Quantum Criticality of Quasi-One-Dimensional Topological Anderson Insulators

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We present an analytic theory of quantum criticality in the quasi-one-dimensional topological Anderson insulators of class AIII and BDI. We describe the systems in terms of two parameters (g, χ) representing localization and topological properties, respectively. Surfaces of half-integer valued χ define phase boundaries between distinct topological sectors. Upon increasing system size, the two parameters exhibit flow similar to the celebrated two-parameter flow describing the class A quantum Hall insulator. However, unlike the quantum Hall system, an exact analytical description of the entire phase diagram can be given. We check the quantitative validity of our theory by comparison to numerical transfer matrix computations.

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The recent discovery of band insulators carrying topologically protected invariants has triggered a surge of research activity [1]. With regard to any potential application of these materials, stability against perturbations, including the inevitable presence of static disorder, is a prime issue. First indications that topology has qualitative effects on the localization properties of disordered electronic systems were, in fact, noticed before the advent of topological materials [2–5]. However, the reverse question, i.e., what happens if disorder is added to a clean topological band insulator (TI), has drawn attention only recently [6–10]. It is now understood that in low dimensions, $d \leq 2$, disorder drives a crossover from a topological band insulator to a topological variant of an Anderson insulator (TAI). The latter has to be topologically charged because the phases carrying different indices in the clean case cannot simply "disappear" even in the presence of disorder strong enough to fill the band gap. This means that the phase transition *points* emerging when a control parameter μ characterizing the clean system is varied must turn into *lines* of phase transitions meandering through a phase plane spanned by μ and the disorder strength w (cf. Fig. 1.) For a number of TIs, the ensuing phase diagrams have been portrayed by numerical methods [6–8], and for one-dimensional topological superconductors, phase transition points have been identified by transfer matrix techniques [9,10].

However, the best studied example of a TAI remains the quantum Hall (QH) insulator. Within the QH context, the crucial role played by disorder with regard to criticality, edge state formation, and other phenomena was appreciated right after the discovery of the effect [11]. Its influence on the universal physics of the QH effect is described by Pruisken's theory [12], a field theory governed by two parameters (g, χ), where g is a measure of longitudinal electric conduction and χ is a topological parameter proportional to the Hall conductance. At the bare level, both g and χ are nonuniversal functions of system parameters, where $g \gg 1$ signifies a "weakly disordered" system, and half-integer values $\chi = n + 1/2$ define the demarcation lines between sectors of different topological index. Increasing the system size, the parameters (g, χ) undergo renormalization group flow either towards TAI states $(0, n)$ with vanishing conductance and integer Hall angle or towards QH transition points $(g^*, n + 1/2)$ at criticality. Unfortunately, the fixed points are buried deep in the realm of strong coupling, $g^* = \mathcal{O}(1)$, and thus far evade analytical treatment. Similar physics is observed for the class C quantum spin Hall system [13], although in this case equivalence to a classical percolation transition allows for more complete analytic treatment [14].

The statement made herein is that a nearly identical scenario repeats itself in quasi-1D disordered topological insulators, viz. the $\mathbb{Z}$ insulators of symmetry classes AIII and BDI. In these cases the topological index n, playing a role analogous to the number of filled Landau levels in QH, counts the number of zero-energy edge states. The addition of disorder to quasi-1D insulators of *finite* length L creates intragap states, which turn the system into a conductor, $g(L) \neq 0$, thus compromising its topological integrity: while the index $n \in \mathbb{Z}$ of a given system continues to be integer valued, its value depends on the impurity configuration, with generally noninteger mean $\chi \equiv \langle n \rangle$. Only upon increasing the system size to infinity localization effects ultimately restore a self-averaging topological index $\chi(L \to \infty) \in \mathbb{Z}$, which is now *stabilized* by the presence of disorder. The function $\chi(L)$, describing the reentrance flow towards an integer, plays a role conceptually analogous to the 2D Hall conductance. Our main results are flow

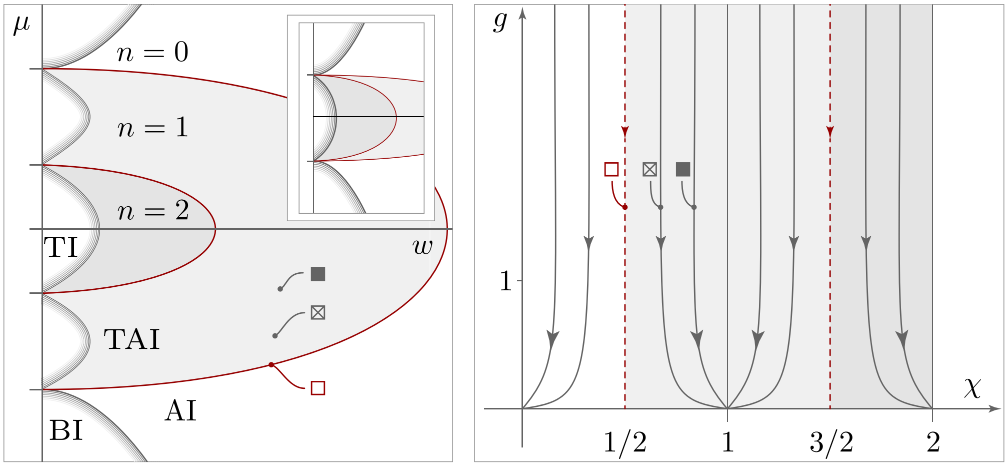

FIG. 1 (color online). Left: (μ, w) phase diagram of TI. Hatched areas denote crossover regions between band (BI) and Anderson insulator. Red lines mark phase transitions between sectors of different topological index n. The inset describes a situation where the clean system, $w = 0$, has degeneracies and phase transition points coalesce (e.g., the case of $N > 1$ uncoupled topological chains where interchain hopping is solely due to disorder). Right: The corresponding (g, χ) phase diagram. For bare parameters corresponding to the bulk of a topological phase (cf. the lines marked by a solid and a crossed box), there is exponentially fast flow in the system size L towards an insulating configuration: $g = 0$ and an integer topological index $\chi = n$. At criticality, $\chi + 1/2$ (open box), the flow towards $g = 0$ is algebraic in L, a signature of a critical state.

diagrams for the two parameters $(g(L), \chi(L))$ strikingly similar to those of the QH systems, but unlike those fully tractable by analytic means.

Before turning to model-specific calculations, let us discuss the structure of the flow in qualitative terms. In the absence of disorder, $\chi = n$ reduces to the standard $\mathbb{Z}$-valued winding number [15], characterizing the clean system. Disorder strong enough to fill the gap, but too weak to localize, $L < \tilde{\xi}$, where $\tilde{\xi} \sim Nl$ is the localization length, $l \sim |w|^2$ the elastic scattering mean free path, and N the number of quantum channels carried by the system, renders the system effectively metallic with Ohmic conductance $g(L) = \tilde{\xi}/L$. The index $\chi(L) = \tilde{\chi}(\mu, w, \ldots) \neq n$ here assumes a nonuniversal real value, where $(\ldots)$ denotes dependence on all other microscopic system parameters. Upon increasing L, the system enters the localized regime, $g(L) \sim \exp[-L/\xi(\tilde{\chi})]$, characterized by an effective length scale $\xi(\tilde{\chi}) \sim |\tilde{\chi} - n - 1/2|^{-\nu}$, where ν is a correlation length exponent. Along with the flow $g(L) \to 0$ towards a TAI configuration, the index $\chi(L)$ flows from its bare value $\tilde{\chi}$ back to the nearest integer $\chi(L) \overset{L\to\infty}{\to} [\tilde{\chi}] = n$. The exponentially fast two-parameter flow $(g(L), \chi(L)) \to (0, n) + \mathcal{O}(e^{-L/\xi(\tilde{\chi})})$ is the quasi-1D analog of the 2D class A QH scaling [12,16] (cf. Fig. 1). Transitions between distinct topological sectors are marked by half-integer values $\tilde{\chi} = n + 1/2$. At criticality the parameter $\chi(L) = \tilde{\chi} = n + 1/2$ remains stationary and algebraic scaling $g(L) \sim L^{-\alpha}$ of the average conductance signifies the presence of a critical delocalized state. Such delocalization at fine-tuned parameter values has been observed first for chiral classes [2] and then for all five symmetry classes [4], which were later understood to possess topologically nontrivial band insulator limits.

The analogy to QH extends to the formal level in that the quasi-1D insulators, too, are described by a two-parameter field theory. On the bare level, the theory is determined by the pair $(\tilde{\xi}, \tilde{\chi})$ describing the system at length scales $l < L < \tilde{\xi}$. Technically, these constants are computed from a self-consistent Born approximation (SCBA) to the Green function, and criticality is detected by probing the half-integerness of $\tilde{\chi}$. We numerically confirm that this procedure accurately describes the phase diagram for given models of disorder. The running observables $(g(L), \chi(L))$ are then extracted by probing the sensitivity of the theory to twists in real-space boundary conditions, an operation analogous to Pruisken's "background field" method [17,18]. Before turning to the BDI insulator, we introduce the approach on the somewhat simpler AIII system.

AIII insulator.—Consider a system of N chains of length L described by the Hamiltonian $\hat{H} = \sum_l \{C_l^\dagger[(\mu + t) + (\mu - t)\hat{P}]C_{l+1} + C_l^\dagger \hat{V}_l C_{l+1}\} + \mathrm{H.c}$, where $C_l = \{c_{l,j}\}$ is a vector of fermion creation operators, $j = 1, \ldots, N$ is the chain index, $l = 1, \ldots, L$ labels chain sites, the intrachain hopping is staggered by the parameter $a = |\mu| - t$, and $\hat{P}c_{l,j} = (-)^l c_{l,j}$ acts as a "parity operator." The matrix $\hat{V}_l$ describes the random interchain hopping, which is Gaussian distributed with correlators $\langle V_l^{ij}(V_{l'}^{i'j'})^*\rangle = (w^2/N)\delta_{ij}\delta_{i'j'}\delta_{ll'}$. The anticommutativity of our (time-reversal noninvariant) Hamiltonian with sublattice parity $[\hat{H}, \hat{P}]_+ = 0$ indicates that the system belongs to symmetry class AIII. In the absence of disorder, a topolgical index, or "winding number," may be defined as [15] $n \equiv -i\sum_{q=1}^{N}\int_0^{2\pi}(dk/2\pi)\mathrm{tr}(Q^{-1}\partial_k Q)$, where $Q \equiv H_{+-}$ is the block of the Hamiltonian connecting sites of positive and negative parity, and k, q are Fourier indices, $c_{k,q} = (1/LN)\sum_{l,j}e^{i(kl/L+q2\pi j/N)}c_{l,j}$. Assuming real hopping amplitude $t > 0$ for simplicity, a straightforward calculation shows that $n = N\Theta(t-|\mu|)$, which identifies the amplitude μ as a topological control parameter triggering a transition from $n = 0$ to $n = N$ at $\mu = \pm t$ [19]. To access the index n in a way not tied to translational invariance, we imagine the chains compactified to a ring of circumference L and pierced by a staggered flux ϕ_0 which affects the fermions as $c_{l,j} \to e^{i\hat{P}\phi_0 l/L}c_{l,j}$ and the momentum representation of the chiral blocks as $Q(k) \to Q(k+\phi_0)$ and $Q^\dagger(k) \to Q^\dagger(k-\phi_0)$, respectively. Now consider the generating function $Z(\phi) = \det(\hat{G}_0^{-1}(\phi_0))/\det(\hat{G}_0^{-1}(-i\phi_1))$, where $\phi = (\phi_0, \phi_1)^T$ and $\hat{G}_\epsilon(\phi_\alpha) = [\epsilon^+ - \hat{H}(\phi_\alpha)]^{-1}$ is the retarded Green function of the gauged Hamiltonian. The transformation of Q then implies the representation $n \equiv \chi = -i\partial_{\phi_0}|_{\phi=(0,0)}Z(\phi)$, which no longer relies on the momentum space language. In this formula we also anticipate that, in nontranslationally invariant systems, n may generalize to a noninteger parameter χ.

Field theory.—We next ask how the system responds to the presence of disorder. The generating function averaged over a Gaussian distribution of the bond amplitudes $V_{j,l}$ affords a representation in terms of a nonlinear σ model, $Z(\phi) = \int \mathcal{D}T \exp(-S[T])$, with the action [3]

$$S[T] = \int_0^L dx\left[\frac{\tilde{\xi}}{4}\mathrm{str}(\partial_x T\partial_x T^{-1}) + \tilde{\chi}\mathrm{str}(T^{-1}\partial_x T)\right]. \quad (1)$$

Here, "str" is the supertrace and

$$T = U\begin{pmatrix} e^{y_1} & \square \\ \square & e^{iy_0}\end{pmatrix}U^{-1}$$

is a 2×2 supermatrix field parametrizing the field space GL(1|1) [20] in terms of two radial coordinates $y_1 \in \mathbb{R}$, $y_0 \in [0, 2\pi]$ and two Grassmann valued angular variables ρ, σ, where

$$U = \exp\begin{pmatrix} \square & \rho \\ \sigma & \square\end{pmatrix}.$$

The theory contains two coupling constants, the localization length $\tilde{\xi}$ and the coefficient of the topological term $\tilde{\chi}$. The latter is computed from the underlying microscopic theory as [3] $\tilde{\chi} = -(i/2)\mathrm{tr}(\hat{G}^+\hat{P}\partial_k\hat{H})$, where $\hat{G}^+$ now stands for the zero-energy Green function averaged over disorder within the self-consistent Born approximation. For vanishingly weak disorder, $G^+ \to -\hat{H}^{-1}$ and $\tilde{\chi} \to n$ reduces to the winding number. The value of $\tilde{\chi}$ in the presence of disorder depends on model specifics and will be discussed in more concrete terms below. The external parameters ϕ enter the theory through a twisted boundary condition, $T(L) = \mathrm{diag}(e^{\phi_1}, e^{i\phi_0})$, and $T(0) = \mathbb{1}$. Finally, we note that the boundary shift in the noncompact "angle" ϕ_1 enables us to extract the conductance of the wire as [21,22] $g = (\partial_0^2 + \partial_1^2)|_{\phi=(0,0)}Z$, where $\partial_\alpha = \partial_{\phi_\alpha}$ for brevity. The rationale behind this expression is that the second derivatives $\partial_\alpha^2 Z \sim \partial_\phi^2 \ln\det(G_0(\phi_\alpha))$ probe the average "curvature" of the ϕ-dependent energy levels which, according to Thouless [23], is a measure of the system's conductance. Since in the chiral classes retarded and advanced zero-energy Green functions are related [3,22], the conductance is indeed expressible through the generating function of a single Green function.

Our goal is to understand the scaling of the observables (g, χ) in dependence on the system size L. In the metallic regime $l < L \ll \tilde{\xi}$, "size quantization" of the operator $\partial_x T\partial_x T^{-1}$ suppresses fluctuations of T. Substitution of the minimal configuration compatible with the boundary conditions $T(x) = \mathrm{diag}(e^{\phi_1 x/L}, e^{i\phi_0 x/L})$ then yields $(g, \chi) = (\tilde{\xi}/L, \tilde{\chi})$, i.e., an Ohmic conductance, and a topological index set by the bare value $\tilde{\chi}$. To understand what happens upon entering the localization regime, $L \gtrsim \tilde{\xi}$, it is convenient to think of x as imaginary time and of $Z(\phi) \equiv \Psi(\phi, L)$ as the path integral describing the free motion [first term in the action Eq. (1)] of a particle moving on the manifold GL(1|1) in the presence of a constant gauge flux (the second term). Central to that picture (cf. Ref. [3] and the Supplemental Material [24] for a more extensive discussion) is an interpretation of $\Psi(\phi, L)$ as a propagator subject to the imaginary time Schrödinger or heat equation

$$\tilde{\xi}\partial_x\Psi(y,x) = \frac{1}{J(y)}(\partial_\alpha - iA_\alpha)J(y)(\partial_\alpha - iA_\alpha)\Psi(y,x), \quad (2)$$

where $J^{-1}\partial_\alpha J\partial_\alpha$ is the radial Laplacian on GL(1|1), the Jacobian $J(y) = \sinh^{-2}[\frac{1}{2}(y_1 - iy_0)]$ accounts for the curvature of that manifold, and the vector potential $A = \tilde{\chi}(1, i)^T$ represents the flux. The eigenfunctions of the gauge-coupled Laplacian are given by $\Psi_l(y) = \sinh[\frac{1}{2}(y_1 - iy_0)]e^{il_\alpha y_\alpha}$, and the corresponding eigenvalues by $\epsilon_l = (l_0 - \tilde{\chi})^2 + (l_1 - i\tilde{\chi})^2$. Given the eigensystem, the time-dependent equation is solved by the spectral decomposition, which, after substitution of the boundary values $y = \phi$, takes the form $\Psi(\phi, L) = 1 + \frac{1}{\pi}\sum_{l_0\in\mathbb{Z}+\frac{1}{2}}\int dl_1 (l_0 + il_1)^{-1}\Psi_l(\phi)e^{-\epsilon_l L/\tilde{\xi}}$, where 1 is by supersymmetric normalization of the partition function $Z(0, x) = 1$, and the l-dependent denominator implements the spectral decomposition of the initial condition $\Psi(\phi, x \to 0)$.

From this representation, it is straightforward to compute the first- and second-order expansions in ϕ to arrive at the result

$$g = \sqrt{\frac{\tilde{\xi}}{\pi L}} \sum_{l_0 \in \mathbb{Z}+1/2} e^{-(l_0-\tilde{\chi})^2 L/\tilde{\xi}},$$

$$\chi = n - \frac{1}{4} \sum_{l_0 \in \mathbb{Z}+1/2} \left[\mathrm{erf}\left(\sqrt{\frac{L}{\tilde{\xi}}} (l_0 - \delta\tilde{\chi}) \right) - (\delta\tilde{\chi} \leftrightarrow -\delta\tilde{\chi}) \right], \quad (3)$$

where $\delta\tilde{\chi} = \tilde{\chi} - n$ is the deviation of $\tilde{\chi}$ off the nearest integer value n. For generic bare values $(\tilde{\xi}, \tilde{\chi})$, the two formulas describe an exponentially fast approach towards an insulating state $(0, n)$ upon increasing length L. At criticality $(\tilde{\xi}, n + 1/2)$, the topological angle remains invariant, while an algebraic decay of the conductance $g(L) \approx (\tilde{\xi}/\pi L)^{1/2}$ signifies the presence of a delocalized state at the band center. The emergence of power law scaling at criticality can be described in terms of an effective correlation length $\xi(\chi) = \tilde{\xi}|\tilde{\chi} - n - 1/2|^{-\nu}$. Comparing the ansatz $g \sim \exp[-L/\xi(\tilde{\chi})]$ to the result above, we identify the correlation length exponent $\nu = 2$ [25]. A number of flow lines are shown graphically in Fig. 2. which is the 1D analog of the two-parameter flow diagram [16] describing criticality in the integer QH system.

Class BDI.—We next extend our discussion to the presence of time-reversal, symmetry class BDI. Systems of this type are realized, e.g., Ref. [26], as lattice p-wave superconductors with Hamiltonian $\hat{H} = \sum_{l=1}^{L} [C_l^\dagger \hat{H}_{0,l} C_l + (C_l^\dagger \hat{H}_{1,l} C_{l+1} + \mathrm{H.c.})]$, where the spinless fermion operators $C_l = (c_{l,j}, c_{l,j}^\dagger)^T$ are vectors in channel and Nambu spaces. The on-site part of the Hamiltonian, $\hat{H}_{0,l} = (\mu/2 + \hat{V}_l)\sigma_3$, contains the chemical potential μ and real symmetric interchain matrices $\hat{V}_l$, with σ_i acting in Nambu space. The contribution $\hat{H}_{1,l} = -\frac{1}{2} t_l \sigma_3 + \frac{i}{2} \hat{\Delta}_l \sigma_2$ contains nearest-neighbor hopping t_l, and the order parameter $\hat{\Delta}_l$ here is assumed to be imaginary for convenience. Quantities carrying a subscript l may contain site-dependent random contributions. The first quantized representation of $\hat{H}$ obeys the chiral symmetry $[\hat{P}, \hat{H}]_+ = 0$, with $\hat{P} = \sigma_1$. The clean system supports $n \leq N$ Majorana end states, where n decreases upon increasing μ. Generalizing to the presence of disorder, we obtain a pattern of phase transition lines similar to the one discussed above. Before turning to field theory, we apply transfer matrix methods to a numerical description of the ensuing phase portrait.

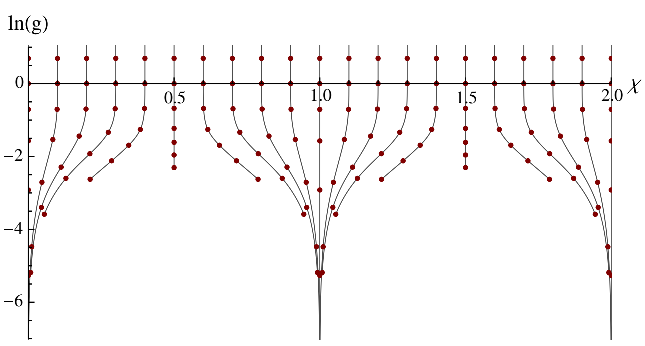

FIG. 2 (color online). Flow of the conductance g and the topological parameter χ as a function of system size. Dots are for values $L/\tilde{\xi} = \frac{1}{4}, \frac{1}{2}, 1, 2, \ldots, 32$.

Transfer matrix.—Defining a doublet $\eta_l = (\psi_{l+1}, \psi_l)^T$, one verifies that the zero-energy eigenfunctions ψ_l of the lattice Hamiltonian obey the recursion relation $\eta_{l+1} = \mathcal{T}_l \eta_l$, where

$$\mathcal{T}_l = \begin{pmatrix} -H_{1,l}^{-1} H_{0,l} & -H_{1,l}^{-1} H_{1,l}^\dagger \\ 1 & 0 \end{pmatrix},$$

and we assumed nondegeneracy of the hopping matrices $\{H_{1,l}\}$. We iterate this equation to obtain $\eta_L = \mathcal{T} \eta_1$, where $\mathcal{T} = \prod_l^L \mathcal{T}_l$ is the transfer matrix. The presence of a chiral structure means that $\mathcal{T} = \mathrm{bdiag}(\mathcal{T}^{11}, \mathcal{T}^{22})$ can be brought to a block-diagonal form. Representing the positive eigenvalues of $\mathcal{T}^{11}$ as $\exp(L\lambda_j)$, $j = 1, \ldots, N$, the index n of individual systems is given by the number of negative Lyapunov exponents, $\lambda_j < 0$ [27]. We numerically compute the average of these numbers by sampling from a Gaussian distribution of on-site potentials $\hat{V}_l$ with correlators $\langle V_l^{ij} V_l^{i'j'} \rangle = (w^2/N)(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'})$ and for a model with $t = \Delta$. The results are shown in Fig. 3, where boxes indicate changes of the topological number.

Field theory.—The field theoretical description of the system parallels that of the AIII insulator. Because of time-reversal invariance, the fields now are 4×4 matrices spanning the coset space $\mathrm{GL}(2|2)/\mathrm{OSp}(2|2)$ [20], and the field action is given by Eq. (1) as before. As in class

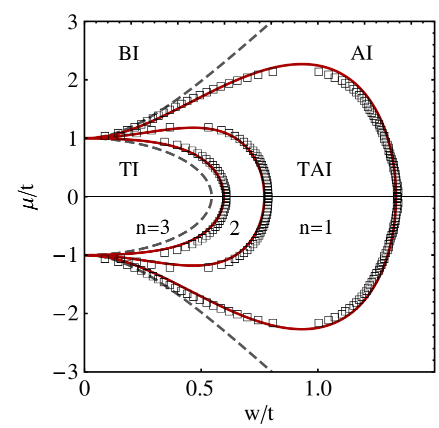

FIG. 3 (color online). Phase diagram of the class BDI 3-channel wire. Dashed lines show crossover regions between BI and AI or TI and TAI phases, derived from the SCBA. Solid lines correspond to half-integer values of the SCBA computed index $\tilde{\chi}$ and mark boundaries between phases of different n. BI and AI have $n = 0$, while $n > 0$ for TI and TAI. Data points are phase boundaries found from a numerical analysis of Lyapunov exponents λ_j.

AIII, the topological coupling constant is given by $\tilde{\chi} = -(i/2)\mathrm{tr}(\hat{G}^+\hat{P}\partial_k\hat{H})$, where the Green function $\hat{G}^+ = (i0 - \hat{H} - \hat{\Sigma})^{-1}$ contains a self-energy $\hat{\Sigma} = -i\Sigma_0\sigma_0 + \Sigma_3\sigma_3$. For the random $\hat{V}_l$ also used in the transfer matrix study, the latter is determined by the SCBA equation $-i\hat{\Sigma}_a = i^a w^2 \mathrm{tr}(\hat{G}^+_{lj,lj}\sigma_a)$. Solving this algebraic equation numerically, one obtains contour lines of half-integer $\tilde{\chi}$ in excellent agreement with the numerical transfer matrix data (cf. Fig. 3). The observable pair (g, χ) can be extracted from the field theory as in the AIII theory. Referring to the Supplemental Material [24] for details, we note that the analysis of a heat equation, somewhat more complicated than the one before, reveals a flow pattern similar to the one shown in Fig. 2: delocalization at half-integer $\tilde{\chi}$, and exponentially fast flow to integer χ away from these critical values.

Edge states.—The analogies to the 2D quantum Hall effects extend to the physics at the boundary. For generic $\tilde{\chi}$ and at length scales large in comparison to the localization $\xi(\tilde{\chi})$ the theory is effectively described by an action $S_{\mathrm{eff}}[T] = \int_0^L dx\, n\, \mathrm{str}(T^{-1}\partial_x T) = n\{\mathrm{str}\ \ln[T(L)] - \mathrm{str}\ \ln[T(0)]\}$. This is the 1D analog of the quantum Hall boundary action. Much as the latter requires integer quantization of the Hall conductivity (i.e., fully developed bulk localization), the multivaluedness of the logarithm requires integer index n. The operators $n\, \mathrm{str}\ \ln(T)$ generate a peak $n\delta(\epsilon)$ in the quasiparticle density of states [3]; i.e., they describe the presence of n edge states at the left and the right boundary of the system, the 1D analog of QH edge states.

Discussion.—In this Letter, we have presented the first fully analytical description of disorder-induced quantum criticality in quasi-one-dimensional $\mathbb{Z}$-topological band insulators. The theory describes the systems in terms of the configurational average of two fundamental variables, the conductance and the topological index. It predicts the evolution of these variables from their initial values, depending on the system's microscopic parameters, to universal values at large system size. Knowledge of both the microscopic bare values of the observables and the flow pattern enables us to describe critical scaling in quantitative agreement with numerical simulations. The main, somewhat counterintuitive, message is the reentrance behavior and stabilization of the topological index by the localization effects. We have discussed this behavior for insulators of class AIII and BDI, and the third $\mathbb{Z}$-topological quantum wire, class CII, can be described in similar terms. Our analysis testifies to a large degree of universality in the low-energy physics of the one- and two-dimensional $\mathbb{Z}$-topological insulators.

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