Fundamental Bounds in Measurements for Estimating Quantum States

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Quantum measurement unavoidably disturbs the state of a quantum system if any information about the system is extracted. Recently, the concept of reversing quantum measurement has been introduced and has attracted much attention. Numerous efforts have thus been devoted to understanding the fundamental relation of the amount of information obtained by measurement to either state disturbance or reversibility. Here, we experimentally prove the trade-off relations in quantum measurement with respect to both state disturbance and reversibility. By demonstrating the quantitative bound of the trade-off relations, we realize an optimal measurement for estimating quantum systems with minimum disturbance and maximum reversibility. Our results offer fundamental insights on quantum measurement and practical guidelines for implementing various quantum information protocols.

Quantum measurement changes the state of a system to another if it provides any information about the system [1,2]. This intriguing feature of quantum mechanics is linked to fundamental quantum properties, such as the uncertainty principle and the no-cloning theorem [3,4]. It has been widely believed that the more information is extracted by measurement, the more disturbed the system is. This implies that there exists a fundamental trade-off relation between the amount of information obtained by measurement and the degree of state disturbance, and it has been the subject of extensive theoretical studies [5–11]. On the other hand, it has recently been reported that quantum measurement may be reversed, if the interaction between the system and the measurement apparatus is weak [12,13]. It has been shown that reversibility, defined by the success probability of reversing a quantum measurement, decreases as the amount of information obtained increases [14]. In this context, the theoretical bound between information gain and reversibility has been recently derived [14]. Note that quantum measurement reversal has been shown to have important applications in quantum information processing [15–18].

In this Letter, we demonstrate trade-off relations in quantum measurement for discrete high-dimensional quantum systems. To the best of our knowledge, our result is the first experimental proof of the information bound with respect to both state disturbance and reversibility in a discrete high-dimensional quantum system, while previous studies have focused separately on disturbance [19–21] or reversibility [22,23] in qubit systems. Furthermore, we realize an optimal measurement for estimating discrete high-dimensional systems with minimum disturbance and maximum reversibility. Specifically, we show that any measurement inducing minimal disturbance is also maximally reversible, while the converse is not true. Moreover,

we show that, for optimally estimating a high-dimensional quantum state, both the choice of measurement and the guessing strategy are crucial [14]. This is in contrast to qubit systems in which the maximum reversibility can always be achieved regardless of the measurement performed if the guessing strategy is optimal. For a continuous variable system, the bound between information gain and state disturbance has so far been demonstrated for coherent states [24].

Consider the measurement process for estimating a quantum state. An arbitrary pure initial state $|\psi\rangle$, i.e., qudit, is prepared in the d-dimensional Hilbert space. Here, we assume that we do not have any prior information about the state. We then perform a measurement, aiming to estimate $|\psi\rangle$. A measurement can be described by a set of operators $\hat{M}_r$, satisfying the completeness relation $\sum_{r=1}^{N} \hat{M}_r^{\dagger}\hat{M}_r = \mathbb{1}$, where $\mathbb{1}$ is an identity and $r = 1, \ldots, N$ corresponds to the measurement outcomes. After the measurement, $|\psi\rangle$ becomes $|\psi_r\rangle = \hat{M}_r|\psi\rangle / \sqrt{\langle\psi|\hat{M}_r^{\dagger}\hat{M}_r|\psi\rangle}$.

The information obtained through the measurement can be quantified by the mean estimation fidelity G [5]. When the measurement outcome is r, we guess that $|\psi\rangle$ was $|\phi_r\rangle$. Then, the quality of the guess can be assessed by the overlap $|\langle\psi|\phi_r\rangle|^2$. By averaging $|\langle\psi|\phi_r\rangle|^2$ over all possible outcomes r for all the pure initial states $|\psi\rangle$, we can evaluate $G = \int d\psi \sum_{r=1}^{N} \langle\psi|\hat{M}_r^{\dagger}\hat{M}_r|\psi\rangle |\langle\psi|\phi_r\rangle|^2$, where $1/d \leq G \leq 2/(d+1)$ and G is determined by both $\hat{M}_r$ and the guessing strategy to choose $|\phi_r\rangle$.

The state disturbance can be quantified by the mean operation fidelity, evaluated by averaging $|\langle\psi|\psi_r\rangle|^2$ for all outcomes r, as $F = \int d\psi \sum_{r=1}^{N} |\langle\psi|\hat{M}_r|\psi\rangle|^2$, where $1/d \leq F \leq 1$ [5]. The general measurement process is illustrated in Fig. 1(a).
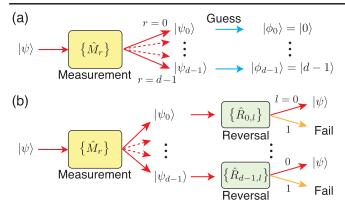
FIG. 1 (color online). (a) Measurement $\{\hat{M}_r\}$ is performed for estimating an unknown initial state $|\psi\rangle$. Information about $|\psi\rangle$ can be extracted by making a guess according to the measurement outcome r. However, $|\psi\rangle$ is inevitably disturbed by the measurement. (b) The post-measurement state $|\psi_r\rangle$ can be probabilistically reversed to the initial state $|\psi\rangle$ by applying the reversing measurement $\{\hat{R}_{r,l}\}$.

We now introduce the reversing operation of $\hat{M}_r$, which can restore $|\psi\rangle$ from $|\psi_r\rangle$. The reversing operator $\hat{R}_{r,0}$ is defined by $\hat{R}_{r,0}\hat{M}_r|\psi\rangle = \eta_r|\psi\rangle$, with a nonzero complex variable η_r. The success probability of the reversing operation is $P_{\mathrm{rev}}(r) = |\eta_r|^2$ [13]. $\hat{R}_{r,0}$ comprises a complete set with $\hat{R}_{r,1}$, $\sum_{l=0}^{1}\hat{R}_{r,l}^{\dagger}\hat{R}_{r,l} = \mathbb{1}$. The general measurement and reversing process is illustrated in Fig. 1(b). The reversibility is defined as the total reversal probability over all the outcomes r, $P_{\mathrm{rev}} = \sum_{r=1}^{N}P_{\mathrm{rev}}(r)$ [14].

Let us now consider the trade-off relations between G, F, and P_{rev}. The trade-off relation between G and F for a qudit is known as [5]

$$\sqrt{F - \frac{1}{d+1}} \leq \sqrt{G - \frac{1}{d+1}} + \sqrt{(d-1)\left(\frac{2}{d+1} - G\right)}. \quad (1)$$

As it provides the fundamental bound of state disturbance for a given information gain, the measurement saturating this inequality is known as the minimum disturbance measurement (MDM) [19–21,24]. Likewise, the trade-off relation between G and P_{rev} is given as [14]

$$d(d+1)G + (d-1)P_{\mathrm{rev}} \leq 2d, \quad (2)$$

providing the quantitative bound of the reversibility for a given information gain, which is fundamentally different from Eq. (1). We will call the measurement saturating this inequality the maximum reversibility measurement (XRM).

Let us now analyze and compare the conditions of MDM and XRM. An arbitrary measurement operator can be represented as $\hat{M}_r = \sum_{i=0}^{d-1}\lambda_i^r\hat{V}_r|\omega_i^r\rangle\langle\omega_i^r|\hat{W}_r$ [5], where $\lambda_0^r \geq \cdots \geq \lambda_{d-1}^r \geq 0$ and $|\omega_i^r\rangle$ are their corresponding eigenvectors. Here, $\hat{V}_r$ and $\hat{W}_r$ are unitary operators.

Vectors $\mathbf{v}_i \equiv (\lambda_i^1, ..., \lambda_i^N)$ for $i = 0, ..., d-1$ can be defined to characterize a set of complete measurement operators $\hat{M}_r$. The MDM set can be obtained if and only if all $\mathbf{v}_i$ are collinear and $|\mathbf{v}_1| = \cdots = |\mathbf{v}_{d-1}|$ [5]. On the other hand, the condition for XRM is different and given as $\hat{M}_r^{\dagger}\hat{M}_r = a_r|\omega_0^r\rangle\langle\omega_0^r| + b_r\mathbb{1}$, with non-negative parameters a_r, b_r [14].

We note here two remarkable observations. First, the maximal reversibility is a necessary condition for the minimal disturbance. We can prove this as follows. Since λ_i^r are arranged in a decreasing order, the MDM condition $|\mathbf{v}_1| = \cdots = |\mathbf{v}_{d-1}|$ means $\lambda_1^r = \cdots = \lambda_{d-1}^r$ for all r. Thus, $\hat{M}_r^{\dagger}\hat{M}_r = \sum_{i=0}^{d-1}(\lambda_i^r)^2|\omega_i^r\rangle\langle\omega_i^r| = \{(\lambda_0^r)^2 - (\lambda_1^r)^2\} \times |\omega_0^r\rangle\langle\omega_0^r| + (\lambda_1^r)^2\mathbb{1}$ so that it satisfies the XRM condition. Therefore, all MDMs are XRMs, but the converse is not true. Second, all the operator sets in two-level systems satisfy the XRM condition, meaning that Eq. (2) becomes an equality. Hence, if we adopt the optimal guessing strategy, we can always achieve the maximal reversibility. However, this is not true for the dimensions higher than two $(d > 2)$, so the choice of the measurement operator set, as well as the guessing strategy, becomes significant to obtain maximal reversibility.

To experimentally demonstrate the above observations, we examine and verify the fundamental quantitative bounds of the $G - F$ and $G - P_{\mathrm{rev}}$ trade-off relations by performing a quantum measurement on a photonic qutrit system. We also demonstrate the implementation of an optimal measurement for estimating a quantum state with minimal disturbance and maximal reversibility. In particular, we consider three cases: (i) $\hat{M}_r^{(1)}$ satisfying both the MDM and XRM conditions, (ii) $\hat{M}_r^{(2)}$ satisfying neither the MDM nor the XRM condition, and (iii) $\hat{M}_r^{(3)}$ satisfying the XRM but not the MDM condition.

The measurement operators satisfying both the MDM and XRM conditions we consider are

$$\hat{M}_r^{(1)} = \sqrt{p}|r\rangle\langle r| + \sqrt{\frac{1-p}{2}}(\mathbb{1} - |r\rangle\langle r|), \quad (3)$$

where $1/3 \leq p \leq 1$ and $r = 0, 1, 2$. Its reversing operator is then given as

$$\hat{R}_{r,0}^{(1)} = \sqrt{\frac{1-p}{2p}}|r\rangle\langle r| + (\mathbb{1} - |r\rangle\langle r|). \quad (4)$$

We assume here to take the optimal guessing strategy, where the initial state is guessed as $|\phi_r\rangle = |r\rangle$ for the outcome r. Then, we obtain $G = (1+p)/4$, $F = (3-p)/4 + \sqrt{p(1-p)/2}$, and the $G - F$ relation, $F = 1 - G + \sqrt{-1 + 6G - 8G^2}$ [5], which saturates Eq. (1) for $d = 3$. In addition, P_{rev} can be calculated from Eq. (4), as

(a) Measurement
State preparation $\hat{M}_r$ Quantum State Tomograpy

(b) Measurement & Reversal
State preparation $\hat{M}_r$ $\hat{R}_{r,l}$ Quantum State Tomograpy

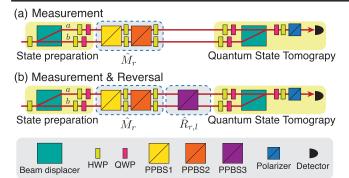Beam displacer   HWP QWP   PPBS1   PPBS2   PPBS3   Polarizer Detector

FIG. 2 (color online). An arbitrary single-photon qutrit state is prepared by using half-wave plates (HWPs), quarter-wave plates (QWPs), and a beam displacer. Quantum measurement and measurement reversal are performed with additional partially polarizing beam splitters (PPBSs), which fully transmit the horizontally polarized photons but partially transmit the vertically polarized photons. The operations of PPBS1, PPBS2, and PPBS3 can be described as $\hat{X}_{PPBS} = |H\rangle\langle H| + \sqrt{t}|V\rangle\langle V|$ with, respectively, $t = p$, $(1-p)/2$, and $(1-p)/2p$. By combining HWPs and PPBSs, a general qutrit measurement operator $\hat{Y} = \sqrt{t_0}|0\rangle\langle 0| + \sqrt{t_1}|1\rangle\langle 1| + \sqrt{t_2}|2\rangle\langle 2|$ can be implemented. Here, (a) and (b) correspond to Figs. 1(a) and 1(b), respectively. G and F are evaluated in (a), while P_{rev} is obtained in (b). This figure shows the experimental setup for implementing $\hat{M}_1^{(1)}$ and $\hat{R}_{1,0}^{(1)}$ described in Eqs. (3) and (4). For other measurement operators, see [27].

$P_{rev} = 3(1-p)/2$. The relation between G and P_{rev} is then $6G + P_{rev} = 3$, which saturates Eq. (2) for $d = 3$ [14].

On the other hand, the measurement operators satisfying neither the MDM nor XRM conditions we consider are

$$\hat{M}_0^{(2)} = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1| + |2\rangle\langle 2|,$$
$$\hat{M}_1^{(2)} = \sqrt{p}|1\rangle\langle 1|, \tag{5}$$

where $0 \leq p \leq 1$. The reversing operator of $\hat{M}_0^{(2)}$ is given by

$$\hat{R}_{0,0}^{(2)} = |1\rangle\langle 1| + \sqrt{1-p}(|0\rangle\langle 0| + |2\rangle\langle 2|). \tag{6}$$

Under the optimal guessing strategy, we obtain $G = (4+p)/12$, $F = (2 + \sqrt{1-p})/3$, $P_{rev} = 1 - p$. Their corresponding relations $F = (2 + \sqrt{5 - 12G})/3$ and $12G + P_{rev} = 5$ do not saturate Eqs. (1) and (2), respectively.

The experimental setup is shown in Fig. 2. We exploit the heralded single-photon qutrit state encoded in the single-photon's path ($|a\rangle$ and $|b\rangle$) and polarization (horizontal $|H\rangle$ and vertical $|V\rangle$). Among the four possible amplitudes, we only excite the three amplitudes $|0\rangle \equiv |a, H\rangle$, $|1\rangle \equiv |a, V\rangle$, and $|2\rangle \equiv |b, H\rangle$ [25,26]. The single-photon source is prepared by the spontaneous parametric down-conversion

process. A 6-mm-thick type I β-BaB$_2$O$_4$ crystal pumped with a 405-nm diode laser generates a pair of 810-nm signal-idler photons.

The experimental setup for implementing $\hat{M}_r$ and $\hat{R}_{r,0}$ is shown in Fig. 2. The final state after the measurement, Fig. 2(a), or the measurement reversal, Fig. 2(b), is analyzed with quantum state tomography (QST). We implement $\hat{M}_r^{(1)}$ and $\hat{M}_r^{(2)}$ for various measurement strengths to demonstrate the trade-off relations. For $\hat{M}_r^{(3)}$, we demonstrate an example in which it is an XRM but not MDM.

To obtain G and F, it is necessary to measure the outcomes for every pure state and average the results. However, if the operation is trace preserving as in this case, the mean fidelities can be calculated by averaging a few pure states which comprise a symmetric basis set [20,28]. For a qutrit, the minimum required number of states is nine, and we used the basis set introduced in Refs. [26,29]. This basis set is also used for QST [26,30,31]. In addition, to confirm that the initial state is retrieved after the reversing operation is carried out, we perform quantum process tomography (QPT) for analyzing the realized operation. See Ref. [27] for the QPT result.

We first examine the trade-off relation between G and F for $\hat{M}_r^{(1)}$ and $\hat{M}_r^{(2)}$. Based on the measurement outcome, we estimate the initial state by using a specific guessing strategy. First, we employ the optimal guessing strategy in which the initial state is guessed as the eigenstate corresponding to the maximal eigenvalues of $\hat{M}_r^{(k)}$ ($k = 1$, 2, 3), i.e., $|\phi_r\rangle = |r\rangle$ [5,22]. The experimental results are plotted in Fig. 3(a), which shows that the more information is extracted by measurement, the more disturbed the state is, irrespectively of the choice of the measurement. The result also verifies the fundamental bound in the $G - F$ relation: While $\hat{M}_r^{(1)}$ saturates the $G - F$ bound, $\hat{M}_r^{(2)}$ does not. When a nonoptimal guessing strategy is adopted [see Fig. 3(b)], the $G - F$ bound cannot be saturated even with $\hat{M}_r^{(1)}$. This remains to be the case for any nonoptimal guessing strategies.

Let us now investigate the trade-off relations between G and P_{rev} for $\hat{M}_r^{(1)}$ and $\hat{M}_r^{(2)}$. The experimental results are presented in Fig. 4(a) for the optimal guessing strategy and in Fig. 4(b) for a nonoptimal one. The results show that P_{rev} decreases as G increases, which is the experimental proof of the $G - P_{rev}$ trade-off relation in a high-dimensional quantum system [14]. We also notice that the fundamental bound for the $G - P_{rev}$ trade-off relation is only reachable by $\hat{M}_r^{(1)}$ with the optimal guessing strategy: an experimental verification of the fact that any MDM operator satisfies the XRM condition.

For the measurement which satisfies the XRM but not the MDM condition, consider the following measurement operator,
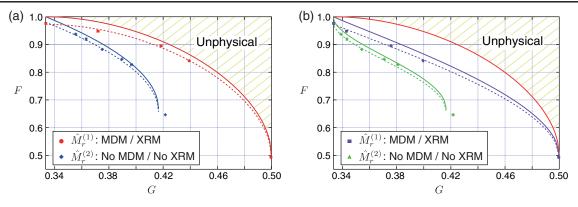
FIG. 3 (color online). An optimal guessing strategy is adopted for (a), and a nonoptimal guessing strategy is adopted for (b). In (b), the state guessing strategy is as follows. For $\hat{M}_r^{(1)}$, we guess the initial state to be $\rho_r^{(\phi)} = p|r\rangle\langle r| + (1-p)/2(\mathbb{1} - |r\rangle\langle r|)$. For $\hat{M}_r^{(2)}$, we guess the initial state to be $\rho_0^{(\phi)} = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ for $r = 0$ and $\rho_1^{(\phi)} = |1\rangle\langle 1|$ for $r = 1$. The solid lines represent the ideal trade-off relations for each operator set. Since the initial states used in experiment are not perfectly pure, the experimental data points lie slightly below the ideal trade-off relations. The dashed lines represent the theoretical trade-off relations assuming a nonideal input state $\rho(r) = r|\psi\rangle\langle\psi| + (1-r)\mathbb{1}/3$. $r = 0.958$ and $r = 0.969$ for $\hat{M}_r^{(1)}$ and $\hat{M}_r^{(2)}$, respectively. The error bars represent the statistical error of ± 1 standard deviation.

$$\hat{M}_0^{(3)} = \sqrt{\frac{1}{3}}|0\rangle\langle 0| + \sqrt{\frac{1}{6}}(|1\rangle\langle 1| + |2\rangle\langle 2|),$$

$$\hat{M}_1^{(3)} = \sqrt{\frac{1}{3}}|1\rangle\langle 1| + \sqrt{\frac{1}{6}}(|2\rangle\langle 2| + |0\rangle\langle 0|),$$

$$\hat{M}_2^{(3)} = \sqrt{\frac{2}{3}}|2\rangle\langle 2| + \sqrt{\frac{1}{2}}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

For $\hat{M}_r^{(3)}$, the theoretical values for information gain, disturbance, and reversibility can be evaluated assuming ideal measurement, and they are calculated to be $G^{\text{th}} = 13/36 \simeq 0.361$, $F^{\text{th}} \simeq 0.988$, and $P_{\text{rev}}^{\text{th}} = 5/6 \simeq 0.833$. For $G^{\text{th}} = 0.361$, the physically allowed maximum disturbance calculated from Eq. (1) is $F \simeq 0.990$. Thus, $\hat{M}_r^{(3)}$ does not saturate the $G-F$ trade-off relation, but it saturates the $G-P_{\text{rev}}$ trade-off relation. The

experimentally obtained values are $G^{\text{exp}} = 0.362 \pm 0.001$, $F^{\text{exp}} = 0.957 \pm 0.002$, and $P_{\text{rev}}^{\text{exp}} = 0.826 \pm 0.003$.

In summary, we have demonstrated the trade-off relations among information gain, disturbance, and reversibility in a qutrit system. To the best of our knowledge, this is the first experimental demonstration of the trade-off relations in discrete variable systems beyond qubits. Our results directly show that state disturbance and reversibility are different. Furthermore, our work shows that both the proper choice of measurement and the optimal guessing strategy are important in achieving an optimal measurement for high-dimensional quantum systems. This is in contrast to qubit systems in which the maximum reversibility can always be achieved regardless of the measurement performed if the guessing strategy is optimal [23]. Our results offer fundamental insights on quantum measurement and practical guidelines for implementing and expanding
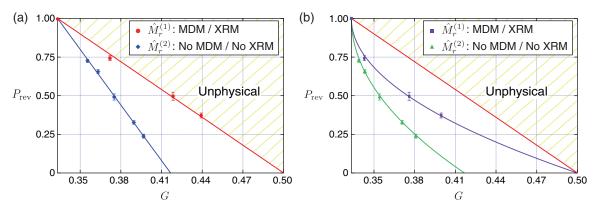

FIG. 4 (color online). (a) When the optimal guessing strategy is adopted, the $G-P_{\text{rev}}$ trade-off relation is linear. The red line is the ideal trade-off relation, $6G + P_{\text{rev}} = 3$, for $\hat{M}_r^{(1)}$. The blue line is the ideal trade-off relation, $12G + P_{\text{rev}} = 5$, for $\hat{M}_r^{(2)}$. (b) When the same nonoptimal guessing strategy adopted in Fig. 3(b) is used, the result shows that P_{rev} cannot reach the bound even with $\hat{M}_r^{(1)}$. The error bars represent the statistical error of ± 1 standard deviation.

various quantum information protocols from qubit to high dimensions.

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