Pure Scaling Operators at the Integer Quantum Hall Plateau Transition

R. Bondesan, D. Wieczorek, and M. R. Zirnbauer
Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany
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Stationary wave functions at the transition between plateaus of the integer quantum Hall effect are known to exhibit multifractal statistics. Here we explore this critical behavior for the case of scattering states of the Chalker-Coddington network model with point contacts. We argue that moments formed from the wave amplitudes of critical scattering states decay as pure powers of the distance between the points of contact and observation. These moments in the continuum limit are proposed to be correlation functions of primary fields of an underlying conformal field theory. We check this proposal numerically by finite-size scaling. We also verify the conformal field theory prediction for a three-point function involving two primary fields.

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Introduction.—A revealing monitor of quantum critical behavior driven by disorder is multifractal wave-function statistics. In this vein, theory and experiment have focused on the multifractality at Anderson localization transitions between different topological phases of disordered electrons in two dimensions, the prime example being the transition between plateaus of the Hall conductance in the integer quantum Hall (IQH) effect [1].

There has long been a consensus that it should be possible to describe the IQH transition by a conformal-invariant effective field theory. Yet, in spite of many efforts [2–4], it remains an unsolved problem to identify that conformal field theory (CFT) description. To make progress with the search for it, one needs to find the conformal fields and determine their scaling dimensions. A step in this direction was taken in [5,6], where the moments of the point-contact conductance were introduced and studied as correlation functions. Alas, these are coherent sums of conformal field correlators and therefore do not give direct access to individual conformal fields in the pure form; see [7] for a recent discussion.

The purpose of this Letter is to put forth a large (and so far unrecognized) class of multifractal observables that correspond directly to correlators of CFT primary fields. Our results are motivated by a recent σ-model based classification of scaling fields at Anderson transitions [8,9]. The new feature here is that we focus on the scattering states of an open system, while the previous work concerned moments of the local density of states for closed systems. For concreteness and simplicity, we work with the Chalker-Coddington (CC) network model.

The CC model is known to be related by a duality transformation to a statistical mechanical system of the vertex-model type [10,11]. The main advance of our work is to construct lattice approximations for pure scaling fields on both sides of the duality—as scattering observables of the CC model and, equivalently, as operators of the vertex model. Both representations serve a purpose. Based on the latter, we argue that our lattice operators indeed are discretizations of pure scaling fields, while the former makes it possible to compute their conformal dimensions numerically by finite-size scaling.

CC model and scattering states.—We begin with a quick review of the CC model [12]. This is a network model for the quantum dynamics of an electron moving in two dimensions under the influence of a strong magnetic field and a random electric potential. Formulated on a square lattice, the model is built from elementary plaquettes with a definite sense of circulation that alternates between neighboring plaquettes. The links of the network are directed accordingly, so that each site has two incoming and two outgoing links. The electron wave function lives on the links and evolves in discrete time as $|\psi(t+1)\rangle = U|\psi(t)\rangle$ by a unitary operator $U = U_s U_r$. The factor U_r is a diagonal matrix modeling the propagation along the links; it assigns to each link a random, independent, and uniformly distributed U(1) phase. The factor U_s is nonrandom and consists of 2×2 matrices that describe the transfer from incoming to outgoing links at each site. When the probabilities for transfer to the left or right are equal, the model is critical and falls into the universality class of the IQH transition [12].

While it is of some interest to study the spectral properties and stationary wave function statistics of the closed network, here we turn to an open network. One major advantage of the open setting is that it allows one to formulate and study CFT correlators right at the critical point. (In contrast, Green's functions of the closed system are defined by introducing a regularization which places the system slightly off criticality.)

The network is opened up by severing a subset of links $C = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\}$, which we call point contacts. Each cut makes for one network-incoming and one network-outgoing link where electric current is injected, respectively, drained by connecting the network to charge reservoirs. The dynamics in the presence of the point contacts is [5]

$$|\psi(t+1)\rangle = U\left(Q|\psi(t)\rangle + \sum_{l=1}^{n} |\boldsymbol{c}_l\rangle a_l\right), \quad (1)$$

where the projector $Q = 1 - \sum_{l=1}^{n} |\boldsymbol{c}_l\rangle\langle\boldsymbol{c}_l|$ implements the draining action at the outgoing open ends, and a_l is the amplitude of the flux per time step fed into the incoming end at $\boldsymbol{c}_l$. We then consider stationary states of this open-network dynamics. Without loss we take the quasienergy to be zero, as the statistical properties of the network model are independent of it. We refer to the solutions of the stationarity condition $|\psi(t+1)\rangle \equiv |\psi(t)\rangle$ as scattering states. For a system with n point contacts, a basis of scattering states is furnished by

$$|\psi_k\rangle \equiv U(1-QU)^{-1}|\boldsymbol{c}_k\rangle \qquad (k = 1, \ldots, n). \quad (2)$$

Note that $\|QU\| < 1$, which ensures that the inverse exists as a convergent power series $(1-QU)^{-1} = \sum_{t=0}^{\infty} (QU)^t$.

Main results and numerics.—The first result to be announced is a statement about two-point functions, allowing one to measure the scaling dimensions of primary fields. Consider a set of links $R = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n\}$ for the purpose of (noninvasive) observation, and define for i, j, $m = 1, \ldots, n$,

$$A_m = \mathrm{Det} K^{(m)}, \qquad K_{ij} = \sum_{k=1}^{n} \psi_k(\boldsymbol{r}_i)\overline{\psi_k(\boldsymbol{r}_j)}, \quad (3)$$

where $K^{(m)}$ denotes the upper-left $m \times m$ submatrix of K. These observables are the open-network counterparts of those considered in [9]. Suppose now that coarse graining of the lattice takes the contact and observation regions (C and R) to single points, i.e., $\boldsymbol{r}_i \to \boldsymbol{r}$ and $\boldsymbol{c}_i \to \boldsymbol{c}$ for all i, while $\boldsymbol{r}$ and $\boldsymbol{c}$ remain distinct. Denoting disorder averages by $\mathbb{E}\{\ldots\}$ and CFT correlators as $\langle\ldots\rangle$, we then claim that [13]

$$\begin{aligned} &\mathbb{E}\{(A_1^{q_1-q_2} A_2^{q_2-q_3} \cdots A_n^{q_n})(R, C)\} \\ &\quad = a^{2\Delta_{q_1,\ldots,q_n}} \langle \varphi_{q_1,\ldots,q_n}(\boldsymbol{r}) \Phi(\boldsymbol{c}) \rangle, \end{aligned} \quad (4)$$

where $q_1, \ldots, q_n$ are complex numbers, $\varphi_{q_1,\ldots,q_n}$ is a CFT primary field with scaling dimension $\Delta_{q_1,\ldots,q_n}$, the operator $\Phi(\boldsymbol{c})$ represents the contacts, and a is the nonuniversal scale parameter of the network. Even though $\Phi(\boldsymbol{c})$ is not a pure scaling field, it here contributes a definite scaling dimension $\Delta_{q_1,\ldots,q_n}$ due to the orthogonality principle for two-point functions. Thus, for an infinite planar network, we predict that the observable in (4) depends on the distance between the contact and observation regions as a pure power $|\boldsymbol{r}-\boldsymbol{c}|^{-2\Delta_{q_1,\ldots,q_n}}$. For the special choice of $q_2 = \cdots = q_n = 0$ this prediction reduces to $\mathbb{E}\{|\psi_{\boldsymbol{c}}(\boldsymbol{r})|^{2q_1}\} \propto |\boldsymbol{r}-\boldsymbol{c}|^{-2\Delta_{q_1,0,\ldots,0}}$, which strongly suggests that $\Delta_{q_1,0,\ldots,0}$ coincides with the multifractality spectrum of the local density of states [1]. The analytical arguments leading to (4) are sketched below.

Next we support our proposal by computing numerically some of the observables above. We consider cylindrical networks of length $L = 400$ (with reflecting boundary conditions) and eight different circumferences $W \in \{19, 22, \ldots, 40\}$. We use Eq. (2) to compute the scattering states for an ensemble of 10^6 disorder realizations. To illustrate the result (4), we focus on the example of $\mathbb{E}\{A_n^q(R,C)\}$ for n contact and n observation links. Assuming (4) and using the CFT prediction for the correlator of (spinless) primary fields on an infinite cylinder of width W (see, e.g., [14]), we have

$$\mathbb{E}\{A_n^q(R,C)\} = \alpha_{q,n}\zeta_{\boldsymbol{rc}}^{-2\Delta_{q,n}}, \quad (5)$$

$$\zeta_{\boldsymbol{rr}'} = \left|\frac{W}{\pi}\sinh\frac{\pi}{W}(\tau - \tau' + i\sigma - i\sigma')\right|, \quad (6)$$

where $\Delta_{q,n} \equiv \Delta_{q,\ldots,q}$, and the form factor $\alpha_{q,n}$ is due to the contact operator $\Phi(\boldsymbol{c})$. The variables σ and τ are the angular and longitudinal cylindrical coordinates of $\boldsymbol{r}$.

Detailed numerical investigations were performed of the correlator $\mathbb{E}\{A_n^q(R,C)\}$ for $n = 1, 2, 3$. For $n > 1$ we place the contacts on equivalent links of n plaquettes next to each other; we checked for $n = 2$ that our results do not change significantly when this choice is modified. Data for the numerically most demanding case of $n = 3$ are shown in Fig. 1(a) for $q = 0.5$ as an example. We see an excellent agreement over the whole range of distances $|\boldsymbol{r}-\boldsymbol{c}|$ between the numerical data (circles) and the functional behavior (solid line) predicted by (5).

In order to extract exponents and make an estimate of the statistical errors, we use the following procedure. Given q and n, we fit the data for $\mathbb{E}\{A_n^q(R,C)\}$ by the prediction (5) for different W. This fit yields eight "raw" exponents $\Delta_{q,n}(W)$ from which we calculate the mean value and the standard deviation. For $n = 3$ and $q = 0.5$, the inset of Fig. 1(a) shows the data collapse by the optimal value of $\Delta_{0.5,3}$ thus obtained. We interpret the excellent quality of these fits as strong evidence that the A_n^q indeed give rise to CFT primary fields in the continuum limit.

Having established the existence of these fields, we now turn to a systematic analysis of the scaling dimensions $\Delta_{q,n}$. These are constrained by $\Delta_{q,n} = \Delta_{n-q,n}$ due to an argument [9] using Weyl group invariance. The simplest ansatz compatible with that constraint would be $\Delta_{q,n} \propto C_2(q,n)$, with $C_2 = nq(n-q)$ the quadratic Casimir eigenvalue of the symmetry group at hand, but it is known from [15,16] that this so-called "parabolic approximation" needs improvement by including in $\Delta_{q,1}$ the square of C_2 with a small coefficient. Focusing on $n = 1$, we find that $\Delta_{q,1}$ is in fact described reasonably well by the parameter set of [15,16]. The situation changes, however, when we take into account our results for $n = 2, 3$, as shown in Fig. 1(b). To get a good

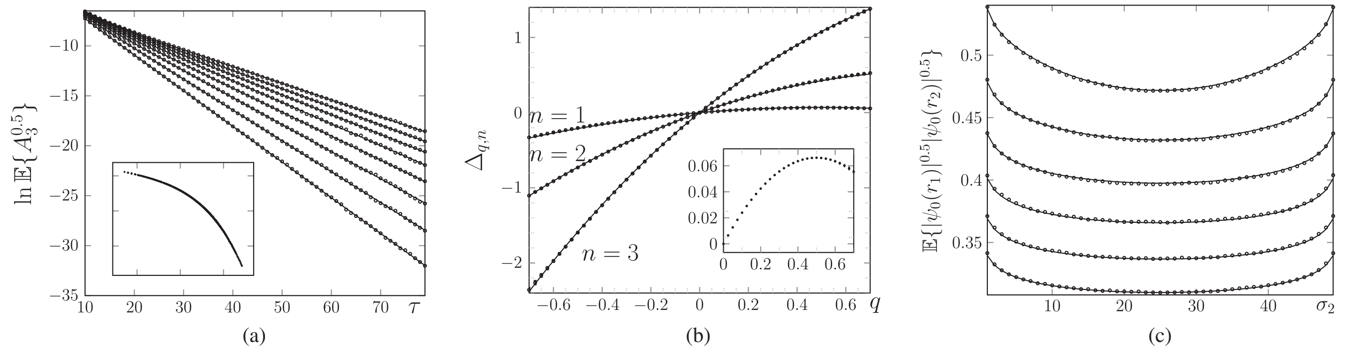

FIG. 1. (a) Fits of $\mathbb{E}\{A_3^{0.5}\}$ by the CFT prediction (5) for $W \in \{19, 22, \ldots, 40\}$ (bottom to top) as a function of $\tau = |\boldsymbol{r} - \boldsymbol{c}|$. Inset: Collapse of rescaled curves $\ln \mathbb{E}\{A_3^{0.5}\} + 2\Delta_{0.5,3} \ln W$ vs $\ln x/W$ using $\Delta_{0.5,3} = 2.15$. (b) Scaling exponents $\Delta_{q,n}$ for $\mathbb{E}\{A_n^q\}$. The solid curves are plots of $\Delta_{q,n} = 0.28C_2 - 0.0011C_2^2 + 0.0014C_4$. Inset: zoom in for $n = 1$. (c) Comparison between the CFT prediction (9) and the angular dependence of the three-point function (8) computed for $W = 50$ and $\tau_1 = \tau_2 = 10, 20, \ldots, 60$ (top to bottom).

fit of all data $n = 1, 2, 3$, simultaneously, we find it necessary to include in $\Delta_{q,n}$ the Casimir eigenvalue of degree 4 [17]. Evaluated on A_n^q this is

$$C_4(q, n) = -n[q(n - q)]^2 + n(n^2 - 1/2)q(n - q). \quad (7)$$

We leave it for future work to decide whether this is a real effect or might have another explanation, e.g., by the presence of irrelevant operators perturbing the CC model away from the CFT fixed point [7].

To strengthen our claim that the operators in (4) behave as CFT primary fields, we present a second result, this time for a three-point function. Here we sacrifice generality for simplicity and open the network at just a single contact link $\boldsymbol{c}_0$ to study the scalar-type observable $A_1(\boldsymbol{r}) = |\psi_0(\boldsymbol{r})|^2$ at two observation links $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Our assertion is that, after coarse graining,

$$\mathbb{E}\{|\psi_0(\boldsymbol{r}_1)|^{2q_1}|\psi_0(\boldsymbol{r}_2)|^{2q_2}\} \propto \langle \varphi_{q_1}(\boldsymbol{r}_1)\varphi_{q_2}(\boldsymbol{r}_2)\Phi(\boldsymbol{c}_0)\rangle \quad (8)$$

depends on $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{c}_0$ as a CFT three-point function. Because of the special nature of the operators φ_{q_1}, φ_{q_2} as "highest-weight vectors" (see below), the contact operator $\Phi(\boldsymbol{c}_0)$ still contributes a definite scaling dimension Δ_{q_0}, where $q_0 = q_1 + q_2$. The CFT prediction for three-point functions [14] then gives

$$\mathbb{E}\{|\psi_0(\boldsymbol{r}_1)|^{2q_1}|\psi_0(\boldsymbol{r}_2)|^{2q_2}\} \propto \times \zeta_{\boldsymbol{r}_1\boldsymbol{r}_2}^{-\Delta_{q_1}-\Delta_{q_2}+\Delta_{q_0}} \zeta_{\boldsymbol{r}_2\boldsymbol{c}_0}^{-\Delta_{q_2}-\Delta_{q_0}+\Delta_{q_1}} \zeta_{\boldsymbol{c}_0\boldsymbol{r}_1}^{-\Delta_{q_0}-\Delta_{q_1}+\Delta_{q_2}}, \quad (9)$$

where $\zeta_{\boldsymbol{r}\boldsymbol{r}'}$ was defined in Eq. (6). In our numerical test we take $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ to have the same τ coordinates and $\boldsymbol{r}_1$ to share the σ coordinate of the point contact $\boldsymbol{c}_0$, while $\boldsymbol{r}_2$ moves along the circumference of the cylinder. Figure 1(c) shows the angular dependence observed for $W = 50$ together with the prediction (9) at $q_1 = q_2 = 1/4$. The exponents $\Delta_{1/4,1}$ and $\Delta_{1/2,1}$ used are those extracted from the study of the two-point function.

Analytical argument.—Finally, we sketch the reasoning that leads to Eqs. (4) and (8); details and generalizations will be discussed in [18]. We use a variant of the Efetov-Wegner method to pass from the CC model to a supersymmetric vertex model (see, e.g., [10,11]) as follows. For each link $\boldsymbol{r}$ of the network we introduce n replicas of charged ($\pm$) canonical bosons and fermions: $b_{\pm,k}(\boldsymbol{r})$, $f_{\pm,k}(\boldsymbol{r})$ ($k = 1, \ldots, n$), acting on a Fock space with vacuum $|0_{\boldsymbol{r}}\rangle$. The time-evolution operator U of the closed network is then replaced by its second quantization $\rho(U)$ acting on the tensor product of all these Fock spaces. This factors as $\rho(U \equiv e^X) = \rho_+(e^X)\rho_-(e^X)$ where, assuming the summation convention,

$$\rho_+(e^X) = e^{b_{+,k}^\dagger(\boldsymbol{r})X_{\boldsymbol{r}\boldsymbol{r}'}b_{+,k}(\boldsymbol{r}') + f_{+,k}^\dagger(\boldsymbol{r})X_{\boldsymbol{r}\boldsymbol{r}'}f_{+,k}(\boldsymbol{r}')},$$
$$\rho_-(e^X) = e^{-b_{-,k}(\boldsymbol{r})X_{\boldsymbol{r}\boldsymbol{r}'}b_{-,k}^\dagger(\boldsymbol{r}') + f_{-,k}(\boldsymbol{r})X_{\boldsymbol{r}\boldsymbol{r}'}f_{-,k}^\dagger(\boldsymbol{r}')}. \quad (10)$$

It is important that second quantization preserves operator products. In particular, $\rho(U_sU_r) = \rho(U_s)\rho(U_r)$. Statistical averages in this Fock representation are defined by $\langle A\rangle_{\mathcal{F}} := \mathrm{STr}\rho(U)A$, where STr is the supertrace over the total Fock space.

For simplicity, we now specialize to $n = 1$ and return to $n \geq 1$ below. Given $Q_\varepsilon = Q + \varepsilon|\boldsymbol{c}\rangle\langle\boldsymbol{c}|$, let

$$\pi_0(\boldsymbol{c}) = |0_{\boldsymbol{c}}\rangle\langle 0_{\boldsymbol{c}}| = \lim_{\varepsilon\to 0+} \rho_+(Q_\varepsilon)\rho_-(Q_\varepsilon^{-1}) \quad (11)$$

be the projector on the vacuum state at $\boldsymbol{c}$. The second-quantized formalism is connected to observables of the first-quantized network model by the basic identities

$$\langle b_+^\dagger(\boldsymbol{r})b_+(\boldsymbol{r})\pi_0(\boldsymbol{c})\rangle_{\mathcal{F}} = \langle \boldsymbol{r}|QU(1 - QU)^{-1}|\boldsymbol{r}\rangle,$$
$$\langle b_-(\boldsymbol{r})b_-^\dagger(\boldsymbol{r})\pi_0(\boldsymbol{c})\rangle_{\mathcal{F}} = \langle \boldsymbol{r}|(1 - U^{-1}Q)^{-1}|\boldsymbol{r}\rangle. \quad (12)$$

To exploit these, we introduce the following key objects for an observation link $\boldsymbol{r}$:

$$\mathcal{Z}_q(\boldsymbol{r},\boldsymbol{c}) = \langle (B^\dagger B)^q(\boldsymbol{r})\pi_0(\boldsymbol{c})\rangle_{\mathcal{F}}, \quad (13)$$

$$B^\dagger = b_+^\dagger - e^{i\alpha}b_-, \qquad B = b_+ - e^{-i\alpha}b_-^\dagger, \quad (14)$$

where $e^{i\alpha}$ is any (fixed) unitary number. Note the vanishing commutator $[B, B^\dagger] = 0$. Note also that neither B nor $B^\dagger$ annihilates any state in Fock space. Therefore, the operator $B^\dagger B$ is strictly positive and $(B^\dagger B)^q$ makes good sense for any $q \in \mathbb{C}$. Moreover, Wick's theorem holds in our noninteracting-particle situation before disorder averaging. Thus for q a positive integer we have

$$\mathcal{Z}_q(\boldsymbol{r},\boldsymbol{c}) = q!\mathcal{Z}_1(\boldsymbol{r},\boldsymbol{c})^q. \quad (15)$$

This extends to complex q by analytic continuation. The basic correlator $\mathcal{Z}_1(\boldsymbol{r},\boldsymbol{c})$ is expressed in terms of the scattering state $|\psi_{\boldsymbol{c}}\rangle$ by the following computation; it is based on the identities (12) and in the last step uses that scattering states with incoming-wave and outgoing-wave boundary conditions are unitarily related to each other:

$$\begin{aligned}\mathcal{Z}_1(\boldsymbol{r},\boldsymbol{c}) &= \langle (b_+^\dagger b_+ + b_- b_-^\dagger)(\boldsymbol{r})\pi_0(\boldsymbol{c})\rangle_{\mathcal{F}} \\ &= \langle \boldsymbol{r}|(1-U^{-1}Q)^{-1}U^{-1}(1-Q)U(1-QU)^{-1}|\boldsymbol{r}\rangle \\ &= |\langle \boldsymbol{r}|(1-U^{-1}Q)^{-1}U^{-1}|\boldsymbol{c}\rangle|^2 = |\psi_{\boldsymbol{c}}(\boldsymbol{r})|^2. \quad (16)\end{aligned}$$

Next we take the disorder average. This is straightforward in the Fock representation since $\rho(U) = \rho(U_s)\rho(U_r)$ and averaging over the random phases in $\rho(U_r)$ simply kills all states with nonzero charge. For any charge-conserving operator A we thus obtain

$$\mathbb{E}\{\langle A\rangle_{\mathcal{F}}\} = \mathbb{E}\{\mathrm{STr}A\rho(U_s)\rho(U_r)\} = \mathrm{STr}' A\rho(U_s), \quad (17)$$

where STr′ is STr restricted to the zero-charge sector. In this way we arrive at what is called a vertex model. We denote vertex-model averages by $\langle A\rangle_{\mathcal{V}} \equiv \mathrm{STr}' A\rho(U_s)$.

The operators $B^\dagger B$ and π_0 conserve charge, so by taking the disorder average of (13) and using (15,16) we get

$$\mathbb{E}\{|\psi_{\boldsymbol{c}}(\boldsymbol{r})|^{2q}\} = q!^{-1}\langle (B^\dagger B)^q(\boldsymbol{r})\pi_0(\boldsymbol{c})\rangle_{\mathcal{V}}. \quad (18)$$

This is an exact result. Although our focus has been on $n = 1$, the general case $n \geq 1$ can be handled in a similar way. The outcome is an exact relation expressing network-model averages as vertex-model averages:

$$\begin{aligned}&\mathbb{E}\{(A_1^{q_1-q_2}A_2^{q_2-q_3}\cdots A_n^{q_n})(R,C)\} \\ &\quad = f(q)\langle (D_1^{q_1-q_2}D_2^{q_2-q_3}\cdots D_n^{q_n})(R)\pi_0(C)\rangle_{\mathcal{V}}, \quad (19)\end{aligned}$$

$$D_m(R) = \mathrm{Det}\left(\sum_{k=1}^{m} B_k^\dagger(\boldsymbol{r}_i)B_k(\boldsymbol{r}_j)\right)_{i,j=1,\ldots,m}. \quad (20)$$

Here, $f(q)$ is a combinatorial factor. The determinants $D_m(R)$ are well defined because the matrix elements $\sum B_k^\dagger(\boldsymbol{r}_i)B_k(\boldsymbol{r}_j)$ all commute. The analytic continuation to complex powers of D_m is well defined because $D_m > 0$.

We will now argue that the expression in (19) becomes a pure scaling function in the continuum limit. The key ingredient here is symmetry: the statistical average $\langle \ldots \rangle_{\mathcal{V}}$ is invariant under the global action of a group with Lie superalgebra $\mathfrak{g} \equiv \mathfrak{gl}(2n|2n)$ generated by all charge-conserving bilinears in $b_{\pm,k}$, $f_{\pm,k}$, and their adjoints.

What is most remarkable about the operators D_m is their property of being highest-weight vectors for $\mathfrak{g}$. By this we mean that there exists a maximal Abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the D_m are (i) eigenoperators with respect to the commutator action by all generators from $\mathfrak{h}$ and (ii) are annihilated by all the raising operators; i.e., the operators from $\mathfrak{g}$ which are positive root vectors for $\mathfrak{h}$. Since the operation of taking the commutator satisfies the Leibniz rule, these properties carry over to products of powers of D_m. Thus $\varphi^{\mathrm{lat}}_{q_1,\ldots,q_n}(R) \equiv D_1^{q_1-q_2}(R)D_2^{q_2-q_3}(R)\cdots D_n^{q_n}(R)$ is a highest-weight vector for $\mathfrak{g}$. The operators $\varphi^{\mathrm{lat}}_{q_1,\ldots,q_n}(R)$ for different $q = (q_1, \ldots, q_n)$ (modulo Weyl transformations) lie in inequivalent representations of $\mathfrak{g}$. Therefore, by a Schur lemma argument they cannot be mixed by the transfer matrix of the $\mathfrak{g}$ invariant vertex model. Thus we expect them to become pure scaling fields of the renormalization flow in the continuum limit. (This has to taken with a grain of salt since our CFT has a logarithmic sector [19].)

Finally, our methods generalize to any multipoint function of the scaling operators given here. In particular, one can derive Eq. (8) by the reasoning above. To arrive at (9) with $\Delta_{q_0} = \Delta_{q_1+q_2}$ one uses that the product of two highest-weight vectors with weights q_1 and q_2 is another highest-weight vector with weight $q_0 = q_1 + q_2$.

Summary and outlook.—We have presented new methods and relations by which to make a systematic study of the IQH plateau transition. For the first time in the context of Anderson transitions, we have identified and studied operators whose correlation functions decay as pure powers at criticality and thus are candidates for CFT primary fields. Although we applied our techniques to the specific case of the CC model, they are rather general and can be used for other situations as well. Future applications will include the spin quantum Hall transition [20] and the study of boundary criticality [21].

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