

Muon Loop Light-by-Light Contribution to Hyperfine Splitting in Muonium

Michael I. Eides*

*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506,
USA and Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Valery A. Shelyuto†

*D. I. Mendeleev Institute for Metrology, St. Petersburg 190005, Russia
(Received 24 February 2014; published 30 April 2014)*

Three-loop corrections to hyperfine splitting in muonium, generated by the gauge-invariant sets of diagrams with muon and tauon loop light-by-light scattering blocks, are calculated. These results complete calculations of all light-by-light scattering contributions to hyperfine splitting in muonium.

DOI: 10.1103/PhysRevLett.112.173004

PACS numbers: 31.30.jf, 32.10.Fn, 36.10.Ee

Calculation of the light-by-light (LBL) scattering contributions to hyperfine splitting (HFS) in muonium has a long history. The nonrecoil contribution generated by the electron LBL scattering block was obtained in [1–3]. Respective recoil contributions are enhanced by the large logarithm of the muon-electron mass ratio. Large logarithm-squared contributions were calculated in [4], and single-logarithmic and nonlogarithmic terms were obtained only recently [5,6]. The LBL scattering contributions due to other particles besides the electron should also be taken into account. The hadron LBL scattering contribution was calculated in [7]. Below we present the results for the only remaining uncalculated LBL scattering contributions to hyperfine splitting in muonium that are due to the virtual muon and tauon loops.

The general expression for the muon loop LBL scattering contribution to HFS in Fig. 1 is similar to the respective electron loop contribution (see, e.g., [5,6]), and can be written in the form

$$\Delta E = [\alpha^2(Z\alpha)/\pi^3](m/M)E_F J, \quad (1)$$

where m is the electron mass, M is the muon mass, $Z = 1$ is the muon charge in terms of the positron charge used for classification of different contributions, the Fermi energy is defined as (m_r is the reduced mass)

$$E_F = (8/3)(Z\alpha)^4(m/M)(m_r/m)^3 mc^2, \quad (2)$$

and J is a dimensionless integral

$$J = -\frac{3M^2}{128} \int \frac{d^4 k}{i\pi^2 k^4} \left(\frac{1}{k^2 + 2mk_0} + \frac{1}{k^2 - 2mk_0} \right) T(k^2, k_0). \quad (3)$$

The dimensionless function $T(k^2, k_0)$ is a sum of the ladder and crossed diagram contributions in Fig. 1,

$$T(k^2, k_0) = 2T_L(k^2, k_0) + T_C(k^2, k_0). \quad (4)$$

Explicit expressions for the functions $T_L(k^2, k_0)$ and $T_C(k^2, k_0)$ can be obtained by the substitution $m \rightarrow M$, $q_\mu \rightarrow k_\mu$ from the respective formulas in [6], where these functions were calculated in the case of the electron LBL scattering block.

Only the even in k_0 terms in the function $T(k^2, k_0)$ contribute to the integral in Eq. (3). After the rescaling of the integration momentum $k \rightarrow kM$, the Wick rotation, and the symmetrization of the function $T(k^2, k_0)$ with respect to k_0 , $T(k^2, k_0) \rightarrow T(k^2, k_0^2)$, the integral in Eq. (3) turns into

$$J = \frac{3}{32\pi} \int_0^\infty \frac{dk^2}{k^2} \int_0^\pi d\theta \sin^2 \theta \frac{T(k^2, \cos^2 \theta)}{k^2 + 16\mu^2 \cos^2 \theta}, \quad (5)$$

where we have parameterized the Euclidean four-vectors as $k_0 = k \cos \theta$, $|\mathbf{k}| = k \sin \theta$, $\mu = m/(2M)$; the function $T(k^2, \cos^2 \theta)$ is the same function as in Eq. (3) but symmetrized with respect to k_0 and with the Wick-rotated momenta. The dimensionless function $T(k^2, \cos^2 \theta)$ after rescaling depends on the dimensionless momentum k and does not contain any parameters with dimension of mass.

We are looking for the μ -independent contributions generated by the integral in Eq. (5). The term with μ^2 in the denominator is irrelevant at large k , and the integral is convergent at large k due to ultraviolet convergence of all diagrams with the LBL insertions. The case of small integration momenta is more involved. Because of gauge

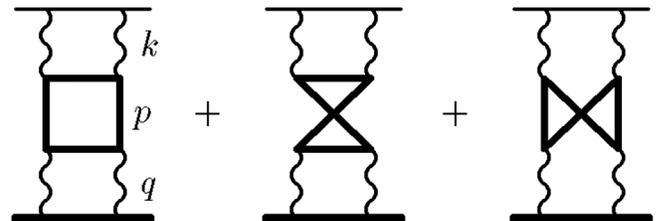


FIG. 1. Diagrams with the muon (tauon) light-by-light scattering block.

invariance, the LBL block is strongly suppressed at $k \rightarrow 0$, and we expect that the integral in Eq. (5) remains finite even at $\mu = 0$. As a result of this finiteness, the diagrams in Fig. 1 do not generate nonrecoil contributions to HFS in accordance with our physical expectations. However, small integration momenta convergence of contributions of individual diagrams at $\mu = 0$ cannot be taken for granted, and we have to consider separate entries in more detail. Using the explicit integral representations for the functions $T_L(k^2, \cos^2\theta)$ and $T_C(k^2, \cos^2\theta)$ (see [6]), we find that these functions, and separate terms in the respective integral representations, decrease not slower than k^2 at $k^2 \rightarrow 0$. The integral in Eq. (5) is logarithmically divergent at $\mu = 0$ if $T(k^2) \sim k^2$ when $k^2 \rightarrow 0$. This means that we cannot omit μ in Eq. (5) calculating the integrals with those terms in $T(k^2)$ that decrease as k^2 when $k^2 \rightarrow 0$. To facilitate further calculations we represented the functions $T_L(k^2, \cos^2\theta)$ and $T_C(k^2, \cos^2\theta)$ in the form

$$T_L(k^2, \cos^2\theta) = T_L^{\text{reg}}(k^2, \cos^2\theta) + T_L^{\text{sing}}(k^2, \cos^2\theta), \quad (6)$$

$$T_C(k^2, \cos^2\theta) = T_C^{\text{reg}}(k^2, \cos^2\theta) + T_C^{\text{sing}}(k^2, \cos^2\theta), \quad (7)$$

where the functions T^{reg} decrease faster than k^2 at small k^2 , and the functions T^{sing} decrease as k^2 at small k^2 .

In these terms the integral in Eq. (5) has the form

$$J = J^{\text{reg}} + J^{\text{sing}}, \quad (8)$$

where

$$J^{\text{reg(sing)}} = \frac{3}{32\pi} \int_0^\infty \frac{dk^2}{k^2} \int_0^\pi d\theta \sin^2\theta \frac{T^{\text{reg(sing)}}(k^2, \cos^2\theta)}{k^2 + 16\mu^2 \cos^2\theta}, \quad (9)$$

and

$$T^{\text{reg(sing)}}(k^2, \cos^2\theta) = 2T_L^{\text{reg(sing)}}(k^2, \cos^2\theta) + T_C^{\text{reg(sing)}}(k^2, \cos^2\theta). \quad (10)$$

We can safely let $\mu = 0$ in the integral J^{reg} , which makes calculation of this integral straightforward. As a result we obtain

$$J^{\text{reg}} = -2.14635(5). \quad (11)$$

Calculation of the integral J^{sing} is more involved. The functions $T_L^{\text{sing}}(k^2, \cos^2\theta)$ and $T_C^{\text{sing}}(k^2, \cos^2\theta)$ decrease as k^2 at low k^2 . As a result, they generate logarithmic contributions to the momentum integral in Eq. (5) that

are cut off at small $k \sim \mu$. We calculated the coefficients before the terms that are proportional to k^2 analytically, and checked that these terms cancel in the sum $2T_L^{\text{sing}}(k^2, \cos^2\theta) + T_C^{\text{sing}}(k^2, \cos^2\theta)$. This cancellation can be used to get rid of the parameter μ in the integral J^{sing} in Eq. (9). To this end we write the momentum integral J^{sing} as a sum of two integrals

$$J^{\text{sing}} = J^{\text{sing}<} + J^{\text{sing}>}, \quad (12)$$

where integration over k^2 goes from zero to 1 in the integral $J^{\text{sing}<}$, and it goes from 1 to infinity in the integral $J^{\text{sing}>}$. The separation point $k^2 = 1$ is arbitrary; the result for the integral J^{sing} does not depend on its choice. We can safely let $\mu = 0$ in the integral $J^{\text{sing}>}$. To facilitate calculation of the integral $J^{\text{sing}<}$ we subtract from the integrand all terms proportional to k^2 at small k . Because of the cancellation mentioned above, this subtraction does not change the value of the integral. After the subtraction we can let $\mu = 0$ before calculation of this integral as well. Calculating the integrals, we obtain

$$J^{\text{sing}<} = 0.17447(2), \quad J^{\text{sing}>} = 1.12951(3). \quad (13)$$

We have checked by direct calculations that the sum

$$J^{\text{sing}<} + J^{\text{sing}>} = 1.30398(4). \quad (14)$$

does not depend on the arbitrary separation point.

Collecting the results in Eq. (11) and Eq. (14), we obtain

$$J = -0.8424(1), \quad (15)$$

and finally

$$\Delta E = -0.8424(1)[\alpha^2(Z\alpha)/\pi^3](m/M)E_F \approx -0.2274 \text{ Hz}. \quad (16)$$

Using the same methods as above, we also calculated a tiny contribution to hyperfine splitting generated by the tauon LBL scattering block in Fig. 1,

$$\Delta E_\tau = -0.00358(1)[\alpha^2(Z\alpha)/\pi^3](m/M)E_F \approx -0.0010 \text{ Hz}. \quad (17)$$

Combining the results in Eq. (16) and Eq. (17) with the other LBL scattering contributions calculated earlier in [1–7], we obtain the total contribution of the LBL scattering block to hyperfine splitting in muonium

$$\begin{aligned} \Delta E &= \frac{\alpha^2(Z\alpha)}{\pi} (1 + a_\mu) E_F [-0.472514(1)] \\ &\quad + \frac{\alpha^2(Z\alpha)}{\pi^3} \frac{m}{M} E_F \left[\frac{9}{4} \ln^2 \frac{M}{m} + \left(-3\zeta(3) - \frac{2\pi^2}{3} + \frac{91}{8} \right) \ln \frac{M}{m} + 5.1525(1) \right] \\ &\quad + \Delta E_{\text{hadr}} + \Delta E_\tau \approx -240.2 \text{ Hz}, \end{aligned} \quad (18)$$

where $\Delta E_{\text{hadr}} = -0.0065$ Hz is the hadronic contribution [7].

Hopefully the new result in Eq. (16) will find applications in the new high-accuracy measurement of the muonium hyperfine splitting planned at J-PARC, Japan [8]. The completion of calculations of all LBL scattering contributions to hyperfine splitting in muonium brings us one step closer to the final goal of reducing the theoretical error of the hyperfine splitting in muonium below 10 Hz [9]. Contributions that are still unknown include three-loop purely radiative corrections of order $\alpha^3(Z\alpha)E_F$, three-loop radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)E_F$, and nonlogarithmic recoil corrections of order $(Z\alpha)^3(m/M)E_F$ (see detailed discussion in [9]); these are the main sources of the theoretical uncertainty. We hope to finish calculation of all remaining three-loop radiative-recoil corrections in the near future.

This work was supported by the NSF Grant No. PHY-1066054. The work of V. S. was also supported in part by the RFBR Grant No. 14-02-00467 and by the DFG Grant No. HA 1457/9-1.

*eides@pa.uky.edu

†shelyuto@vniim.ru

- [1] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, *Phys. Lett. B* **268**, 433 (1991); **316**, 631(E) (1993); **319**, 545(E) (1993); *Yad. Fiz.* **55**, 466 (1992); **57**, 1343(E) (1994) [*Sov. J. Nucl. Phys.* **55**, 257 (1992); **57**, 1275(E) (1994)].
- [2] T. Kinoshita and M. Nio, *Phys. Rev. Lett.* **72**, 3803 (1994).
- [3] T. Kinoshita and M. Nio, *Phys. Rev. D* **53**, 4909 (1996).
- [4] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, *Phys. Lett. B* **216**, 405 (1989); *Yad. Fiz.* **49**, 493 (1989) [*Sov. J. Nucl. Phys.* **49**, 309 (1989)].
- [5] M. I. Eides and V. A. Shelyuto, *Phys. Rev. D* **87**, 013005 (2013).
- [6] M. I. Eides and V. A. Shelyuto, *Phys. Rev. D* **89**, 014034 (2014).
- [7] S. G. Karshenboim, V. A. Shelyuto, and A. I. Vainshtein, *Phys. Rev. D* **78**, 065036 (2008).
- [8] K. Shimomura, *AIP Conf. Proc.* **1382**, 245 (2011).
- [9] M. I. Eides, H. Grotch, and V. A. Shelyuto, *Theory of Light Hydrogenic Bound States* (Springer, New York, 2007).