Wave Function and Strange Correlator of Short-Range Entangled States

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We demonstrate the following conclusion: If $|\Psi\rangle$ is a one-dimensional (1D) or two-dimensional (2D) nontrivial short-range entangled state and $|\Omega\rangle$ is a trivial disordered state defined on the same Hilbert space, then the following quantity (so-called "strange correlator") $C(r,r') = \langle\Omega|\phi(r)\phi(r')|\Psi\rangle/\langle\Omega|\Psi\rangle$ either saturates to a constant or decays as a power law in the limit $|r-r'| \to +\infty$, even though both $|\Omega\rangle$ and $|\Psi\rangle$ are quantum disordered states with short-range correlation; $\phi(r)$ is some local operator in the Hilbert space. This result is obtained based on both field theory analysis and an explicit computation of $C(r,r')$ for four different examples: 1D Haldane phase of spin-1 chain, 2D quantum spin Hall insulator with a strong Rashba spin-orbit coupling, 2D spin-2 Affleck-Kennedy-Lieb-Tasaki state on the square lattice, and the 2D bosonic symmetry-protected topological phase with Z_2 symmetry. This result can be used as a diagnosis for short-range entangled states in 1D and 2D.

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A short-range entangled (SRE) state is a ground state of a quantum many-body system that does not have ground-state degeneracy or bulk topological entanglement entropy. But a SRE state (for example, the integer quantum Hall state) can still have protected stable gapless edge states. Thus, it appears that the bulk of all the SRE states are identically trivial, and a nontrivial SRE state is usually characterized by its edge states [1]. In this Letter, we propose a diagnosis to determine whether a given many-body wave function defined on a lattice without boundary is a nontrivial SRE state or a trivial one. This diagnosis is based on the following quantity called a "strange correlator": [2]

$$C(r,r') = \frac{\langle\Omega|\phi(r)\phi(r')|\Psi\rangle}{\langle\Omega|\Psi\rangle}. \tag{1}$$

Here, $|\Psi\rangle$ is the wave function that needs diagnosis, and $|\Omega\rangle$ is a direct product trivial disordered state defined on the same Hilbert space. Our conclusion is that if $|\Psi\rangle$ is a nontrivial SRE state in one or two spatial dimensions, then for some local operator $\phi(r)$, $C(r,r')$ will either saturate to a *constant* or decay as a *power law* in the limit $|r-r'| \to +\infty$, even though both $|\Omega\rangle$ and $|\Psi\rangle$ are disordered states with short-range correlation.

Another possible way of diagnosing a SRE wave function is through its entanglement spectrum [3]. If a SRE state has nontrivial edge states, an analogue of its edge states should also exist in its entanglement spectrum [4]. However, many SRE states are protected by certain symmetry, and some SRE states are protected by lattice symmetries [for example, the spin-2 Affleck-Kennedy-Lieb-Tasaki (AKLT) state on the square lattice requires translation symmetry]. If the cut we make to compute the entanglement spectrum breaks such lattice symmetry, then the entanglement spectrum would be trivial, even if the original state is a nontrivial SRE state. By contrast, the strange correlator in Eq. (1) is defined on a lattice without edge; thus, it already preserves all the symmetries of the system, including all the lattice symmetries. Thus, the strange correlator can reliably diagnose SRE states protected by lattice symmetries as well.

The strange correlator can be roughly understood as follows: $|\Psi\rangle$ can be viewed as a generic initial state evolved with a constant nontrivial SRE Hamiltonian from $\tau = -\infty$ to 0; $\langle\Omega|$ is a state evolved from $\tau = +\infty$ to 0 with a trivial Hamiltonian, and thus, the strange correlator can be viewed as a "correlation function" at a temporal domain wall of the quantum field theories (QFTs) at $\tau = 0$; see Fig. 1(a). If there is an approximate Lorentz invariant description of the system, a space-time rotation can transform Eq. (1) to a space-time correlation at a spatial interface between nontrivial and trivial SRE systems; see Fig. 1(b). And for one and two spatial dimensions, a spatial interface between trivial and nontrivial SRE states should have either long-range or power-law correlation between certain local operators, which after Lorentz rotation will lead to the conclusion of this Letter (see the Supplemental Material [5]). A similar observation of Lorentz rotation was used to derive the bulk wave function of topological superconductors [6].

For bosonic SRE states that are protected by certain symmetry [so-called symmetry-protected topological (SPT) states [7,8]], the argument above can be demonstrated more explicitly. In Ref. [9], it was demonstrated that a large class of one-dimensional (1D) and two-dimensional (2D) bosonic SPT states can be described by the following two nonlinear sigma model field theories:

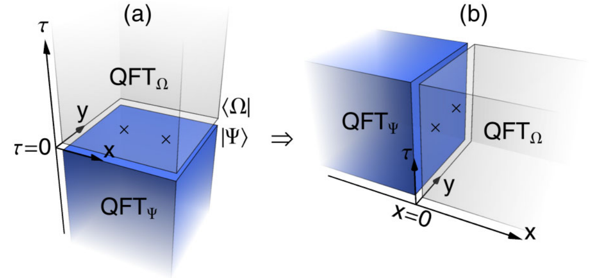

FIG. 1 (color online). (a) $|\Psi\rangle$ and $\langle\Omega|$ are given by infinite time evolution of their QFTs from below and above, respectively. The strange correlator can be viewed as the correlator at the $\tau = 0$ interface. (b) Under the Lorentz rotation, the two QFTs are separated by the $x = 0$ interface, and the strange correlator can be interpreted as the correlation function on this spatial interface.

$$\mathcal{S}_{1\mathrm{D}} = \int dx d\tau \frac{1}{g}(\partial_\mu \boldsymbol{n})^2 + \frac{i2\pi}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c, \tag{2}$$

$$\mathcal{S}_{2\mathrm{D}} = \int d^2x d\tau \frac{1}{g}(\partial_\mu \boldsymbol{n})^2 + \frac{i2\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}n^a\partial_\mu n^b\partial_\nu n^c\partial_\rho n^d. \tag{3}$$

Here, $\boldsymbol{n}(x)$ is an O(3) or O(4) vector order parameter with unit length constraint: $\boldsymbol{n}^2 = 1$. Different SPT phases are distinguished from each other based on different implementations of the symmetry group on the vector order parameter $\boldsymbol{n}$. In both 1D and 2D, ground-state wave functions of SPT phases can be derived based on Eqs. (2) and (3) (see Ref. [10]),

$$|\Psi\rangle_d \sim \int D\boldsymbol{n}(x)e^{-\int_{S^d} d^dx(1/G)(\nabla\boldsymbol{n})^2 + \mathrm{WZW}_d[\boldsymbol{n}]}|\boldsymbol{n}(x)\rangle, \tag{4}$$

where S^d is the compactified real space manifold, and $\mathrm{WZW}_d[\boldsymbol{n}]$ is a real space Wess-Zumino-Witten term

$$\mathrm{WZW}_1[\boldsymbol{n}] = \int_0^1 du \frac{i2\pi}{8\pi}\epsilon_{\mu\nu}\epsilon_{ab}n^a\partial_\mu n^b\partial_\nu n^c, \quad \mu,\nu = x,u,$$
$$\mathrm{WZW}_2[\boldsymbol{n}] = \int_0^1 du \frac{i2\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}n^a\partial_\mu n^b\partial_\nu n^c\partial_\rho n^d, \quad \mu,\nu,\rho = x,y,u. \tag{5}$$

In contrast, the trivial state wave function is a superposition of all configurations of $|\boldsymbol{n}(x)\rangle$ without a Wess-Zumino-Witten (WZW) term. On the basis of the wave functions in Eq. (4), the strange correlator of order parameter $\boldsymbol{n}(x)$ reads

$$C(r,r') = \frac{\int D\boldsymbol{n}(x)n^a(r)n^b(r')e^{-\int_{S^d} d^dx(1/G)(\nabla\boldsymbol{n})^2 + \mathrm{WZW}_d[\boldsymbol{n}]}}{\int D\boldsymbol{n}(x)e^{-\int_{S^d} d^dx(1/G)(\nabla\boldsymbol{n})^2 + \mathrm{WZW}_d[\boldsymbol{n}]}}. \tag{6}$$

Mathematically, this strange correlator can be viewed as an ordinary space-time correlation function of a $[(d-1)+1]$-dimensional field theory with a WZW term, as long as we view one of the spatial coordinate as the time direction. When $d = 1$, this strange correlator is effectively a spin-spin correlation of one isolated free spin 1/2, and the correlation is always long range. When $d = 2$, this strange correlator is effectively a space-time correlation function of a $(1+1)$D O(4) nonlinear sigma model with a WZW term, and when this model has a full SO(4) symmetry, this theory is an $\mathrm{SU}(2)_1$ conformal field theory with power-law correlation [11,12]; when the symmetry of the system is a subgroup of SO(4), as long as the residual symmetry prohibits any linear Zeeman coupling to order parameter $\boldsymbol{n}$, this $(1+1)$D system either remains gapless or spontaneously breaks the symmetry and develops long-range order. Thus, the strange correlator is either long range or decays with a power law.

The two arguments above both rely on a certain continuum limit description of the SRE state. However, for a fully gapped system, when the gap is comparable with the ultraviolet energy scale of the system, a continuum limit description may not be appropriate. In the rest of the Letter, we will compute the strange correlator for several examples of SRE states *far from* the continuum limit; i.e., the gap of the SRE states is comparable with UV cutoff. We will see that in some examples, the strange correlator is indeed different from the physical edge of the SRE state, but our qualitative conclusion is still valid.

The first example we study is the AKLT state [13,14] of the Haldane phase of spin-1 chain. In the S^z basis, the AKLT wave function is a "dilute" Néel state; namely, it is an equal weight superposition of all the S^z configurations with an alternate distribution of $|+\rangle = |S^z = +1\rangle$ and $|-\rangle = |S^z = -1\rangle$, sandwiched with arbitrary numbers of $|0\rangle = |S^z = 0\rangle$ [15]

$$|\Psi\rangle = \sum \frac{1}{N}|+0\cdots0-0\cdots0+\cdots\rangle. \tag{7}$$

We choose the trivial state to be $|\Omega\rangle = |000\cdots\rangle$. Straightforward calculation leads to the following answer of the strange correlator:

$$C(r,r') = \frac{\langle\Omega|S_r^+S_{r'}^-|\Psi\rangle}{\langle\Omega|\Psi\rangle} = 2, \tag{8}$$

which is the expected long-range correlation.

The second example we study is the two-dimensional quantum spin Hall (QSH) insulator with a Rashba spin orbit coupling. We will use the same notation as in Ref. [16]. The QSH insulator Hamiltonian reads

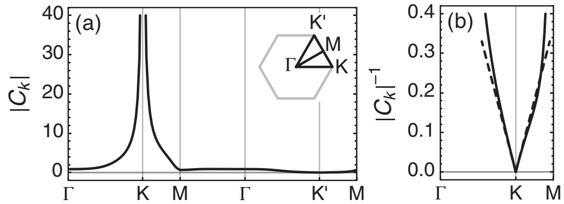

FIG. 2. (a) The amplitude of strange correlator in the momentum space. The inset shows the Brillouin zone and the high symmetry points. (b) $|C_k|^{-1}$ exhibits nice linearity around the K point, establishing the $1/|k|$ divergence in $|C_k|$.

$$H = t\sum_{\langle i,j\rangle} c_i^\dagger c_j + i\lambda_{\rm SO}\sum_{\langle\langle i,j\rangle\rangle} \nu_{ij} c_i^\dagger s^z c_j + \lambda_R \sum_{\langle i,j\rangle} c_i^\dagger (\boldsymbol{s}\times\hat{\boldsymbol{d}}_{ij})_z c_j + \lambda_v \sum_i \xi_i c_i^\dagger c_i. \quad (9)$$

$\lambda_{\rm SO}$ is the spin-orbit coupling that leads to the QSH topological band structure, λ_R is the Rashba coupling that breaks the conservation of s^z, and λ_v is a staggered potential that leads to charge density wave. The electron operator c_i carries spin and sublattice indices; thus, the strange correlator $C(r, r')$ is a 4×4 matrix. For QSH state $|\Psi\rangle$, we choose $\lambda_{\rm SO} = t$, $\lambda_R = 0.5t$, $\lambda_v = 0$; trivial state $|\Omega\rangle$ is chosen to be a strong CDW state with $\lambda_{\rm SO} = t$, $\lambda_R = 0.5t$, $\lambda_v = 10t$. These two states are far from the continuum limit; namely, the gap is comparable with the UV cutoff.

Figure 2(a) shows the amplitude of strange correlator $|C_{\boldsymbol{k}}| = |\langle\Omega|c^\dagger_{A,\uparrow,\boldsymbol{k}} c_{B,\uparrow,\boldsymbol{k}}|\Psi\rangle/\langle\Omega|\Psi\rangle|$ plotted in the momentum space. There is one clear singularity at the corner of the Brillouin zone, which diverges as $\sim 1/|\boldsymbol{k}|$, as demonstrated in Fig. 2(b) (see the Supplemental Material [17]). This implies that in the real space the strange correlator decays as $|C(\boldsymbol{r},\boldsymbol{r}')| \sim 1/|\boldsymbol{r}-\boldsymbol{r}'|$, which is consistent with the result obtained from Lorentz transformation, despite the large bulk gap.

The third example we will study is the spin-2 AKLT state on the square lattice [14,18], which is a SPT state protected by the on-site $\mathbb{Z}_2\times\mathbb{Z}_2$ and the lattice translation symmetry [19], whose wave function has a tensor product state representation [20,21]

$$|\Psi\rangle = \sum_{\{m_i\}} {\rm tTr}(\otimes_i T^{m_i})|\{m_i\}\rangle. \quad (10)$$

Here, $m_i = 0, \pm 1, \pm 2$ labels the S^z quantum number of the spin-2 object on site i, and $|\{m_i\}\rangle$ is the state for the configuration $\{m_i\}$ over the lattice. tTr traces out the internal legs in the tensor network shown Fig. 3(a), in which the vertex tensor is given by

$$T^m_{s_1 s_2 s_3 s_4} = \begin{cases} 4 s_1 s_2 & : -s_1 - s_2 + s_3 + s_4 = m, \\ 0 & : \text{otherwise}, \end{cases} \quad (11)$$

with $s_j = \pm 1/2$ labeling the spin-1/2 internal degrees of freedom, whereas the trivial state $|\Omega\rangle = |\{\forall i : m_i = 0\}\rangle$ is chosen to be the direct product state of $S^z = 0$ on every site. We look into the strange correlator

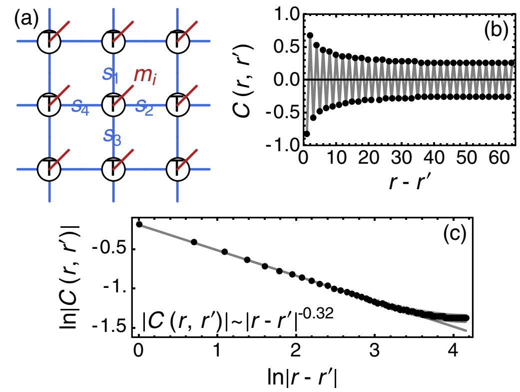

FIG. 3 (color online). (a) Tensor network representation of the 2D AKLT state. The red (blue) legs represent the physical (internal) degrees of freedom. (b) Strange correlator of the 2D AKLT state measured along the horizontal direction. (c) The amplitude follows a power-law behavior in the log-log plot. The final deviation is due to the finite-size effect.

$$C(r, r') = \frac{\langle\Omega|S_r^+ S_{r'}^-|\Psi\rangle}{\langle\Omega|\Psi\rangle} = \frac{{\rm tTr}[T^0\cdots T^1(r)T^{-1}(r')\cdots]}{{\rm tTr}(T^0\cdots)}, \quad (12)$$

which can be expressed as a ratio between two tensor networks: the denominator is a uniform network of the tensor T^0 on each site, and the numerator is the same network except for impurity tensors $T^{\pm 1}$ on site r and r', respectively.

The evaluation of the tensor trace in Eq. (12) over the 2D lattice can be reformulated as a $(1+1)$-dimensional quantum mechanics problem in terms of the transfer matrix for each row, which can then be studied by the density matrix renormalization group method [22,23]. The calculation is performed on an $128\times\infty$ lattice with periodic boundary condition along both directions. We found that the strange correlator decays with oscillation [as in Fig. 3(b)], and its amplitude follows a power-law behavior $|C(r, r')| \sim |r - r'|^{-\eta}$ with the exponent $\eta \simeq 0.32$ [see Fig. 3(c)], which is consistent with our previous field theory argument. The last example we will study is the two-dimensional bosonic SPT phase with $\mathbb{Z}_2$ symmetry, which was first studied in Ref. [24]. The ground-state wave function of this SPT phase is

$$|\Psi\rangle = \sum_{\{\sigma_i\}} (-1)^{N_d} \exp\left(-\frac{\beta}{2}\sum_{\langle i,j\rangle}\sigma_i\sigma_j\right)|\{\sigma_i\}\rangle, \quad (13)$$

which is a superposition of all the configurations of the Ising degree of freedom $|\{\sigma_i\}\rangle$ with a factor (-1) associated with each closed Ising domain wall (with N_d being the number of domain wall loops). The trivial state $|\Omega\rangle$ is simply an Ising paramagnet, whose wave function is similar to Eq. (13) but without the domain wall sign structure $(-1)^{N_d}$. Compared with Ref. [24], we have added a factor $\exp[(-\beta/2)\sum_{\langle i,j\rangle}\sigma_i\sigma_j]$ to each Ising configuration to adjust the spin correlation length.

The strange correlator of the $\mathbb{Z}_2$ bosonic SPT phase can be viewed as a correlation function of a "classical statistical mechanics model"

$$C(r,r') = \frac{\sum_{\{\sigma_i\}}\sigma_r\sigma_{r'}(-1)^{N_d}e^{-\beta\sum_{\langle i,j\rangle}\sigma_i\sigma_j}}{\sum_{\{\sigma_i\}}(-1)^{N_d}e^{-\beta\sum_{\langle i,j\rangle}\sigma_i\sigma_j}}. \quad (14)$$

Our goal is to show that this is either a long range or power-law correlation for arbitrary β. In other words, Eq. (14) is less likely to disorder than the ordinary 2D Ising model. This result can be naively understood as follows: the ordinary 2D Ising model is disordered at high temperature (small β) due to the proliferation of Ising domain walls. But in the current model, due to the (-1) sign associated with each domain wall, the proliferation of domain walls is suppressed, and thus, eventually the current Ising model Eq. (14) is not completely disordered even for small β.

This Ising model is dual to a loop model with the following partition function:

$$Z = \sum_{\mathcal{C}} K^L n^{N_d}, \quad (15)$$

where loops are the domain walls of the original Ising model, $K = \exp(-2\beta)$ is the loop tension, $n = -1$ is the loop fugacity, L is the total length of loops, and N_d is the total number of closed loops. If the loops do not cross, then according to Ref. [25], by tuning K there is a phase transition between a small loop phase (which corresponds to the Ising ordered phase) for small K and a dense loop phase for large K. The critical point and the dense loop phase are both critical with power-law correlations, and they correspond to two different conformal field theories with central charges $c = -3/5$ and -7, respectively. If the loops are allowed to cross, the dense loop phase is driven to a different conformal field theory with $c = -2$, which is described by free symplectic fermions [26].

The Ising order parameter σ_i corresponds to the "twist" operator of the loop model, because σ_i changes its sign when it crosses a loop. The twist operator is well-studied at the critical point of loop models, and in our case with $n = -1$, at the critical point between small and dense loop phases the scaling dimension of the twist operator is $-1/10$ [27], which is confirmed by our numerical calculation.

The tensor renormalization group method [28,29] has been applied to loop models in Ref. [30]. Here we use the same approach to study the twist operator correlations for the loop model in Eq. (15). For simplicity we forbid the loops to cross, so the model never develops antiferromagnetic order even for negative β. For positive large β, the strange correlator is long ranged; see Fig. 4(a). As β decreases, the correlator grows and diverges at the critical point $\beta_c \simeq 0.521$ with a power-law $C(r,r') \sim |r-r'|^{0.199}$ as shown in Fig. 4(b), which confirms the theoretical prediction of scaling dimension $-1/10$ of twist operator [27]. Theoretically, the entire dense loop phase (when $\beta < \beta_c$) should be controlled by one stable conformal field theory fixed point. Our numerical results suggest that this fixed point is around $\beta \sim -0.1816$; the power-law behavior of $C(r,r')$ at this point (Fig. 4) is consistent with the conclusion of this Letter [31].

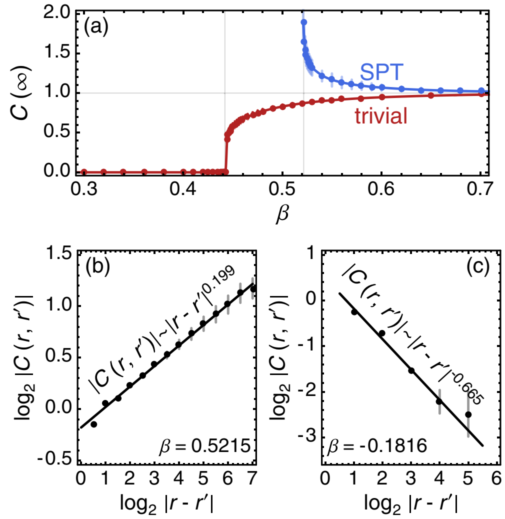

FIG. 4 (color online). (a) The strange correlator of the SPT state (in blue) at infinite distance $|r-r'| \to \infty$, in comparison with that of the trivial state (in red). The SPT strange correlator follows the power-law behavior (b) at the critical point and (c) in the dense loop phase.

We have checked that the ordinary free electron three-dimensional (3D) topological insulator also gives us a very clear power-law decay of strange correlator. However, in general a strongly interacting SRE state in three-dimensional space can be more complicated, because its two -dimensional edge can be (1) a gapless $(2+1)$D conformal field theory, (2) long-range order that spontaneously breaks symmetry, or (3) two-dimensional topological phase [32]. On the basis of our Lorentz transformation argument, it is possible that $\langle\Omega|\Psi\rangle$ is mapped to the partition function of a topological phase, and then in this case the strange correlator $C(r,r')$ may also be short ranged. Thus, for 3D SRE states, besides the strange correlator, we also need another method that diagnoses the situation when $\langle\Omega|\Psi\rangle$ corresponds to a topological phase partition function. We will propose a method to diagnose 3D SRE states in another paper.

In summary, we have proposed a general method to diagnose 1D and 2D SRE states based on their bulk ground-state wave functions. We expect our method to be useful for future numerical studies of SRE states. In

Refs. [33–36], it was proposed that interacting fermionic topological insulators and topological superconductors can be characterized by the full fermion Green's function; Ref. [37] proposed a method to diagnose bosonic SPT states characterized by group cohomology. The method proposed in our current Letter is applicable to both fermionic and bosonic SRE states.

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[1] The 1D Haldane phase, which is a SRE state, can also be characterized by a nonlocal order parameter [15,38], but it is unclear whether this nonlocal order parameter can be generalized to all SRE states.
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