

Information Theoretical Analysis of Quantum Optimal Control

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We study the relations between classical information and the feasibility of accurate manipulation of quantum system dynamics. We show that if an efficient classical representation of the dynamics exists, optimal control problems on many-body quantum systems can be solved efficiently with finite precision. In particular, one-dimensional slightly entangled dynamics can be efficiently controlled. We provide a bound for the minimal time necessary to perform the optimal process given the bandwidth of the control pulse, which is the continuous version of the Solovay-Kitaev theorem. Finally, we quantify how noise affects the presented results.

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Quantum optimal control lies at the heart of the modern quantum revolution, as it allows us to match the stringent requirements needed to develop quantum technologies, to develop novel quantum protocols, and to improve their performances [1]. Along with the increased numerical and experimental capabilities developed in recent years, problems of increasing complexity have been explored and recently a lot of attention has been devoted to the application of optimal control (OC) to many-body quantum dynamics: OC has been applied to information processing in quantum wires [2], the crossing of quantum phase transitions [3], the generation of many-body squeezed or entangled states [4], chaotic dynamics [5], unitary transformations [6]. Recent studies have been devoted to the understanding of the fundamental limits of OC in terms of energy-time relations (time-optimal) [7] and its robustness against perturbations [8,9].

These exciting developments call for a general theoretical framework to understand when and under which conditions is it possible to solve a given OC problem in a many-body quantum system. The aim of this work is to introduce an unified framework to characterize the complexity of OC problems in many-body quantum systems. At present, very little is known about the conditions under which is possible to drive a many-body quantum systems and if these conditions are due to physical, algorithmical or other fundamental limitations that Nature might impose. Indeed, due to the exponential growth of the Hilbert space with the number of constituents, solving exactly an OC problem on a many-body system is in general highly inefficient: the algorithmic complexity (AC) of exact time-optimal problems can be super-exponential [6]. However, limited precision, errors and practical limitations naturally introduce a finite precision both in the functional to be minimized and on the total time of the transformation. The smoothed complexity (SC) has been introduced recently to cope with this situation and to describe the “practical” complexity of solving a problem in the real world with finite precision (see Supplemental Material, Sec. 1 [10]). It has been shown that the SC can be

drastically different from the AC: indeed the AC—which is defined by the scaling of the worst case instance—might be practically irrelevant as this scaling might never be found in practice [12]. A paradigmatic case is that of the simplex algorithm applied to linear programming problems: it is characterized by an exponential AC in the dimension of the searched space; however, the SC is only polynomial; that is the worst case disappears in the presence of perturbations [13].

In this Letter, we perform an information theoretical analysis that allows us to define and discern between different limitations of our capability to control many-body quantum systems: algorithmical, informational, or physical. We identify the complexity of OC problems, and present some interesting classes of problems that can be efficiently solved. We quantify the effects of noise in the control field on these results relating the channel capacity to the minimal possible error that can be achieved. We finally provide an information-time bound, relating the bandwidth of the control field with the minimal time necessary to achieve the optimal transformation.

A quantum OC problem can be stated from a dynamical equation,

$$\dot{\rho} = \mathcal{L}(\rho, \gamma(t)), \quad (1)$$

with boundary condition $\rho(t=0) = \rho_0$, where ρ is the density matrix describing a quantum system defined on a Hilbert space \mathcal{H} , and \mathcal{L} the Liouvillian operator with the unitary part generated by a Hamiltonian,

$$H = H_D + \gamma(t)H_C, \quad (2)$$

where $\gamma(t)$ is a time-dependent control field, and H_D and H_C the drift and control Hamiltonian, respectively. For simplicity here we consider the case where only a single control field is present (the generalization is straightforward) and we work in adimensional units ($\hbar = 1$). As any quantum system with limited energy and limited in space is

effectively finite-dimensional, from now on we focus on finite-dimension Hilbert space of dimension, i.e., $\mathcal{H} = \mathbb{C}^N$, and on the set of density matrices (of dimension D_ρ) defined over \mathcal{H} . Equation (1) generates a set of states depending on the control field $\gamma(t)$ and on the initial state ρ_0 : the manifold that is generated for every $\gamma(t)$ defines the set \mathcal{W} of reachable states from ρ_0 with dimension $\dim \mathcal{W} \equiv D_{\mathcal{W}}(N)$ [14]. A system is said to be controllable if the manifold \mathcal{W} is the complete space of density matrix operators; that is, $D_{\mathcal{W}} = D_\rho$ [15]. However, in general the set of reachable states is smaller than the whole set of density matrices; that is, $D_{\mathcal{W}} \leq D_\rho$.

Given a goal state $\bar{\rho}$ the problem to be solved is to find a control pulse $\bar{\gamma}(t)$ that drives the system from a reference state ρ_0 within an ϵ -ball around the goal state $\bar{\rho}$. Equivalently, the OC problem can be expressed as a functional minimization of the form

$$\min_{\gamma(t)} \mathcal{F}(\rho_0, \bar{\rho}, \gamma(t), [\lambda_i]), \quad (3)$$

where the functional \mathcal{F} might also include constraints introduced via Lagrange multipliers λ_i . The problem is solved by a (not necessarily unique) optimal $\bar{\gamma}(t)$, that identifies a final state ρ_f such that $\|\rho_f - \bar{\rho}\| < \epsilon$ in some norm $\|\cdot\|$.

We now recall the definition of the information content of the control pulse $\gamma(t)$ as we show in the following that it is intimately related to the complexity of the OC problem. The information (number of bits b_γ) carried by the control pulse $\gamma(t)$ is given by the classical channel capacity C times the pulse duration T . In the simple case of a noiseless channel, the channel capacity is given by Hartley's law; thus,

$$b_\gamma = T\Delta\Omega\kappa_s, \quad (4)$$

where $\Delta\Omega$ is the bandwidth, and $\kappa_s = \log(1 + \Delta\gamma/\delta\gamma)$ is the bit depth of the control pulse $\gamma(t)$, and $\Delta\gamma = \gamma_{\max} - \gamma_{\min}$ and $\delta\gamma$ are the maximal and minimal allowed variation of the field [16]. Note that given an uniform sampling rate of the signal δt , $T\Delta\Omega = T/\delta t = n_s$ where n_s is the number of sampling points. Any optimization method of choice depends on n_s variables—the values of the control field in those points; i.e., n_s defines the dimension of the input of the optimization problem. We thus define the dimension of the quantum OC problem \mathcal{D} as follows: Given a dynamical law of the form of Eq. (1), a reference initial state ρ_0 and any possible goal state in the set reachable states \mathcal{W} , the dimension of the quantum OC problem is defined by the minimal number of independent degrees of freedom \mathcal{D} in the OC field necessary to achieve the desired transformation up to precision ϵ . Notice that \mathcal{D} might be the minimal number of sampling points n_s (i.e., $\mathcal{D} = T\Delta\Omega$), of independent bang-bang controls, of frequencies present in the control field, or the dimension of the subspace of functions the control field has nonzero projection on.

From now on we consider the physical situations where the control is performed in some finite time $t \in [0, T]$, with

bounded control field and bounded Hamiltonians, e.g., $\|H_D\| = \|H_C\| = 1$ and $\gamma(t) \in [\gamma_{\min}; \gamma_{\max}] \forall t$. The aforementioned physical constraints naturally introduce a new class of states: The set of time-polynomial reachable states $\mathcal{W}^+ \subseteq \mathcal{W}$ is the set of states (such that $\dim(\mathcal{W}^+) = D_{\mathcal{W}^+}$) that can be reached with finite energy with precision ϵ in polynomial time as a function of the Hilbert space dimension N , $D_{\mathcal{W}^+}(N) \leq D_{\mathcal{W}} \leq D_\rho$. This is the class of interesting states from the point of view of OC, as if a state can be reached only in exponential time the optimal control problem cannot be solved efficiently (see [17] for a counting argument of the dimension of the manifold \mathcal{W}^+ for local Hamiltonians). Similarly to standard definitions, we define a time-polynomial reachable system if all states can be reached (with precision ϵ) in polynomial time by means of at least one path (i.e., $D_{\mathcal{W}^+} = D_{\mathcal{W}}$) and a time-polynomially controllable system if \mathcal{W}^+ is equal to the whole set of density matrices (in this case $D_{\mathcal{W}^+} = D_{\mathcal{W}} = D_\rho$). Notice that if the bound on the strength of the control γ_{\max} is relaxed we have always that $D_{\mathcal{W}^+} = D_{\mathcal{W}}$, however $D_{\mathcal{W}} \leq D_\rho$. Given the above definitions, we can state the following:

Theorem: The dimension \mathcal{D} of a quantum OC problem in \mathcal{W}^+ up to precision ϵ is a polynomial function of the dimension of the manifold of the time-polynomial reachable states $D_{\mathcal{W}^+}$.

Proof: We first prove that the dimension of the problem is bounded from below by $D_{\mathcal{W}^+}$ and then that it is bounded from above by a polynomial function of $D_{\mathcal{W}^+}$.

Lower bound: We divide the complete set of time-polynomial reachable states \mathcal{W}^+ in balls of size $\epsilon^{D_{\mathcal{W}^+}}$. The number of ϵ -balls necessary to cover the whole set \mathcal{W}^+ is $\epsilon^{-D_{\mathcal{W}^+}}$ and one of them identifies the set of states that live around the state $\bar{\rho}$ within a radius ϵ . The information content of the OC field must be at least sufficient to specify the ϵ -ball surrounding the goal state, that is $b_\gamma \geq b_\epsilon^-$, where $b_\epsilon^- = \log \epsilon^{-D_{\mathcal{W}^+}}$. Finally one obtains

$$\epsilon \geq 2^{\frac{T\Delta\Omega\kappa_s}{D_{\mathcal{W}^+}}}. \quad (5)$$

Setting a maximal precision (e.g., machine precision) expressed in bits $\kappa_\epsilon = -\log_2 \epsilon$ results in $T\Delta\Omega\kappa_s/D_{\mathcal{W}^+} = \mathcal{D}\kappa_s/D_{\mathcal{W}^+} = \kappa_\epsilon$, and imposing $\kappa_\epsilon = \kappa_s$ we obtain

$$\mathcal{D} \geq D_{\mathcal{W}^+}. \quad (6)$$

Upper bound: The goal state belongs to the set of time-polynomial states $\bar{\rho} \in \mathcal{W}^+$; thus, the path of finite length L that connects the initial and goal states in polynomial time exists. The maximum of (nonredundant) information that provides the solution to the problem is the information needed to describe the complete path b_ϵ^+ . Setting the desired precision ϵ , this is equal to $\log \epsilon^{-D_{\mathcal{W}^+}}$ bit of information for each ϵ -ball needed to cover the path times the number of balls n_ϵ . The latter is given by

$$n_\epsilon = L/\epsilon \leq T v_{\max}/\epsilon = \text{Poly}(D_{\mathcal{W}^+}) v_{\max}/\epsilon, \quad (7)$$

where L is the length of the path and v_{\max} is the maximal allowed velocity along the path due to the bounded energy. In conclusion, we obtain that

$$b_S^+ = \frac{\text{Poly}(D_{\mathcal{W}^+})v_{\max}}{\varepsilon} \log \varepsilon^{-D_{\mathcal{W}^+}}, \quad (8)$$

which implies, together with the condition $b_\gamma \leq b_S^+$,

$$\text{Poly}'(D_{\mathcal{W}^+})v_{\max}/\varepsilon \geq \mathcal{D}. \quad (9)$$

As \mathcal{D} is bounded by a polynomial function of $D_{\mathcal{W}^+}$, thus $\mathcal{D} = \text{Poly}(D_{\mathcal{W}^+})$ ■

Notice that the lower bound holds in general for any reachable state in \mathcal{W} and can be saturated, as recently shown in [18,19]. On the other hand, the upper bound diverges for $\varepsilon \rightarrow 0$, as finding the exact solution of the control problem might be as difficult as super exponential [6]. The theorem has a number of interesting practical and theoretical implications that we present in the rest of the paper.

Complexity.—The aforementioned theorem poses the basis to set the SC of solving the OC problem. An algorithm recently introduced to solve complex quantum OC problems, the chopped random basis (CRAB) optimization, builds on the fact that the space of the control pulse $\bar{\gamma}(t)$ is limited from the very beginning to some (small) value \mathcal{D} , and then solves the problem by means of a direct search method as the simplex algorithm (see Supplemental Material, Sec. 3 [20]). Recently, numerical evidence has been presented that this algorithm efficiently finds exponentially precise solutions as soon as $\mathcal{D} \geq D_{\mathcal{W}}$ [28]. This result can be put now on solid ground as under fairly general conditions OC problems are equivalent to linear programming [29] and linear programming can be solved via simplex algorithm with polynomial SC (see Supplemental Material, Sec. 1 [10] [12]; thus, the CRAB optimization solves with polynomial SC OC problems with dimension \mathcal{D} . More formally, one can make the following statement: The class of OC problems that satisfy the hypothesis (H1–H3) of Ref. [29] is characterized by a polynomial SC in the dimension of the problem \mathcal{D} . In conclusion, studying the scaling of the dimension of the control problem \mathcal{D} with the system size N is of fundamental interest to understand and classify our capability of efficiently control quantum systems. The polynomial relation that we have proven between \mathcal{D} and $D_{\mathcal{W}^+}$ allows us to focus from now on to the latter quantity which can be directly investigated in different settings. In particular, for many-body quantum systems, the scaling of $D_{\mathcal{W}^+}$ with the number of constituents $n = \log_d N$ is fundamental to discriminate between feasible and unfeasible OC dynamics.

The first results in this direction can be obtained observing the influence of the integrability of the quantum system on $D_{\mathcal{W}^+}$, resulting in the following properties:

(i) The dimension \mathcal{D} of a generic OC problem defined a time-polynomial controllable nonintegrable n -body quantum system is exponential with the number of constituents n . Indeed, in this case the dynamics explores the whole

accessible space and the set of time-polynomial reachable states is $D_{\mathcal{W}^+} \propto N^2$ ($D_{\mathcal{W}^+} = N$ for pure states).

On the contrary, despite the exponential growth of the Hilbert space, the dimension of \mathcal{W}^+ for integrable systems is at most linear in the number n of constituents of the system, that implies together with the theorem above that:

(ii) The dimension \mathcal{D} of OC problems defined on time-polynomially controllable integrable many-body quantum system, is polynomial with n and thus this class of problems can be solved efficiently. Notice that this statement generalizes a theorem that has been proven for the particular case of tridiagonal Hamiltonian systems presented in [30].

Finally, there exists a class of intermediate dynamics that despite in principle might explore an exponentially big Hilbert space, are confined in a corner of it and can thus be efficiently represented. The simplest example of this class of problems is mean-field dynamics; however, more generally, to this class of dynamics belong those that can be represented efficiently by means of a tensor network as t-DMRG (see Supplemental Material, Sec. II [31] [32]). We can thus state the following:

(iii) The dimension \mathcal{D} of an OC problem defined on a dynamical process that can be described efficiently by a tensor network; e.g., in one dimension a matrix product state is polynomial in n . The dimension of the set of states that can be efficiently represented by a tensor network scales as $\text{Poly}(n)T \geq D_{\mathcal{W}} \geq D_{\mathcal{W}^+}$, where T is the total time of the evolution and $\text{Poly}(n)$ is the dimension of the biggest tensor network state represented during the time evolution. Notice that, although the previous statement is in principle valid in all dimensions, it has practical implications mostly in one-dimensional systems as much less efficient representations of the dynamics are known in dimensions bigger than one [33]. The previous property can be rephrased as

(iv) Time evolution of slightly entangled one-dimensional many-body quantum systems can be efficiently represented via matrix product states with $D_{\mathcal{W}^+} \leq D_{\mathcal{W}} = O(Td\chi^2n)$ parameters, where χ is the maximal Schmidt rank of any bipartition present in the system [34]. Thus, systems with $\chi \propto \log(N)$ for every time can be efficiently controlled.

We stress that the entanglement present in the system is not uniquely correlated with the complexity of the OC problem: indeed due to the previous results, integrable systems (also highly entangled) are efficiently controllable, as shown recently in [18]. On the contrary, as said before, highly entangled dynamics of non integrable systems, for which it does not exist an efficient representation, are exponentially difficult to control. In conclusion, the dimension of the control problem \mathcal{D} depends on the dimension of the manifold over which the dynamics takes place, and this dimension sets the complexity of the OC problem. This can be simply understood by considering the scenario where the dynamics over which the control problem is defined is restricted to the space of two eigenstates of a complex many-body Hamiltonian (i.e., $\mathcal{D} = 2$), each of them highly entangled with respect to some local basis. If one has access

to a direct coupling between them, the complexity of the OC problem is not more than that of a simple Landau-Zener process (independently from the entanglement present in the system) as the manifold is effectively two dimensional. However, this is not generally the case, as one has usually access to some local (or global) operator, and the dynamic of the system is not in general restricted to two states. In the case of nonintegrable systems, a generic couple of initial and goal states projects on exponentially many basis states independently of the chosen basis, while for integrable states it exists a base where the states have a simple representation. Thus, the minimal amount of information needed to solve the quantum OC problem is exponential and polynomial, respectively. In between, there is the class of TN-efficiently representable dynamics, for which we know how to build an efficient representation and correspondingly we know how to efficiently solve the OC problem.

Time bounds.—Manipulating Eq. (5) applied to the whole set of reachable states \mathcal{W} we achieve a bound for the minimal time needed to achieve the desired transformation as a function of the control bandwidth: The minimal time needed to reach a given final state in $D_{\mathcal{W}}$ with precision ε at finite bandwidth is

$$T \geq \frac{D_{\mathcal{W}}}{\Delta\Omega\kappa_S} \log(1/\varepsilon), \quad (10)$$

or again, under the assumption that $\kappa_\varepsilon = \kappa_S$,

$$T \geq \frac{D_{\mathcal{W}}}{\Delta\Omega}. \quad (11)$$

The previous relation is a continuous version of the Solovay-Kitaev theorem: it provides an estimate of the minimal time needed to perform an optimal process given a finite band-width (see Supplemental Material, Sec. IV [35]). Notice also that the bandwidth provides the average bits rate per second; thus, these results coincide with the intuitive expectation that the minimal time needed to perform an optimal quantum process is the time necessary to “inform” the system about the goal state given that the control field has only a finite bit transmission rate.

We recall that there is a time-energy bound, known as the quantum speed limit, that in its general form is

$$T_{QSL} \geq \frac{d(\rho_0, \bar{\rho})}{\bar{\Lambda}}, \quad (12)$$

where $d(\cdot, \cdot)$ is the distance and $\bar{\Lambda} = \int_0^T \|\mathcal{L}\|_p dt / T$ with $\|\cdot\|_p$ the p-norm [7] (see Supplemental Material, Sec. V [37]). The most efficient process saturates both bounds, which implies $\Delta\Omega \propto D_{\mathcal{W}}$; thus the bandwidth of the time-optimal pulse in general should scale as the dimension of the space \mathcal{W} , requiring exponential higher frequencies for nonintegrable many-body quantum systems and thus practically preventing its physical realization.

Noise.—In the presence of noise, Eq. (4) has to be modified. In the following we consider a common scenario; however, this analysis can be adapted to the specific noise considered. For Gaussian white noise, according to the Shannon-Hartley theorem the channel capacity is $k_s = \log(1 + S)$, where S is the signal to noise power ratio [16]. Thus, following the same steps as before, we obtain that

$$\varepsilon \geq (1 + S)^{-\frac{p}{D_{\mathcal{W}}}}, \quad (13)$$

and similarly

$$T \geq \frac{D_{\mathcal{W}} \log(1/\varepsilon)}{\Delta\Omega \log(1 + S)}. \quad (14)$$

For small noise-to-signal ratio ($1/S \ll 1$), the previous bound results in $\varepsilon \gtrsim (1/S)^{D_{\mathcal{W}}}$, which together with the fact that \mathcal{D} has to be a polynomial function of $D_{\mathcal{W}}$ show that the control problem is in general exponentially sensitive to the problem dimension. However, if one saturates the lower bound on the complexity of the optimal field, i.e., $\mathcal{D} = D_{\mathcal{W}}$, the sensitivity to Gaussian white noise becomes linear in the noise-to-signal ratio. That is, the effects of the noise on the optimal transformation are negligible if the noise level is below the error, $1/S \lesssim \varepsilon$. As requiring the optimal transformation to be more precise than the error on the control signal is somehow unnatural, this relation demonstrates that OC transformations are in general robust with respect to noise, as recently observed [39]. At the same time, for $\varepsilon \lesssim 1/S$ this result agrees with the scaling for exact optimal transformations recently found in [9].

Control of unitaries.—The aforementioned statements also hold for the generation of unitaries as the differential equation governing the evolution of the time evolution operator $iU(t) = H(t)U(t)$ is formally equivalent to Eq. (1) replacing the density matrix with the time evolution operator $U(t)$, the reference state with the identity operator, and the goal state with the unitary to be generated.

Observability.—As any controllable system is also observable by a coherent controller [40], the previous definitions and results can be straightforwardly applied to the complexity of observing a many-body quantum system with precision ε .

In conclusion, we have shown that if one allows a finite error (both in the goal state and in time) as it typically occurs in any practical application of OC, what can be efficiently simulated can also be optimally controlled, and the optimal solution is in general robust with respect to perturbation on the control field. Notice that the presented results are valid both for open and closed loop OC.

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