Superdiffusive Modes in Two-Species Driven Diffusive Systems

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Using mode-coupling theory and dynamical Monte Carlo simulations, we investigate the scaling behavor of the dynamical structure function of a two-species asymmetric simple exclusion process, consisting of two coupled single-lane asymmetric simple exclusion processes. We demonstrate the appearance of a superdiffusive mode with dynamical exponent $z = 5/3$ in the density fluctuations, along with a Kardar-Parisi-Zhang mode with $z = 3/2$; we argue that this phenomenon is generic for short-ranged driven diffusive systems with more than one conserved density. When the dynamics is symmetric under the interchange of the two lanes, a diffusive mode with $z = 2$ appears instead of the non-Kardar-Parisi-Zhang superdiffusive mode.

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Transport in one dimension has, for a long time, been known to usually be anomalous [1,2]. Signatures of this behavior are a superdiffusive dynamical structure function and a power-law divergence of transport coefficients with system size, characterized by universal critical exponents. Unfortunately, however, despite a vast body of work, analytical results for model systems have remained scarce and numerical results are often inconclusive. Therefore, some ten years ago, the exact calculation [3] of the dynamic structure function for the universality class of the Kardar-Parisi-Zhang (KPZ) equation [4] with dynamical exponent $z = 3/2$ came as a major breakthrough. This function was obtained for a specific driven diffusive system, the asymmetric simple exclusion process, which has a single conserved density and hence a single mode, the KPZ mode. By virtue of universality, then, systems such as growth models [5,6] or one-dimensional driven fluids [7] can also be described in terms of the KPZ universality class.

More recently, in the context of anharmonic chains [8] and very general short-ranged one-dimensional Hamiltonian systems [9], it was established that in the presence of more than one conserved quantity, the dynamics is richer and other modes have to be expected. In particular, in systems with three conservation laws a heat mode with $z = 5/3$ may be present, in addition to two KPZ modes. The main assumption underlying these conclusions is that the relevant slow variables are the long-wavelength Fourier components of the conserved quantities, and products of these Fourier components [2,9].

Going back to driven diffusive systems, we note that—somewhat surprisingly—there is little information about the dynamical structure functions in driven diffusive systems with more than one conservation law. In one dimension, these systems are known to exhibit extremely rich stationary and dynamical behavior; they serve widely as paradigmatic models for the detailed study of nonequilibrium fluctuation phenomena such as shocks [10–12], spontaneous symmetry breaking and hysteresis [13,14], phase separation and coarsening [15–17], or dynamical phase transitions [18]. In view of this, it is of interest to explore the transport properties of such systems; in particular, to explore which modes govern the fluctuations of the locally conserved slow modes and how these decay in time.

In this spirit, Ferrari *et al.* [19] very recently studied a two-species exclusion process, using mode-coupling theory and Monte Carlo simulations, and found two very clean KPZ-modes, but no other modes. For a similar model, exact finite-size scaling analysis of the spectrum indicates a dynamical exponent $z = 3/2$ [20]. In older work on other lattice gas models with two conservation laws, the presence of a KPZ mode and a diffusive mode was observed [11,21]. So far, there has been no indication of the existence of a heatlike mode with $z = 5/3$. In the light of the work on short-ranged Hamiltonian systems [8,9], this is intriguing; it raises the question whether a heatlike mode can exist in driven diffusive systems, and, if yes, how many conservation laws are required to generate it. In this Letter we answer these questions by using the mode-coupling theory developed in [8,19] for nonlinear fluctuating hydrodynamics and by confirming the analytical findings with Monte Carlo simulations of a two-species asymmetric simple exclusion process. It will transpire that a superdiffusive $z = 5/3$ mode exists, along with a KPZ mode, and that two conservation laws are sufficient to generate the phenomenon. Also, a KPZ mode along with a diffusive mode can occur on a line of higher symmetry, a phenomenon which cannot appear in Hamiltonian systems [22].

We consider the following stochastic lattice gas model. Particles hop randomly on two parallel chains with N sites each, without exchanging the lane, unidirectionally and with a hard-core exclusion and periodic boundary conditions. We denote the particle occupation number on site k in lane i by $n_k^{(i)}$. A hopping event from site k to site $k+1$ on the same lane may happen if site k is occupied and site $k+1$ is empty. The rate of hopping r_i in lane i depends on the sum of particle numbers at sites k, $k+1$ on the adjacent lane as follows (Fig. 1): Let us denote the sum of particles on the sites k, $k+1$ of lane i as $n^{(i)} := n_k^{(i)} + n_{k+1}^{(i)}$. The rates r_i of hopping from site k to site $k+1$ on lane i are then given by

$$r_1 = 1 + \gamma n^{(2)}/2, \qquad r_2 = b + \gamma n^{(1)}/2, \tag{1}$$

where $\gamma \geq -\min(1, b)$ is a coupling parameter. For $b = 1$ we recover the two-lane model of [23]. Since the hopping is only within lanes, the total number of particles M_i in each lane is conserved.

The model in a more general multilane geometry was introduced in [24]. It was shown that the choice of rates (1) results in a stationary distribution that is a product measure, both between lanes and between the sites. For the two-lane system that we study here, this leads to stationary currents

$$\begin{aligned} j_1(\rho_1, \rho_2) &= \rho_1(1-\rho_1)(1+\gamma\rho_2) \\ j_2(\rho_1, \rho_2) &= \rho_2(1-\rho_2)(b+\gamma\rho_1), \end{aligned} \tag{2}$$

where $\rho_{1,2} = M_{1,2}/N$ are the densities of particles in the first and second lane, respectively. Notice that a product measure corresponds to a grand canonical ensemble with a fluctuating particle number. These fluctuations are described by the symmetric compressibility matrix C with matrix elements

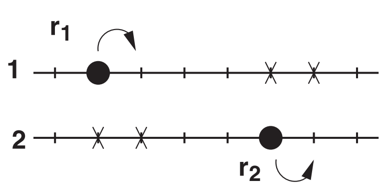

FIG. 1. Schematic representation of the two-lane totally asymmetric simple exclusion process. A particle on lane i hops to the neighboring site (provided this target site is empty) on the same lane, with rate r_i that depends on the number of particles on the adjacent sites of the other lane (marked by a cross).

$$C_{ij} = 1/N\langle (M_i - \rho_i N)(M_j - \rho_j N)\rangle = \rho_i(1-\rho_i)\delta_{i,j}. \tag{3}$$

The starting point for investigating the large-scale dynamics of this microscopic model is the system of conservation laws $\partial_t \rho_i(x,t) + \partial_x j_i(x,t) = 0$ [25], where $\rho_i(x,t)$ is the coarse-grained local density field of component i and $j_i(x,t)$ is the associated current, given as a function of the local densities by (2). With the vector $\vec{\rho}$ of densities $\rho_i(x,t)$, these equations can be written

$$\frac{\partial}{\partial t}\vec{\rho} + A\frac{\partial}{\partial x}\vec{\rho} = 0, \tag{4}$$

where A is the Jacobian with matrix elements $A_{ij} = \partial j_i/\partial \rho_j$. Its eigenvalues, c_i, are the characteristic velocities that are the speeds of local perturbations on microscopic scale [23]. The matrix AC is symmetric as a function of the ρ_i [26], which guarantees that the system (4) is hyperbolic [27].

The hydrodynamic equations (4) describe the deterministic time evolution of the density $\rho_i(x,t)$ under Eulerian scaling [25]. The effect of fluctuations, which occur on finer space-time scales, can be captured by adding phenomenological white noise terms ξ_i and taking the nonlinear fluctuating hydrodynamics approach together with a mode-coupling analysis of the nonlinear equation [19]. In this framework, one expands the local densities around their stationary values $\rho_i(x,t) = \rho_i + u_i(x,t)$ and transforms to normal modes $\vec{\phi} = R\vec{u}$ where A is diagonal. The transformation matrix R is uniquely defined by $RAR^{-1} = \mathrm{diag}(c_i)$ and the normalization condition $RCR^T = 1$. Keeping terms to first nonlinear order yields

$$\partial_t \phi_i = -\partial_x[c_i\phi_i + (1/2)\langle\vec{\phi}, G^{(i)}\vec{\phi}\rangle - \partial_x(D\vec{\phi})_i + (B\vec{\xi})_i]. \tag{5}$$

Here the angular brackets denote the inner product in component space and

$$G^{(i)} = (1/2)\sum_j R_{ij}(R^{-1})^T H^{(j)} R^{-1} \tag{6}$$

are the mode-coupling matrices obtained from the Hessian $H^{(i)}$ with matrix elements $\partial^2 j_i/(\partial\rho_j\partial\rho_k)$. The matrices D (transformed diffusion matrix) and B (transformed noise strength) do not appear explicitly below.

For strictly hyperbolic systems the normal modes have different speeds; hence, their interaction becomes very weak for long times. Thus, by identifying ϕ_i with the gradient of a height variable, one has to leading order generically two decoupled KPZ equations with nonlinearity coefficients $G_{ii}^{(i)}$. The other diagonal elements, $G_{jj}^{(i)}$, provide the leading corrections to the KPZ modes; the off-diagonal elements result in subleading corrections. We point out the scenarios relevant to our model, as predicted by mode-coupling theory. (i) If both $G_{11}^{(1)}$ and $G_{22}^{(2)}$ are nonzero, we expect two KPZ modes with $z = 3/2$. (ii) On the other hand, if, e.g., $G_{11}^{(1)} = 0$, but $G_{22}^{(1)} \neq 0$ and $G_{22}^{(2)} \neq 0$, then mode-coupling theory predicts mode 1 to be a superdiffusive mode with $z = 5/3$ and mode 2 to be KPZ. (iii) Finally, if both $G_{11}^{(1)} = G_{22}^{(1)} = 0$, but $G_{22}^{(2)} \neq 0$, then mode 2 remains KPZ while mode 1 becomes diffusive (up to possible logarithmic corrections, which may arise from cubic couplings to triplets of modes [9,28]).

For our system, the explicit forms of A and $H^{(i)}$ are

$$A = \begin{pmatrix} (1+\gamma\rho_2)(1-2\rho_1) & \gamma\rho_1(1-\rho_1) \\ \gamma\rho_2(1-\rho_2) & (b+\gamma\rho_1)(1-2\rho_2) \end{pmatrix}, \tag{7}$$

$$H^{(1)} = \begin{pmatrix} -2(1+\gamma\rho_2) & \gamma(1-2\rho_1) \\ \gamma(1-2\rho_1) & 0 \end{pmatrix}, \tag{8}$$

$$H^{(2)} = \begin{pmatrix} 0 & \gamma(1-2\rho_2) \\ \gamma(1-2\rho_2) & -2(b+\gamma\rho_1) \end{pmatrix}. \tag{9}$$

To prove that scenarios (i)–(iii) can be realized, we choose $\rho_1 = \rho_2 =: \rho$ and we set $\gamma = 1$ for convenience.

Consider first $b = 2$. Then

$$R = R_0 \begin{pmatrix} 1-\rho & -\rho \\ \rho & 1-\rho \end{pmatrix}, \tag{10}$$

where $R_0^{-1} = \sqrt{\rho(1-\rho)[\rho^2 + (1-\rho)^2)]}$. The characteristic velocities are

$$c_1 = 1 - \rho - 3\rho^2, \qquad c_2 = 2 - 3\rho - \rho^2. \tag{11}$$

The matrices $G^{(1)}$, $G^{(2)}$ are symmetric and have matrix elements $G_{11}^{(1)} = -2g_0(6\rho^4 - 8\rho^3 + 5\rho^2 + \rho - 1)$, $G_{12}^{(1)} = G_{21}^{(1)} = g_0(4\rho^3 - 10\rho^2 + 8\rho - 1)$, $G_{22}^{(1)} = -2g_0\rho(1-\rho)(2\rho^2 - 6\rho + 3)$ and $G_{11}^{(2)} = 4g_0\rho(1-\rho)$, $G_{12}^{(2)} = G_{21}^{(2)} = -g_0(1-2\rho^2)^2$, $G_{22}^{(2)} = 4g_0[1 - 3\rho(1-\rho)]$ with $g_0 = -1/2\{\rho(1-\rho)/[1-2\rho(1-\rho)]^3\}^{1/2}$. Therefore, generically, condition (i) for the presence of two KPZ modes is satisfied. However, while $G_{11}^{(2)} \neq 0$ and $G_{22}^{(2)} \neq 0\ \forall \rho \in (0,1)$, the self-coupling coefficient $G_{11}^{(1)}$ changes sign at $\rho^* = 0.45721\ldots$. Since $G_{22}^{(1)}(\rho^*) \neq 0$, the condition for case (ii), KPZ mode plus superdiffusive non-KPZ mode, is thus satisfied at density $\rho = \rho^*$. In fact, by diagonalizing A for arbitrary densities ρ_1, ρ_2, one can show that for $b \neq 1$ there is a curve in the space of densities where condition (ii) is satisfied. On the other hand, there is no density where condition (iii), $G_{11}^{(1)} = G_{22}^{(1)} = 0$, is satisfied. Indeed, numerical inspection of the mode-coupling matrices for several other parameter choices of γ and b suggests that condition (iii) cannot be satisfied when $b \neq 1$.

Next we study $b = 1$. In this case, the system is symmetric under the interchange of the two lanes, which is reflected in the relation $j_2(\rho_1, \rho_2) = j_1(\rho_2, \rho_1)$ for the currents (2). Calculating the mode-coupling matrices for $\rho_1 = \rho_2 =: \rho$ and $\gamma = 1$ yields

$$G^{(1)} = \tilde{g}_0(1+\rho)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad G^{(2)} = \tilde{g}_0\begin{pmatrix} 2-\rho & 0 \\ 0 & 3\rho \end{pmatrix}, \tag{12}$$

with $\tilde{g}_0 = -\sqrt{2\rho(1-\rho)}$. Interestingly, in this case, condition (iii) is satisfied for all ρ; i.e., mode 1 is expected to be diffusive and mode 2 is KPZ. The occurrence of a diffusive mode is somewhat counterintuitive, as both particle species interact and hop in a totally asymmetric fashion; in the case of the Arndt-Heinzel-Rittenberg model, this prevents the existence of a diffusive mode [19].

In order to check the predictions of mode-coupling theory, we performed dynamical Monte Carlo simulations, using a random sequential update where a random site is chosen uniformly in each step and jumps are performed, provided the target site is empty, with probabilities obtained from normalizing the jump rates (1) by the largest jump rate. A full Monte Carlo time step, then, corresponds to $2N$ such update steps. The initial distribution is sampled from the uniform distribution, except for the occupation number at site $N/2$, which is determined so that the normal modes can be studied separately [23]. Averages are performed over up to 10^8 realizations of the process and $N = 300\ldots600$. In order to measure the dynamical exponent, we compute the first and second moment of the dynamical structure function, from which we obtain the variance $\sigma(t) \propto t^{2/z}$ of the density distribution as a function

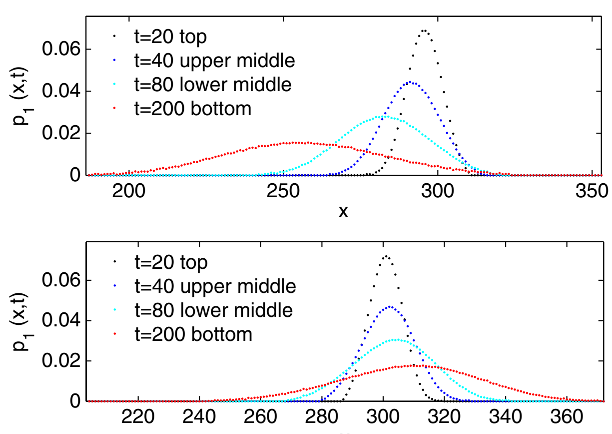

FIG. 2 (color online). Case (ii): Dynamical structure functions for particles on chain 1 for the KPZ mode (top) and the non-KPZ mode (bottom), at different times t, from Monte Carlo simulations, averaged over $2.7 \times 10^7 (5 \times 10^7)$ histories for the KPZ (non-KPZ) mode for $N = 600$. Statistical errors are smaller than symbol size.

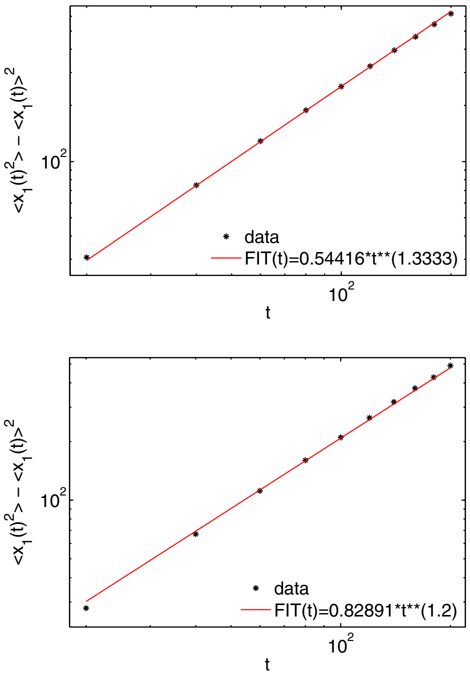

FIG. 3 (color online). Case (ii): Variance of the dynamical structure function shown in Fig. 2 as function of time. The lines with the predicted universal slopes $2/z = 4/3$ for KPZ mode (top) and $2/z = 6/5$ for the non-KPZ mode (bottom) are guides to the eye. Error bars (not shown) are approximately symbol size.

of time, for times where $\sigma(t)$ is small enough to avoid finite-size effects.

In order to test the existence of a superdiffusive non-KPZ mode, we have chosen $\gamma = -0.52588$ and $b = 1.3$. This yields $G_{22}^{(2)} = 0$ at $\rho_1^* = \rho_2^* = 0.5500003 \approx 55/100$. The matrices $G^{(1)}$, $G^{(2)}$ become

$$G^{(1)} = \begin{pmatrix} 0.2950 & 0.0717 \\ 0.0717 & 0.3157 \end{pmatrix}, \qquad G^{(2)} = \begin{pmatrix} 0.0706 & 0.2972 \\ 0.2972 & 0 \end{pmatrix},$$

which means that mode 2 is expected to be a non-KPZ mode and mode 1 is KPZ. The corresponding characteristic velocities are $c_1(\rho^*) = -0.2171$, $c_2(\rho^*) = 0.0449$, and the eigenvectors are $(-0.7465, 0.6654)^T$ for c_2 (non-KPZ mode) and $(0.6654, 0.7465)^T$ for c_1 (KPZ mode).

The simulations confirm the predictions, see Figs. 2 and 3. For both modes, the measured velocity differs from the theoretical prediction by less than 0.003. A linear least-square fit on a log-log scale of the simulation results for the variance of the non-KPZ mode 2 yields $2/z_2^{MC} = 1.19 \pm 0.02$, very close to the mode-coupling value $2/z_2 = 6/5 = 1.2$. For the amplitude $\propto t^{-1/z}$ at the maximum as a function of time, we find $1/z_2^{MC} = 0.58$, also in good agreement with $1/z_2 = 0.6$. The fitted exponent $2/z_1^{MC} = 1.302$ of the KPZ mode 1 deviates slightly from $4/3$, which is consistent with the strong coupling to the

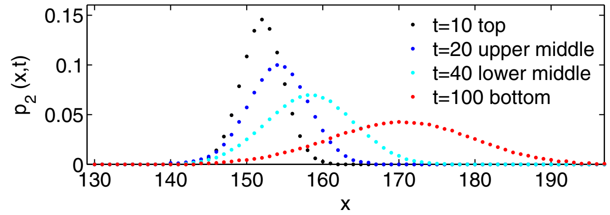

FIG. 4 (color online). Case (iii): Dynamical structure function of the diffusive mode for particles on chain 2 with $N = 300$, with $c_2 = 0.2$ (bottom) at different times t, from Monte Carlo simulations, averaged over 10^7 histories. Statistical errors are smaller than symbol size.

non-KPZ mode: The matrix element $G_{22}^{(1)} \approx 0.3157$ is larger than the KPZ self-coupling constant $G_{11}^{(1)} \approx 0.2950$.

In order to test case (iii) (KPZ and diffusive mode), we choose $\gamma = -0.8$, $b = 1$, $\rho_1 = \rho_2 = 0.5$. The characteristic velocities are $c_1 = -0.2$ [eigenvector $(1,1)^T/\sqrt(2)$] and $c_1 = 0.2$ [eigenvector $(1,-1)^T/\sqrt(2)$]. The mode-coupling matrices are given by

$$G^1 = 0.2121 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G^2 = 0.2121 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

corresponding to a KPZ mode 1 and diffusive mode 2, see Figs. 4 and 5. The characteristic velocities agree with the theoretical prediction with an accuracy of better than 1%; also, the measured scaling exponents, $2/z_1^{MC} = 1.343$, $2/z_2^{MC} = 1.030$, are in good agreement with the theoretical prediction, $2/z_1 = 4/3$ and $2/z_2 = 1$. We have also verified numerically the occurrence of two KPZ modes for generic values of the densities. This behavior is expected and the data are not shown here.

In summary, we have shown that the two-lane asymmetric simple exclusion process with two conservation laws exhibits anomalous transport, and that it has a superdiffusive non-KPZ mode with dynamical exponent $z = 5/3$ on a line in the space of conserved densities

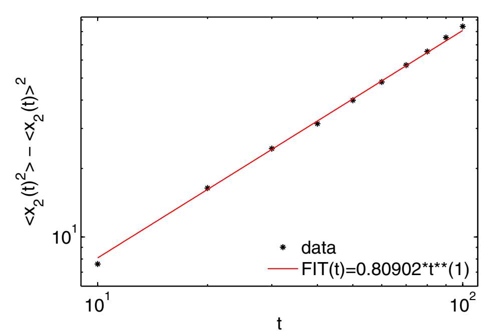

FIG. 5 (color online). Case (iii): Variance of the dynamical structure function shown in Fig. 4 as function of time. The line with the predicted universal slope $2/z = 1$ for the diffusive mode (bottom) is a guide to the eye. Error bars (not shown) are approximately symbol size.

(ρ_1, ρ_2). The heat mode in Hamiltonian systems [9] is associated with a symmetry for the velocities of the sound modes, in contrast to our two-component scenario where there is no such symmetry. In the case of higher internal symmetry, where our model is invariant under lane change, a diffusive mode can occur instead of the non-KPZ mode. This is surprising, as the hopping of both particle species is totally asymmetric. We did not find any point in parameter space where the KPZ mode would be completely absent. We argue that the existence of a superdiffusive non-KPZ mode is generic for driven diffusive systems with more than one conservation law, and it will generally occur at some specific manifold in the space of conserved densities ρ_i. This new universality class for anomalous transport in driven diffusive systems is expected to result in a novel exponent for the stationary density profile in open systems [29]. An interesting open problem raised by our findings is the role of symmetries for the suppression of the non-KPZ mode and the occurrence of a diffusive mode.

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