Increasing Sensing Resolution with Error Correction

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The signal to noise ratio of quantum sensing protocols scales with the square root of the coherence time. Thus, increasing this time is a key goal in the field. By utilizing quantum error correction, we present a novel way of prolonging such coherence times beyond the fundamental limits of current techniques. We develop an implementable sensing protocol that incorporates error correction, and discuss the characteristics of these protocols in different noise and measurement scenarios. We examine the use of entangled versus untangled states, and error correction's reach of the Heisenberg limit. The effects of error correction on coherence times are calculated and we show that measurement precision can be enhanced for both one-directional and general noise.

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Quantum sensing and metrology [1] are key goals of quantum technologies. Impressive achievements have been made in both fields in recent years. The frequency uncertainty of atomic clocks has decreased dramatically [2,3], the signal to noise ratio of magnetic field measurements has considerably increased [4–8], and the contrast of spin imaging has improved [9,10]. Since the sensitivity of quantum sensing scales as $\frac{1}{\sqrt{T_2}}$, where T_2 is the coherence time [11], a great deal of effort has been devoted to the design and realization of protocols that increase this time while also maintaining the sensing signal. The utilization of error correction (EC) for quantum sensing objectives could increase the coherence time substantially, and thus, enhance the sensitivity of field measurement and the contrast of imaging.

State of the art methods of noise reduction by dynamical decoupling (DD) face three main challenges [12,13], for which EC could significantly enhance precision: (1) when the correlation time of the noise, denoted t_c, is very short (e.g., white noise), DD and similar methods fail, since they have to be faster than t_c (This limitation can be overcome by EC, since it only needs to be faster than the noise effects, which are typically much slower.); (2) when there is noise in the control, in which DD requires special pulse sequences that constitute obstacles in various scenarios; (3) in the case of T_1 noise and noise in all three directions.

EC tackles high frequency noise by using redundant qubits. An EC scheme is composed of a code subspace $\{|\psi_1, \psi_2, \ldots \psi_N\rangle\}$, in which all the pertinent information is found; i.e., the sensing signal should work inside the code (e.g., $H_s = g|\psi_i\rangle\langle\psi_j| + \mathrm{H.c.}$, where g is the signal). The code is susceptible to errors, which map it to orthogonal subspaces. Correction of the errors is accomplished by means of projective measurements, i.e., projecting the state onto one of the orthogonal subspaces, and then applying a correction sequence [14,15]. The basic idea is shown in Fig. 1. In the past few decades, following Shor's work [16], various EC protocols have been proposed (including Stean's code [17]), and various fault tolerant methods [15] have been put forward. Recently, several protocols have been implemented [18,19].

There are three main differences between using EC for quantum computing and using EC for sensing purposes. While a logic operation can be done by arbitrary pulses connecting different code states which were chosen solely to facilitate EC, the sensing signal usually has simple form (generally, it operates on each qubit separately), which largely constrains the code states that can be chosen, since the signal has to operate inside the code. Moreover, logic operations can be realized by fast pulses, which allow for the design of complicated Hamiltonians by controlling the pulse sequence, but the sensing signal for most scenarios is weak and continuous, which is an obstacle to using this technique effectively. These two differences complicate the problem.

However, EC mechanisms for sensing can benefit from the use of fully protected qubits, which are not sensitive to either noise or signal and help simplify the EC schemes. We

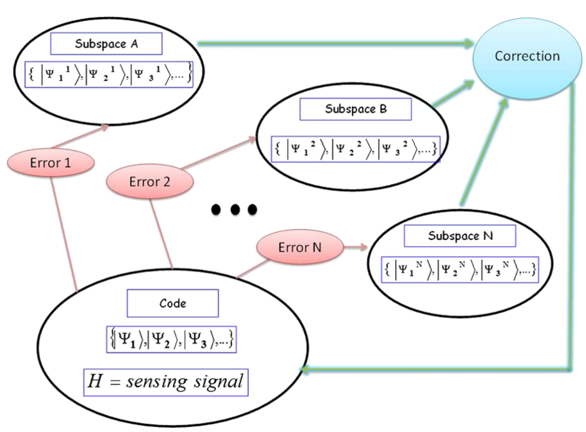

FIG. 1 (color online). The basic mechanism of the combination of EC with sensing. Errors map the code subspace to orthogonal subspaces, and are followed by an EC sequence, composed of a detection and then a correction, that brings the state back to the code.

denote these as "good" qubits, as compared to those that are "sensitive" to the signal and noise. These good qubits can be produced by clock states or robust nuclear spins. There is no analog for this in the case of quantum computing since in that case we use, by design, the most robust qubits available.

Improving dynamical decoupling with error correction.— We now consider a representative sensing model and assume that, in this setup, DD fails to prolong the coherence time due to noisy control parameters or due to fast noise correlations. We will present a combined DD and error-correction procedure aimed at extending the model's coherence time and analyze the limits in which this procedure is efficient.

The model we examine is composed of a single two level system (TLS), spanned by the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, which are chosen to be the eigenstates of σ_z, and are separated by an energy gap ω_0. The coupling of the sensing signal g, which we want to measure, to the TLS is described by the Hamiltonian

$$H = \left[\frac{\omega_0}{2} + f(t)\right]\sigma_z + [\Omega + \delta\Omega(t)]\sigma_x \cos(\omega_0 t) + g\sigma_z \cos(\Omega t). \tag{1}$$

Here $f(t)$ and $\delta\Omega$ represent the slow external and fast control noise, respectively, and σ_k is the Pauli operator in the kth direction. This Hamiltonian represents the main magnetometery scenario which has been realized in nitrogen vacancy (NV) centers [4,7], ions [8], and atoms [20]. We transform H first to the interaction picture with respect to $\omega_0\sigma_z/2$ and then to the interaction picture with respect to $\Omega\sigma_x$, taking advantage of the rotating-wave approximation when possible. Assuming that $f(t) \ll \Omega \ll \omega_0$ and since f is slow, the σ_z noise $[f(t)]$ can be neglected (this is the DD). We are left with the Hamiltonian: $H_{\rm I} = \frac{g}{2}\sigma_z + \delta\Omega\sigma_x$.

In addition, we assume that we have a good qubit $\{|0\rangle, |1\rangle\}$, that can be controlled at will. In NV centers, this may be realized by the ^{13}C, which has a much longer coherence time than that of the electron. We define the code as

$$\{|\downarrow, 0\rangle, |\uparrow, 1\rangle\}, \tag{2}$$

the signal g inflicts a phase shift between the two states, which can be measured as follows: assuming that, at time $t = 0$, we have initialized the state at

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\downarrow, 0\rangle + \frac{1}{\sqrt{2}}|\uparrow, 1\rangle, \tag{3}$$

the probability of measuring the same state at time t will be proportional to $\cos^2(gt)$ [11], allowing for the extraction of g. The effect of the noise $\delta\Omega\cdot\sigma_x^1$ operation is a flip of the first TLS state (e.g., the NV electron), which maps the system onto the error states

$$\{|\uparrow, 0\rangle, |\downarrow, 1\rangle\}. \tag{4}$$

The rate at which such a bit flip occurs (denoted $\Gamma_{\delta\Omega}$) is governed by the magnitude of $\delta\Omega$ regardless of its correlation time.

The proposed EC procedure then closely follows the three-qubit code of quantum computing: the measurement of the $\sigma_z^1\sigma_z^2$ spin correlation operator detects whether an error has occurred [eigenvalue of (-1)] or not (eigenvalue of 1). In the case where an error was detected, we correct it by operating with σ_x^1, thus, mapping the system back onto the code subspace. A similar effect can also be achieved without measurement by application of appropriate logical gates. This procedure can be realized in NV centers by applying the gates between the NV electrons and the ^{13}C [19,21], and relying on the ability to efficiently polarize the electrons and read their state, or it can be implemented in ion chains by exploiting clock transitions. Assuming that a CNOT gate is available between the good and the sensing qubits, the procedure may be implemented by first applying this gate, mapping the code states on to $\{|\downarrow, 0\rangle, |\downarrow, 1\rangle\}$ and the error states on to $\{|\uparrow, 0\rangle, |\uparrow, 1\rangle\}$. Following this, by polarizing the sensing qubit back to the down state and applying the CNOT gate again, the EC procedure can be completed without measuring the system [22].

To understand the effect of the noise, we view the noise as inducing bit flips at random times. Since the sensing signal is continuously rotating the code states and the corresponding error states in opposite directions, we get a phase difference depending on the time of the flip; thus, we get phase change proportional to $e^{i2g(\tau - t_{\rm err})}$ (where τ is the time interval between EC sequences). On average, $t_{\rm err} = \tau/2$ since the noise is self homogeneous in time we simply get a delay of length τ in the measurement each time we detect an error. This delay cannot be corrected but can be accounted for since we know the exact number of detected errors during each run. On the other hand, terms proportional to the standard deviation of $t_{\rm err}$ cannot be accounted for, and will cause decoherence proportional to $\Gamma_{\delta\Omega}g^2\tau^3$.

To understand this effect better, note that the noise and the signal do not commute, and thus, the signal g rotates the noise term and effectively generates noise in all directions (as terms proportional to σ_y and σ_z appear in the time propagation). This noise is not correctable by the proposed protocol as can be seen from the master equation of the system. Transforming H_I to the interaction picture with respect to g, we get $H_{\rm II} = \delta\Omega(t)[\cos(gt)\sigma_x + \sin(gt)\sigma_y]$. Looking at the correlation term of the two error states $|\uparrow 0\rangle\rho^{\rm err}_{\uparrow 0,\downarrow 1}\langle\downarrow 1|$, we see that the first order in g, $\Gamma_{\delta\Omega}$ we have the master equation $\dot{\rho}_{\uparrow 0,\downarrow 1}(t) = 2\Gamma_{\delta\Omega}(1 - ig\tau)\rho_{\uparrow 1,\downarrow 0}(t)$. Applying EC, we measure the error with probability $\Gamma_{\delta\Omega}\tau$. After normalization of the state, and correcting as described above, we have $\rho^1_{\uparrow 1,\downarrow 0} \simeq (1 - ig\tau)/2$ which corresponds to an almost pure state. This correlation shows a phase change proportional to $g\tau$ that cannot be corrected but can be accounted for as mentioned above. Decoherence (Dec) appears in the second order as ${\rm Dec} = 1/2 - |\rho^1_{\uparrow 1,\downarrow 0}| \simeq -g^2\tau^2/12$. The entire derivation is valid when single-flip errors are dominant, that is, $\Gamma_{\delta\Omega}\tau \ll 1$.

After n errors, the decoherence behaves as ${\rm Dec} \propto 1 - (1 - g^2\tau^2)^n \simeq g^2\tau^2 n$. We define the coherence time T_2 as the typical time after which decoherence reaches an order of unity, and use $\langle n\rangle = \Gamma_{\delta\Omega} t$ to obtain

$$T_2 = (g\tau)^{-2}\Gamma_{\delta\Omega}^{-1} \quad [\Gamma_{\delta\Omega}\tau \ll 1, g\tau \ll 1]. \tag{5}$$

In brackets are quoted the EC efficiency (and, also, derivation validity) conditions. For more details, see Sec. 5 of the Supplemental Material [22]. For slow noise of correlation time η_0, there are additional contributions proportional to $(g\eta_0)^{-1}$, which do not appear for fast noise because of destructive interference. Numerical simulations with very fast noise, as presented in Fig. 2, have shown that such errors are correctable as long as the stated condition is fulfilled.

Despite the minimal nature of the Hamiltonian in Eq. (1), the EC procedure described here is suitable for dealing with a wide range of sensing setups susceptible to noise. Of particular interest is the measurement of an interaction between two TLSs, described by $g\sigma_x^1\sigma_x^2$. The generalization of the EC procedure we just described to the case of two TLSs, thus coupled, where we allow for local operations only, is straightforward by defining the code subspace $\{[|\downarrow_1, 0; \uparrow_2, 1\rangle - |\uparrow_1, 0; \downarrow_2, 1\rangle]/\sqrt{2}, [|\downarrow_1, 1; \uparrow_2, 0\rangle + |\uparrow_1, 1; \downarrow_2, 0\rangle]/\sqrt{2}\}$, and then identifying a z-directional noise as effective spin flips [22].

Collective enhancement.—Measurements made with N entangled qubits are of particular importance, since it was shown that, in a noiseless system, these can surpass the untangled system's precision by a factor of $\sqrt{N}$, reaching

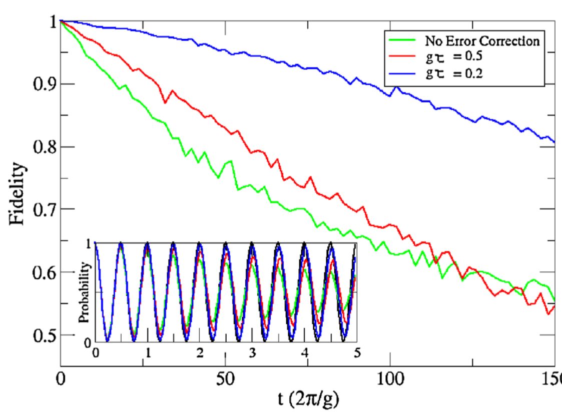

FIG. 2 (color online). The fidelity of a state as a function of time, for different time differences between EC operations. Here, each point represents an average over $N = 1024$ simulations, and the fidelity of the system is plotted for run times that are integer multiplications of $2\pi/g$. In each run, either no EC procedure was applied (green), an EC procedure was applied each time interval of $g\tau = 0.5$ (red) or $g\tau = 0.2$ (blue). In each case, the range of the randomly chosen noise was $(-g/2, g/2)$. Inset: The probability of measuring the initial state of the system, as a function of time. Here, the black line represents the case with no noise, and the different lines show the cases where no EC was made (green), or where an EC was made at intervals of $g\tau = 0.5$ (red) and $g\tau = 0.2$ (blue). Here, we considered a stronger noise chosen randomly within the range of $(-2g, 2g)$.

the Heisenberg limit [23,24]. However, it was also shown that when parallel noise is present, the best achievable enhancement in precision is by a constant (numerical) factor achieved for asymptotically large N [11,25,26]. In the presence of orthogonal noise, the use of entanglement may improve the precision by a parametric factor of $N^{1/3}$ [27]. This makes an interesting goal for EC to overcome the noise of such entangled systems.

We suggest a simple setup for such a measurement using N sensitive qubits, and only one good qubit. It should be stressed that for untangled systems with N sensitive qubits we need at least N good qubits. The code states will be $|1N\rangle = |\uparrow_1 \ldots \uparrow_N; 1\rangle$, $|0N\rangle = |\downarrow_1 \ldots \downarrow_N; 0\rangle$, where the number after the ";" is the state of the good qubit. The Hamiltonian here will be $H = g\sum_{i=1}^N \sigma_z^i + \sum_{i=1}^N \xi_i(t)\sigma_x^i - gO_N$, where g is the signal, O_N is an interaction part, and $\xi_i(t)$ are orthogonal noise terms which scale like Γ [e.g., for white uncorrelated noise, we have $\langle\xi_j(t)\xi_l(t')\rangle = \delta_{jl}\delta(t-t')\Gamma/2$]. The measurement procedure will follow a Ramsey measurement, the system will be initiated into the GHZ state $(|1N\rangle + |0N\rangle)/\sqrt{2}$, and then evolve under H for time t.

It can be shown that, for certain nonvanishing O_N, the Heisenberg limit can be reached [22] as shown in the Supplemental Material Sec. 2.1. But the required interactions for such enhancement are specific and restrictive, and do not naturally occur in experiments. Indeed, these interactions can be generated by means of virtual transitions, but those are problematic, as will be shown towards the end of this Letter (an iteration based version that avoids this problem is also presented in the Supplemental Material Sec. 2.3).

We now turn to the $O_N = 0$ case. In this case, the errors are bit-flip errors, and because the good qubit never flips, we can correct any number of subsequent flips by the same EC sequence as in the last section, only here we measure N two-spin-correlation operators $\{\sigma_i\sigma_{N+1}\}_{i=1}^N$. The corresponding increase in the coherence time is limited by the mixing rate of the noise by the signal, from which we can derive the equivalent of Eq. (5) [22]

$$T_2 = \frac{T_2^*}{(g\cdot\tau)^2} = \frac{T_2^{*\mathrm{SQL}}}{N(g\cdot\tau)^2} \gg T_2^*, \quad \left[\text{for } \tau\Gamma \ll \frac{1}{N}\right], \tag{6}$$

where T_2^* is the coherence time without EC, and SQL denotes a single qubit system (unlike our entangled system). As can be seen, the noise in the entangled system works on all the qubits at once, causing a faster decoherence, and thus, $T_2^* = (N\Gamma)^{-1} = T_2^{*\mathrm{SQL}}/N$. The condition on τ is the condition that single-flip errors are dominant over multiple-flip errors; note that this condition means we have to go to very short measurements times as N grows. The effective time delay here is $\frac{\tau}{N}$ for a single-flip error, and $\frac{2\tau}{N}$ for a double-flip and so forth. This is a rather small effect.

Finally, we compare the entangled protocol precision, δg_{ent}, to that of the untangled (SQL) protocol when using EC, δg_{SQL}. Notice that the measured frequency of the entangled protocol is $\omega = Ng$ whereas for the untangled protocol the frequency is simply g, but the measurement is conducted on N systems in parallel. The precision of the measurement goes as $\delta\omega \propto (ntT_2)^{-1/2}$, where t is the total time and n is the number of parallel systems [11]. Plugging in Eq. (6), we get

$$\delta g_{\text{ent}} = \frac{N^{-1}}{\sqrt{1tT_2}} = \frac{1}{N}\frac{g\tau\sqrt{N}}{\sqrt{tT_2^{*\text{SQL}}}} = \frac{g\tau}{\sqrt{NtT_2^{*\text{SQL}}}} = \delta g_{\text{SQL}}. \quad (7)$$

There is no improvement over the untangled EC procedure; thus, we do not achieve a super-classical precision in this naive approach. However, these results can be improved by using nonzero interaction terms, as mentioned earlier. Furthermore, the bounds on precision [26], mentioned earlier, become trivial for parallel noise and short evolution times and, thus, do not apply to our system when working in the limit $\Gamma\tau \ll 1/N$. Hence, it is plausible that super-classical precision can be attained by use of more sophisticated EC protocols, even without interactions. Nevertheless, our system does achieve a substantial improvement over the no-EC case (by factor $g\tau \ll 1$), under the EC efficiency condition in Eq. (6).

Turning our attention to the correction of noise in multiple directions, we apparently need to change our code to something of the form $|\uparrow_{x1}\ldots\uparrow_{xN};1\rangle, |\downarrow_{x1}\ldots\downarrow_{xN};0\rangle$, where $|\uparrow_x\rangle = (|\uparrow;1\rangle + |\downarrow;0\rangle)/\sqrt{2}$ is a state composed of one good qubit and one sensitive qubit, so the above-mentioned efficiency advantage over the untangled case is lost. Note, once again, the need for two body interactions proportional to the signal. The problem with such interactions, as well as ways to circumvent it, are presented below.

Multidirectional noise, T_1, and interaction terms.—We move to correcting noise in all directions (general errors), and T_1 related errors. As a model for T_1 errors, we take decay errors, and note that the decay of different qubits is usually not coherent; hence, different decays should bring the state into orthogonal subspaces. These error types are relatively hard to correct.

To enable correction of these kinds of noise, the code states must differ by the state of many qubits. However, the measured field will, in general, only connect states that are relatively "close" (i.e., similar), typically changing the state of only one sensitive qubit at a time and, thus, cannot connect the distant code states. This means that to measure the field we have to connect the code states by a virtual passage through a state outside the code (e.g., by Raman transition). Because this state is outside the code, errors on it will not be correctable.

It is important to note that many ways of generating "effective multiqubit Hamiltonians" work by similar methods, and thus, our description here applies to those as well [28].

Assume a three qubit code where the first qubit is a sensitive qubit with limited T_1 time; in other words, it is susceptible to decay—whereas the other two are good qubits. The Hamiltonian of the decay is $H_d = \gamma(a^\dagger\sigma_-^1 + a\sigma_+^1)$, where a is the annihilation operator of an environment photon mode, matching the decay of the qubit from the $|\uparrow\rangle$ state to the $|\downarrow\rangle$ state. We assume that the photonic mode is unoccupied to begin with, and also, that photons travel away from the system very rapidly after their creation; under these conditions, we are left with the decay operator $H_d \propto \gamma(\sigma_-^1)$.

The sensing Hamiltonian is $H = g\sigma_z^1$, where g is the signal, and the proposed code is $|A\rangle = |\downarrow 0 + \uparrow 1\rangle|0\rangle$; $|C\rangle = |\downarrow 0 - \uparrow 1\rangle|1\rangle$. Let us look at the following procedure: by opening a gap between the code state and a utility state $|B\rangle = |\downarrow 0 - \uparrow 1\rangle|0\rangle$ and adding a laser, we can generate the interaction picture Hamiltonian

$$H = g|B\rangle\langle A| + \Omega|B\rangle\langle C| + \text{H.c.} + \delta|B\rangle\langle B|. \quad (8)$$

When Ω, $g \ll \delta$, this is a Raman transition mediating the sensing signal between the code states through the utility state.

In this scheme, the utility state is occupied by a small amplitude proportional to $\varepsilon = \frac{\Omega}{\delta} \ll 1$. It can be shown that decay errors on the first bit are correctable, yielding a prolonged coherence time $T_2 = \frac{T_1^*}{\varepsilon^2}$. But it can also be shown that, due to the reduced frequency, the precision of the measurement undergoes a reduction by a factor of ε, yielding a total of no benefit: $\delta g_{\text{Raman}} \propto \frac{1}{\varepsilon}\frac{1}{\sqrt{T_2}} = \frac{1}{\sqrt{T_1^*}} \propto \delta g_0$.

This is an "approximate EC" [29] scheme, consistent with general field measurements, that can prolong T_1. This method is metrically equivalent to the no-EC scheme, in that it will not enhance precision in the general case, but neither will it hinder precision. Thus, it may be useful in certain cases in which T_1 is of even greater importance.

Other schemes we checked achieve similar results, suggesting that this might be the best achievable result in the case of a general field with multidirectional noise. In addition this effect restricts the "multiparticle Hamiltonian generation methods" that may be used for sensing with EC. Below, we show how this result can be improved if we allow for a more specific field.

Flip flop interactions.—One scenario that overcomes this problem involves measuring the coupling strength between different qubits. The sensed interaction allows for a more general sensing Hamiltonian, which enables us to connect code states that differ by the state of more than one qubit. The simplest of these signals is the coupling strength between two TLSs. Below, we show how to correct general errors when measuring two particle coupling.

More specifically we want to measure the interaction strength between a dissipative TLS and a stable one. We assume an interaction that will induce flip flops between the two TLSs. This interaction model is fairly general, and describes, among other things, dipole-dipole interactions, sideband interaction, and hyperfine coupling. Sideband interaction is of special interest, because it enables measurement of Rabi frequencies, a quality that makes it very general.

Here, we use the code: $|A\rangle = (|\uparrow 00\rangle + |\downarrow 11\rangle)/\sqrt{2}$, $|C\rangle = (|\downarrow 10\rangle + |\uparrow 01\rangle)/\sqrt{2}$, where the first qubit is the sensing qubit and the other two are good qubits. The flip flop interaction, described by the Hamiltonian $H_s = g(\sigma_+^1\sigma_-^2 + \mathrm{H.c.})$ (g is the signal), couples the two code states directly, and thus, we do not use a utility state that lies outside the code.

This code enables us to correct general errors on the first qubit (i.e., any error that might occur on this bit). Correction of general errors is equivalent to correcting phase flips (i.e., σ_z $\pi/2$ pulse) and bit flips (i.e., σ_x π pulse). These corrections are demonstrated below.

Beginning in some general state $|\psi\rangle = a|A\rangle + c|C\rangle$, a bit flip on the first qubit results in the state

$$|\psi_{E_x}\rangle = a\frac{|\downarrow 00\rangle + |\uparrow 11\rangle}{\sqrt{2}} + c\frac{|\uparrow 10\rangle + |\downarrow 01\rangle}{\sqrt{2}}. \qquad (9)$$

By measuring the spin correlations operator $\sigma_z^1\sigma_z^2$, and applying a bit flip if the outcome is $+1$, we return to the original state and the error is corrected. Now a phase flip occurs, resulting in the state

$$|\psi_{E_z}\rangle = a\frac{|\uparrow 00\rangle - |\downarrow 11\rangle}{\sqrt{2}} + c\frac{-|\downarrow 10\rangle + |\uparrow 01\rangle}{\sqrt{2}}. \qquad (10)$$

We measure the appropriate operator and apply a phase flip as needed, resulting in full correction and a return to the original $|\psi\rangle$ state.

Here, the time delay when an error is detected is equal to $\frac{\tau}{2}$, and again, it can be accounted for. Writing the equivalent of Eqs. (6) and (7), we get the following enhancement of the precision (relative to the no-EC case):

$$\delta g_{\mathrm{EC}} = \delta g_0\sqrt{\frac{T_1^*}{T^2}} = (g\tau)\delta g_0 \ll \delta g_0. \qquad (11)$$

Conclusions and perspectives.—We proposed and analyzed the use of EC to increase the signal to noise ratio of various sensing protocols. Because of the very specific characteristics of the sensing signals and the noise model, special EC protocols were designed. We showed that EC is a powerful method that could have considerable implications for quantum-technologies goals and precision measurements. Finally, we showed that EC can be used in certain cases to increase the precision of multidirectional limited measurements, and may even help approach the Heisenberg limit.

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Note added.—Recently, we have become aware of other papers on the same topic [30–32].

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