Fluctuating Interfaces Subject to Stochastic Resetting

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We study one-dimensional fluctuating interfaces of length L, where the interface stochastically resets to a fixed initial profile at a constant rate r. For finite r in the limit $L \to \infty$, the system settles into a nonequilibrium stationary state with *non-Gaussian* interface fluctuations, which we characterize analytically for the Kardar-Parisi-Zhang and Edwards-Wilkinson universality class. Our results are corroborated by numerical simulations. We also discuss the generality of our results for a fluctuating interface in a generic universality class.

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Fluctuating interfaces are paradigmatic nonequilibrium systems commonly encountered in diverse physical situations, e.g., propagation of flame fronts in paper sheets, fluid flow in porous media, vortex lines in disordered superconductors, liquid-crystal turbulence, and many others. Study of such interfaces has many practical applications in the field of molecular beam epitaxy, crystal growth, fluctuating steps on metals, growing bacterial colonies or tumor, etc [1–3]. A well-studied model of fluctuating interfaces is the Kardar-Parisi-Zhang (KPZ) equation [4], which is believed to describe a wide class of such out-of-equilibrium growth processes.

Earlier studies of the KPZ equation focused on the universal behavior of the interface roughness, a property which, for instance, in $1+1$ space-time dimensions is characterized by the interface width $W(L,t)$ at time t for an interface growing over a substrate of linear size L. It is then known that $W(L,t)$ grows algebraically with time as t^β for times $t \ll L^z$ where z is the dynamic exponent, and saturates for times $t \gg L^z$ to a L-dependent value $\sim L^\alpha$. Here, α is the roughness exponent, while $\beta = \alpha/z$ is the growth exponent. For the KPZ universality class in $1+1$ dimensions, one has $\alpha = 1/2$ and $z = 3/2$ [1–3]. More recently, in this case, significant theoretical progress has shown that in the growing regime (i.e., for times $t \ll L^z$), the notion of universality extends beyond the interface width and holds even for the full interface height distribution at late times [5–9]. For example, the scaled cumulative distribution of the interface height fluctuations in a curved (respectively, flat) geometry is described by the so-called Tracy-Widom (TW) distribution $F_\beta(x)$, with $\beta = 2$ (respectively, $\beta = 1$). The distribution $F_2(x)$ (respectively, $F_1(x)$) characterizes fluctuations at the edge of the spectrum of random matrices in the Gaussian unitary ensemble [respectively, Gaussian orthogonal ensemble (GOE)] [10,11]. Height fluctuations measured in experiments on nematic liquid crystals with both curved and flat geometries demonstrated a very good agreement with the TW distributions [12,13].

One of the first models of interface growth is the Eden model, aimed at addressing the growth of bacterial colonies or tumors in mammals [14]. Such growth typically proceeds through stochastic cell division, and generates an almost compact cell cluster bounded by a rough interface that within the Eden model has scaling properties in the KPZ universality class of interfaces with a curved geometry. The growth however may be abruptly interrupted with the cell cluster reduced to its initial size by application of chemicals, as is done, e.g., in chemotherapy to stop the spread of a tumor before it becomes life threatening. It is then interesting to study how such random interruptions affect the growth process. In this Letter, we show that random interruptions, or random resettings, yield novel steady states with non-Gaussian fluctuations which we characterize analytically. We focus on the simpler case of flat interfaces, but our results can be generalized to those with a curved geometry.

We consider a $1+1$-dimensional interface characterized by a height field $H(x,t)$ at position x and time t. Starting

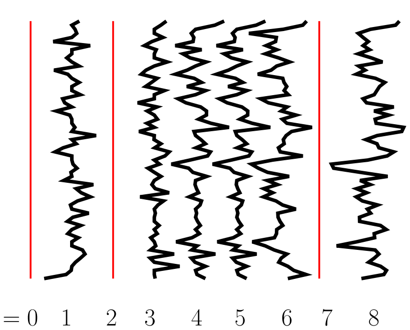

FIG. 1 (color online). Schematic interface evolution with resetting: Starting from a flat profile, the evolution is interrupted at random times by resetting to the initial configuration from which it recommences.

from an initially flat profile, $H(x,0) = 0\,\forall x$, the heights evolve according to the KPZ equation [4]:

$$\frac{\partial H}{\partial t} = \nu \frac{\partial^2 H}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \eta(x,t), \qquad (1)$$

where ν is the diffusivity, λ accounts for the nonlinearities, while $\eta(x,t)$ is a Gaussian noise of zero mean and correlations $\langle \eta(x,t)\eta(x',t')\rangle = 2D\delta(x-x')\delta(t-t')$. For an interface of length L evolving according to (1), the spatially averaged height $\overline{H(x,t)} = \int_0^L dx H(x,t)/L$ grows with time with velocity $v_\infty = (\lambda/2)\int_0^L dx \langle (\partial H/\partial x)^2\rangle$. The interface width $W \equiv W(L,t)$ is defined as the standard deviation of the height about $\overline{H(x,t)}$. For times larger than a nonuniversal microscopic time scale $T_{\mathrm{micro}} \sim O(1)$, the width exhibits Family-Vicsek scaling [15], $W(L,t) \sim L^\alpha \mathcal{W}(t/T^*)$, with the crossover time scale $T^* \sim L^z$ corresponding to the scale over which height fluctuations spreading laterally correlate the entire interface. The scaling function $\mathcal{W}(s)$ behaves as a constant as $s \to \infty$, and as s^β as $s \to 0$. At long times $t \gg T^*$, the dynamics of height fluctuations $h(x,t) \equiv H(x,t) - \overline{H(x,t)}$ reaches a nonequilibrium stationary state (NESS) in a *finite* system, in which the height distribution $P_{\mathrm{st}}(h)$ is a simple Gaussian [1].

Motivated by the situation where the growth is randomly interrupted, e.g., by chemicals, as discussed above, we study the case where the interface is reset at a fixed rate r to the initial flat configuration. The dynamics with random interruptions, shown schematically in Fig. 1, raises a natural question: Does it lead to a steady state and if so, can one characterize the distribution of the steady state height fluctuations? Here, we show that indeed random interruptions lead, quite generically, to a nontrivial steady state even in the thermodynamic limit $L \to \infty$, and for a class of $1+1$-dimensional models including the KPZ interface, we compute analytically the height distribution in this steady state.

A recent series of work has shown that resetting dynamics has a rich and dramatic effect even on a single particle diffusing in a one-dimensional space x [16–19]. The system is a random walker diffusing in presence of resetting, whereby the walker returns to its starting position $x = x_0$ at a constant rate r. The dynamics models the natural search strategy in which a search for misplaced belongings after continuing in vain for a while recommences by returning to the starting point. While with no resetting, the spatial distribution of the walker is a Gaussian centered at x_0 with width growing diffusively with time as $\sqrt{t}$, a nonzero r leads to a NESS, with an exponentially decaying profile centered at x_0 [16–19]. Thus, resetting makes an otherwise diverging mean search time finite, increasing the efficiency of the search strategy. Random walks with restarts have been used in computer science as a useful strategy to optimize search algorithms in hard combinatorial problems [20–22].

Our model of resetting of $1+1$-dimensional interface dynamics is a natural extension of the above mentioned single-particle studies to the case of an extended system comprising many interacting degrees of freedom. We note that a recent work also addresses resetting in an extended system, namely, a one-dimensional coagulation-diffusion process, albeit with a different resetting strategy [23].

While in the absence of resetting, the dynamics of interface fluctuations has no steady state in the thermodynamic limit, we demonstrate here that a nonzero resetting rate drives the system to a nontrivial NESS, even in the thermodynamic limit. The NESS obtained is characterized by non-Gaussian interface fluctuations, as we demonstrate analytically. In particular, the stationary interface width W_r does not scale with the system size, but instead remains bounded, scaling algebraically with r, $W_r \sim r^{-\beta}$, as $r \to 0$. We discuss our results for fluctuating interfaces belonging to a generic universality class, including the KPZ and the Edwards-Wilkinson (EW) class. Without resetting, the steady state distribution of height fluctuations for both the EW and the KPZ class are identical, and are Gaussian. In contrast, this is not anymore the case in presence of resetting, which carries the information of the different growth dynamics of the KPZ and the EW class into the steady state. Our analysis is supported by numerical simulations.

We now turn to a derivation of our results. We start with the observation that for times $t \gg T_{\mathrm{micro}}$, when universal scaling behaviors are expected, the dynamics involves two time scales: (i) $T_{\mathrm{r}} \sim 1/r$, the average time between two consecutive resets, and (ii) the crossover time $T^* \sim L^z$. Here we consider the case $T_{\mathrm{r}} \ll T^*$ (but still $T_{\mathrm{r}} \gg T_{\mathrm{micro}}$), which is easily achieved in the limit of an infinite system, $L \to \infty$, with finite r. In what follows, we set $L \to \infty$, or equivalently consider time scales $t \ll T^* = L^z$, such that the asymptotic dynamics is completely governed by the resetting process.

In order to compute the height distribution $P^{\mathrm{reset}}(h,t)$ at time t in presence of resetting, we note that the dynamics in the space of configurations is a Markov process. Indeed, let us denote by $\mathcal{C} = \{h(x,t)\}_{0\le x\le L}$ a configuration of the whole system of size L. The KPZ equation (1) implies that in the time interval between two successive resetting events, the dynamics of the "vector" $\mathcal{C}$, with entries labelled by the space position x, is Markovian. The dynamics in configuration space is thus a renewal process. Then, at a fixed time t, let the time elapsed since the last renewal be in $[\tau, \tau + d\tau]$, with $0 \le \tau \le t$. Noting that the probability for this event is $re^{-r\tau}d\tau$, we have

$$P^{\mathrm{reset}}(\mathcal{C},t) = \int_0^t d\tau r e^{-r\tau} P(\mathcal{C},\tau) + e^{-rt}P(\mathcal{C},t). \qquad (2)$$

Here, $P^{\rm reset}(\mathcal{C},t)$ [respectively, $P(\mathcal{C},t)$] is the probability to be in configuration $\mathcal{C}$ at time t, starting from an initially flat interface in the presence (respectively, absence) of any resetting. The last term on the right-hand side (rhs) of (2) accounts for the event when there has not been a single resetting event in the time interval [0, t]. Integrating both sides of Eq. (2) over all the possible configurations $\mathcal{C}$, and noting that $P(\mathcal{C},\tau)$ is normalized to unity for every τ, we check that $P^{\rm reset}(\mathcal{C},t)$ for every t is also normalized. The dynamics is Markovian in the configuration space, but is not so for the relative height $h(x,t)$ at a given point x due to the presence of space derivatives of the height field on the rhs of (1) [24]. Nevertheless, Eq. (2) being linear, one gets the marginal distribution $P^{\rm reset}(h,t)$ of the height field $h(x,t)$ by integrating Eq. (2) over heights $h(y,t)$ at positions $y\neq x$,

$$P^{\rm reset}(h,t)=\int_0^t {\rm d}\tau r e^{-r\tau}P(h,\tau)+e^{-rt}P(h,t), \qquad (3)$$

where $P(h,t)$ is the height distribution in the absence of resetting, starting from a flat initial configuration. In the limit $t\to\infty$, we see from (3) that the system reaches a steady state characterized by the distribution

$$P_{\rm r}(h)=P^{\rm reset}(h,t\to\infty)=\int_0^\infty {\rm d}\tau r e^{-r\tau}P(h,\tau), \qquad (4)$$

an exact result valid for any r and h. Note that due to resetting, a *nonlocal* probability flux exists only from all $h\neq 0$ values to $h=0$. This leads to a circulation of probability between a source at $h=0$ and several sinks at $h\neq 0$, so that the steady state reached is a NESS.

We consider first the simpler EW equation which corresponds to $\lambda=0$ in Eq. (1), thereby leading to an evolution linear in h [25]. In this case, $v_\infty=0$ and the Family-Vicsek scaling holds with the EW exponents $\alpha=1/2$ and $z=2$. Without resetting, the steady state distribution $P_{\rm st}(h)$ at times $t\gg T^*$ in a finite system is Gaussian, and is in equilibrium, in contrast to the NESS of a KPZ interface. For times $t\ll T^*$, the interface distribution is also Gaussian, but with a nonstationary width $W^{\rm EW}(t)=D\sqrt{2t/(\pi\nu)}$, for all t. Hence, plugging this Gaussian form for $P(h,\tau)$ into Eq. (4), we find that $P_{\rm r}^{\rm EW}(h)$ has the scaling form

$$P_{\rm r}^{\rm EW}(h)\sim\sqrt{\gamma}r^{1/4}G^{\rm EW}(h\sqrt{\gamma}r^{1/4}), \qquad (5)$$

where $\gamma=\sqrt{\pi\nu}/(D2^{3/2})$ and $G^{\rm EW}(x)$ is given by

$$G^{\rm EW}(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{{\rm d}y}{y^{1/4}}\exp\left(-y-\frac{x^2}{\sqrt{y}}\right), \qquad (6)$$

which is symmetric in x, $G^{\rm EW}(-x)=G^{\rm EW}(x)$, yielding zero mean, and variance $\int x^2G^{\rm EW}(x)dx=\sqrt{\pi/4}$. From the scaling form in (5), one obtains the scaling of the stationary width with r as $W_{\rm r}^{\rm EW}\sim r^{-1/4}$ [26]. The integral in (6) can be expressed in terms of hypergeometric series. In particular, $G^{\rm EW}(x)$ behaves asymptotically as

$$G^{\rm EW}(x)\sim\begin{cases}\frac{1}{\sqrt{\pi}}[\Gamma(\frac{3}{4})-x^2\Gamma(\frac{1}{4})+\frac{8}{3}\sqrt{\pi}|x|^3], & x\to 0,\\ c|x|\exp[-3/2^{2/3}|x|^{4/3}], & x\to\pm\infty,\end{cases} \qquad (7)$$

where $\Gamma(x)$ is the gamma function and c is a computable constant. Interestingly, due to the $|x|^3$ term in (7), $G^{\rm EW}(x)$ is nonanalytic close to $x=0$. In the limit $x\to\pm\infty$, the stretched exponential behavior (7) is significantly different from a Gaussian tail.

In order to check our prediction (5), we now report on results of numerical simulations performed for a discrete one-dimensional periodic interface $\{H_i(t)\}_{i=1,2,\ldots,L}$ evolving in times $t_n=n\Delta t$, with n an integer and $\Delta t\ll 1$. Starting from a flat interface, $H_i(0)=0\,\forall i$, the interface at time step t_n is reset to its initial configuration with probability $r\Delta t$, and updated according to the EW dynamics with probability $1-r\Delta t$. The results shown in Fig. 2 illustrate a very good agreement with theory. Evidently, $P_{\rm r}^{\rm EW}(h)$ is highly non-Gaussian.

We now turn to the KPZ case. Here it is known that for times $T_{\rm micro}\ll t\ll T^*$, and for a flat initial profile, the interface height $H(x,t)$ has a deterministic linear growth with stochastic $t^{1/3}$ fluctuations [9]:

$$H(x,t)=v_\infty t+(\Gamma t)^{1/3}\chi(x). \qquad (8)$$

Here, $\Gamma\equiv\Gamma(\nu,\lambda,D)$ is a constant, while χ is a time-independent random variable distributed according to the TW distribution corresponding to GOE, $f_1(\chi)=F_1{}'(\chi)$, which can be written explicitly in terms of the Hastings-McLeod solution of the Painlevé II equation [10]. In particular, $f_1(\chi)$ has asymmetric non-Gaussian tails [10,27]: $f_1(\chi)\approx\exp(-|\chi|^3/24)$ as $\chi\to-\infty$, while $f_1(\chi)\approx\exp(-2\chi^{3/2}/3)$ as $\chi\to\infty$ [28].

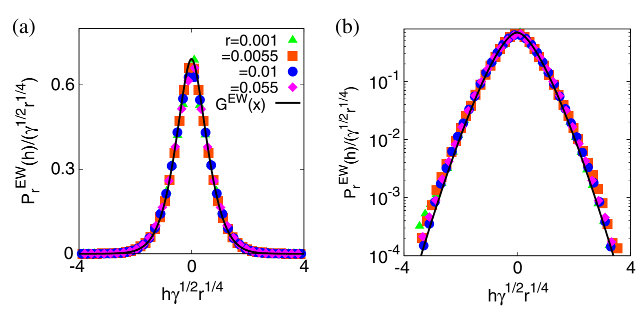

FIG. 2 (color online). EW interface with resetting: Scaling of the distribution of interface height fluctuations according to Eq. (5), on linear (a) and log-linear (b) scales. With $D=\nu=1$, giving $\gamma=\sqrt{\pi}/2^{3/2}$, the points are simulation data for $L=2^{14}$, while solid lines for $G^{\rm EW}(x)$ denote analytical results given by Eq. (6).

Equation (8) gives

$$h = (\Gamma t)^{1/3}\left[\chi - (1/L)\int_0^L \mathrm{d}x\chi(x)\right]. \quad (9)$$

Knowing that $f_1(\chi)$ has a finite mean $\langle\chi\rangle < 0$, it follows from the law of large numbers that in the limit $L \to \infty$, the second term on the rhs converges to $\langle\chi\rangle$, so that $\langle h\rangle = 0$. In this case, in the limit $\tau \to \infty$, $h \to \infty$, keeping $h/\tau^{1/3}$ fixed, $P(h,\tau)$ takes the scaling form

$$P(h,\tau) \sim \frac{1}{(\Gamma\tau)^{1/3}}\tilde{f}_1\left(\frac{h}{(\Gamma\tau)^{1/3}}\right), \quad (10)$$

where $\tilde{f}_1(x) \equiv f_1(x + \langle\chi\rangle)$. From Eq. (4), we see that in the limit $r \to 0$, $P_r^{\mathrm{KPZ}}(h)$ has a universal scaling form, as follows from the fact that the integral (4) in this limit is dominated by the limit $\tau \to \infty$ where $P(h,\tau)$ can be replaced by its scaling form (10) for $h \to \infty$, with $h/\tau^{1/3}$ fixed. Hence, for $r \to 0$, $h \to \infty$, with $hr^{1/3}$ fixed, we get

$$P_r^{\mathrm{KPZ}}(h) \sim (r\Gamma^{-1})^{1/3} G^{\mathrm{KPZ}}[(r\Gamma^{-1})^{1/3}h], \quad (11)$$

where the scaling function $G^{\mathrm{KPZ}}(x)$ is given by

$$G^{\mathrm{KPZ}}(x) = \int_0^\infty \mathrm{d}y \frac{e^{-y}}{y^{1/3}}\tilde{f}_1\left(\frac{x}{y^{1/3}}\right). \quad (12)$$

In contrast to $G^{\mathrm{EW}}(x)$, $G^{\mathrm{KPZ}}(x)$ is not symmetric in x. Since $\tilde{f}_1$ has zero mean, it follows that G^{KPZ} has also vanishing mean, but is still asymmetric, with a variance $\int x^2 G^{\mathrm{KPZ}}(x)dx \approx 1.44$. From (11), the stationary width scales as $W_r^{\mathrm{KPZ}} \sim r^{-1/3}$. The asymptotic behaviors of $G^{\mathrm{KPZ}}(x)$ for $x \to \pm\infty$, obtained from the corresponding behaviors of $\tilde{f}_1(x)$ combined with a saddle point analysis, are

$$G^{\mathrm{KPZ}}(x) \approx \begin{cases} \exp(-|x|^{3/2}/\sqrt{6}), & x \to -\infty \\ \exp(-3^{1/3}x), & x \to +\infty. \end{cases} \quad (13)$$

Equation (12) implies that $G^{\mathrm{KPZ}}(x)$ has a nonanalytic behavior as $x \to 0$: $G^{\mathrm{KPZ}}(x) \sim A + Bx + Cx^2 \ln x$, with A, B, C being constants. Nonanalyticity at the resetting value was also observed for the EW interfaces, Eq. (7), and, hence, is a generic feature of stochastic resetting.

To confirm the scaling form (11), we performed simulations of a discrete one-dimensional periodic interface $\{H_i(t)\}_{1\le i\le L}$ evolving in discrete times t. The interface resets to the initial flat configuration with probability r, and is updated with probability $1 - r$ according to the dynamics of the ballistic deposition model in the KPZ class [1–3]: $H_i(t+1) = \max[H_{i-1}(t), H_i(t)+1, H_{i+1}(t)]$. Comparison between simulations and theory in Fig. 3 shows a very good agreement. The comparison requires computing the

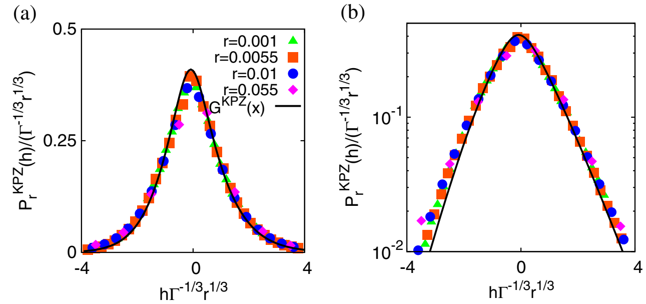

FIG. 3 (color online). KPZ interface with resetting: Scaling of the distribution of interface height fluctuations according to Eq. (11), on linear (a) and log-linear (b) scales. Here, $\Gamma = 1.05$, points are simulation data for $L = 2^{15}$, while solid lines for $G^{\mathrm{KPZ}}(x)$ denote analytical results given by Eq. (12).

integral in (12) by using the TW GOE distribution with mean shifted to zero, and scaling the data by a model-dependent fitting parameter equivalent to Γ in (8). As for the EW case, $P_r(h)$ is non-Gaussian.

For a general interface with scaling exponents α, z, and $\beta = \alpha/z$, we now give scaling arguments for $P_r(h)$. In the limit $\tau \to \infty$ and $h \to \infty$, with h/τ^β fixed, the distribution $P(h,\tau)$ has the general scaling form $P(h,\tau) \sim \tau^{-\beta} g(h\tau^{-\beta})$. In this case, the distribution $P_r(h)$ is universal in the limit $r \to 0$ and $h \to \infty$, keeping hr^β fixed [see the discussion following Eq. (10)]. The associated scaling function is obtained by using the above form for $P(h,\tau)$ in Eq. (4), giving

$$P_r(h) \sim r^\beta G(hr^\beta), \qquad G(x) = \int_0^\infty \frac{dy}{y^\beta} e^{-y} g\left(\frac{x}{y^\beta}\right), \quad (14)$$

implying in particular the behavior of the stationary width $W_r \sim r^{-\beta}$ as $r \to 0$. The EW and KPZ interfaces correspond to $\beta = 1/4$ and $\beta = 1/3$ respectively. In the generic case when $g(x) \sim \exp(-ax^{\gamma_\pm})$ as $x \to \pm\infty$, one obtains by a saddle point analysis of (14) that $G(x) \sim \exp(-bx^{\nu_\pm})$ as $x \to \pm\infty$ with $\nu_\pm = \gamma_\pm/(1 + \beta\gamma_\pm)$. Note also that (14) implies that $G(x)$ is generically nonanalytic at $x = 0$.

One of the interesting conclusions of our study is the following. In $1 + 1$ dimensions and without resetting, the stationary height distribution in a finite system is identical (Gaussian) for both EW and KPZ interfaces, despite the fact that their dynamics are entirely different. This stationary state carries no information about the dynamics. However, with resetting, the resulting NESS is very different in the two cases even in $1 + 1$ dimensions. Thus, the information about the dynamics is carried forward to the stationary state via resetting events. Hence, the resetting-induced NESS is much richer than the ordinary stationary state induced by the finite system size.

In this Letter, we addressed a very general interesting question: What happens when a many-body interacting system evolving under its own stochastic dynamics is subject to repeated interruption and recommencement at random times? Does it lead to a NESS? If so, can one characterize this state? Related questions came up recently in biological contexts, where stochastic resetting or switching between different phenotypic states allows organisms to survive in randomly fluctuating environments [29–31]. As a first step towards answering this generic question, we studied in this work one-dimensional interfaces subject to stochastic resetting, for which we were able to completely characterize the NESS analytically. Our analysis, where the resetting occurs to a flat configuration, is readily extendible to cases where the resetting configuration is sampled from a given distribution. Our results may thus serve as a genesis and a benchmark for future studies on stochastic resetting of more complex many-particle systems. Moreover, the results are amenable to possible verification in experiments, in particular, in the recent ones on liquid crystals [12,13].

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