsymbol{G}_p^2(\tau',\boldsymbol{r})\right], \quad (10)$$

where $\Sigma = -i\gamma^1\gamma^2\gamma^3\boldsymbol{\gamma}$, where u_{p_0,σ_0} is the usual constant free bispinor [2], and where a unity quantization volume is assumed. In the Supplemental Material [53], it is shown that in the relevant case of a strong ($\xi_0 \gg 1$), tightly focused ($w_0 \approx \lambda_0$), and short ($T \sim \lambda_0$) optical ($\lambda_0 \sim 1\ \mu$m) laser field, the only restrictive condition for the validity of the wave function $\psi^{(1)}_{p_0,\sigma_0}(\tau,\boldsymbol{r})$ is the classical one $m\xi_0 \ll \varepsilon_0$. The wave function $\psi^{(1)}_{p_0,\sigma_0}(\tau,\boldsymbol{r})$ reduces to the one obtained in Ref. [54] in the particular case of a background time-independent scalar potential (see also Ref. [55]). Also, ultrarelativistic wave functions for scalar particles [56] and two-particles scattering amplitudes [57,58] have been derived in the context of high-energy scattering in QED in the leading-order eikonal approximation, which corresponds in our notation to neglect terms proportional to $1/p_{0,+}$. However, keeping these terms is essential here, e.g., to recover the plane wave results. In fact, if the background field is a plane wave field depending on τ, the state in Eq. (9) coincides within our approximations with the corresponding Volkov state [2]. In this respect, we note that the average spin $\hbar\boldsymbol{\zeta}$ [see the discussion below Eq. (11) in the Supplemental Material [53]] in a Volkov state describing an electron initially counterpropagating with respect to a linearly polarized plane wave never acquires a component along the magnetic field of the plane wave if it is initially along the electron momentum [2]. Whereas, by employing the wave function $\psi^{(1)}_{p_0,\sigma_0}(\tau,\boldsymbol{r})$, this does occur in the case of a focused laser field. In the particular setup mentioned below Eq. (8), the average spin acquires an x component $\hbar\zeta^{(1)}_x(\tau,\boldsymbol{r}) = (e\hbar/p_{0,+})\int_{\tau_0}^{\tau}d\tau'[E_x(\tau',\boldsymbol{r}) - B_z(\tau',\boldsymbol{r})]$, which vanishes identically in the corresponding plane wave case.

The negative-energy electron states $\psi^{(1)}_{-p_0,-\sigma_0}(\tau,\boldsymbol{r})$ can be obtained via the substitutions $p_0^\mu \to -p_0^\mu$ and $\sigma_0 \to -\sigma_0$ in Eq. (9) except that in $1/\sqrt{2\varepsilon_0}$, with the resulting quantity $u_{-p_0,-\sigma_0}$ being the free negative-energy constant bispinor [2]. In or out states are obtained by performing the limit $\tau_0 \to \mp\infty$ in Eq. (9) and in the action $S_{p_0}(\tau,\boldsymbol{r})$, with the quantum numbers p_0 and σ_0 corresponding to the asymptotic four-momentum and spin outside the field at $\tau_0 \to \mp\infty$.

Once the single-particle positive and negative energy, in and out states $\psi^{(\mathrm{in/out})}_{\pm p,\pm\sigma}(x)$ in ordinary coordinates have been determined (the upper index (1) from the single-particle states has been removed for the sake of notational simplicity), the matrix element $M_{f,i}$ of a typical process as nonlinear Compton scattering can be calculated as (see, e.g., Eq. (4.1.32) in [59])

$$M_{f,i} = -ie\sqrt{\frac{2\pi}{\omega}}\int d^4x\bar{\psi}^{(\mathrm{out})}_{p_f,\sigma_f}(x)\hat{e}^*_{k,\lambda}\psi^{(\mathrm{in})}_{p_i,\sigma_i}(x)e^{i(kx)}. \quad (11)$$

Here, the quantities $p_{i/f}$ and $\sigma_{i/f}$ characterize the initial/final electron, whereas the emitted photon has four-momentum $k^\mu = (\omega,\boldsymbol{k})$ and polarization four-vector $(e_{k,\lambda})^\mu$ ($\hat{e}^*_{k,\lambda} = \gamma^\mu(e^*_{k,\lambda})_\mu$). A semiquantitative analysis of the matrix element $M_{f,i}$ already reveals new features in the focused-field case with respect to the plane wave one (and also to the locally constant-crossed field one, which is relevant for PIC codes). First, unlike that in the plane wave case, we can introduce here the concept of transverse formation region(s) of radiation with respect to the laser propagation direction, analogous to the concept of "impact parameter" in, e.g., electron-nucleus collision [54]. In the quasiclassical limit, this can be physically understood as, unlike that in a plane wave, electron trajectories in a focused field differing only by the initial transverse position contribute in general with different phases to the radiation process. Now, Eqs. (9)–(10) show that $\psi^{(\mathrm{in/out})}_{p_{i/f},\sigma_{i/f}}(x) \sim \exp[iS^{(\mathrm{in/out})}_{p_{i/f}}(x)/\hbar]\, u_{p_{i/f},\sigma_{i/f}}$, with $\lim_{t\to\mp\infty}S^{(\mathrm{in/out})}_{p_{i/f}}(x) = -(p_{i/f}x)$. In the quasiclassical, ultrarelativistic regime at $\xi_0 \gg 1$, the matrix element $M_{f,i}$ can be evaluated approximately via the saddle-point method (see, e.g., [32]). For any saddle point x_l characterized by the conditions $\Delta\pi^\mu(x_l) = \pi_f^{(\mathrm{out})\mu}(x_l) + \hbar k^\mu - \pi_i^{(\mathrm{in})\mu}(x_l) = 0$, with $\pi^{(\mathrm{in/out})\mu}_{i/f}(x) = -\partial^\mu[S^{(\mathrm{in/out})}_{p_{i/f}}(x)]$, one can estimate the transverse formation regions l_w, with $w = \{x,z\}$, from the resulting quadratic term in $s(x_w - x_{l,w})^2$ in the exponent as $l_w \sim \sqrt{\hbar/|\partial\Delta\pi_w/\partial x_w|}$, where all quantities are calculated at x_l. Moreover, contrary to a plane wave, a focused field can transfer momentum to the electron in principle along any direction. The four-momentum transfer $\Delta\pi^{\mu}_{\text{field}}(x_l)$ at each emission point x_l from the field can be estimated from the relations $0 = \Delta\pi^{\mu}(x_l) = p^{\mu}_f + \hbar k^{\mu} - p^{\mu}_i - \Delta\pi^{\mu}_{\text{field}}(x_l)$ and by employing the classical solution in Eqs. (6)–(8). Finally, the focusing of the laser is expected to alter also the electron emission spectrum. This can be already anticipated by estimating the "instantaneous" classical cutoff emission frequency $\omega_c \sim \varepsilon^3/m^3\rho$, where ρ is the curvature radius of the electron trajectory at the instant of emission [11]. By calculating ρ from Eqs. (6)–(8), one estimates $\omega_c \sim (|e|\varepsilon_0^2/m^5)\times \sqrt{(E_z + B_x)^2 + (E_x - B_z)^2}$, with the second term inside the square root vanishing identically in the plane wave case.

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