

Chimeralike States in an Ensemble of Globally Coupled Oscillators

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We demonstrate the emergence of a complex state in a homogeneous ensemble of globally coupled identical oscillators, reminiscent of chimera states in nonlocally coupled oscillator lattices. In this regime some part of the ensemble forms a regularly evolving cluster, while all other units irregularly oscillate and remain asynchronous. We argue that the chimera emerges because of effective bistability, which dynamically appears in the originally monostable system due to internal delayed feedback in individual units. Additionally, we present two examples of chimeras in bistable systems with frequency-dependent phase shift in the global coupling.

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In spite of more than 40 years of research pioneered by A. Winfree [1] and Y. Kuramoto [2], the dynamics of globally coupled oscillator populations remains a challenging issue, with applications ranging from laser and Josephson junction arrays to problems of bridge engineering and modeling of brain waves [3]. In addition to the well-studied self-synchronization transition, of particular recent interest are complex states between synchrony and asynchrony [4]. On the other hand, a lot of attention has been attracted to the regimes of the coexistence of coherence and incoherence in oscillator lattices [5]. These states, also known as “chimeras,” have been addressed in numerous theoretical studies [6] and demonstrated in an experiment [7]. Furthermore, it has been shown that already two interacting populations of globally coupled identical oscillators can for some initial conditions exhibit symmetry-breaking of synchrony, so that one population synchronizes whereas the other remains asynchronous [8]; the existence of such chimeras has also been confirmed experimentally [9]. A natural question, addressed in this Letter, is under which conditions can such a symmetry-breaking into synchronous and asynchronous groups be observed in a completely homogeneous globally coupled population of identical oscillators.

In case of global coupling all oscillators are subject to the same force. Therefore, if the units are identical, one may expect that they should evolve similarly. This expectation is rather natural and is indeed true for simple systems like the standard Kuramoto model as well as for many other examples from the literature. However, in a system of identical globally coupled chaotic maps, K. Kaneko observed one large synchronized cluster and a cloud of scattered units (see Fig. 2(b) in [10])—a state reminiscent of a chimera. For periodic units, namely for nonisochronous Stuart-Landau oscillators, such a state has been reported by Daido and Nakanishi [11] and Schmidt *et al.* [12], who studied the cases of linear and nonlinear global

coupling, respectively, see also [13]. These observations of identical nonlinear elements behaving differently in spite of being driven by the same force, indicate presence of bi or multistability. Here we demonstrate that chimeralike states naturally appear for a minimal generalization of the popular Kuramoto-Sakaguchi phase model to the case of globally coupled identical phase oscillators with internal delayed feedback, and discuss the underlying mechanism of dynamically sustained bistability.

Globally coupled self-sustained oscillators can be quite generally treated in the phase approximation [2]. In the simplest case of identical sine-coupled units such an ensemble of N units is described by the Kuramoto-Sakaguchi model [14]:

$$\dot{\varphi}_k = \omega + \frac{\varepsilon}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_k + \beta) = \omega + \varepsilon \text{Im}(e^{i\beta} Z e^{-i\varphi_k}),$$

where φ are the oscillators’ phases, $\varepsilon > 0$ is the coupling strength, β is the phase shift in the coupling, and $Z = Re^{i\Theta} = N^{-1} \sum_{k=1}^N e^{i\varphi_k}$ is the complex Kuramoto order parameter (complex mean field). The system is known to tend to the fully synchronous state $\varphi_1 = \varphi_2 = \dots = \varphi_N$, if the coupling is attractive, i.e., $|\beta| < \pi/2$, and to remain asynchronous otherwise.

We consider a similar setup for oscillators with an internal delayed feedback loop. The latter is a natural ingredient, e.g., of lasers with external optical feedback [15] and of numerous biological systems where signal transmission in the feedback pathway may be rather slow [16]. It is known, that phase dynamics of an autonomous oscillator with a delayed feedback loop can be in the simplest case represented as $\dot{\varphi} = \omega + \alpha \sin(\varphi_\tau - \varphi)$, where $\varphi_\tau \equiv \varphi(t - \tau)$, τ is the delay, and α quantifies the feedback strength [15,17,18]. Assuming the global coupling to be of the Kuramoto-Sakaguchi type as above, we write our basic model as

$$\dot{\varphi}_k = \omega + \alpha \sin(\varphi_{\tau,k} - \varphi_k) + \varepsilon \text{Im}(e^{i\beta} Z e^{-i\varphi_k}). \quad (1)$$

We start by numerical demonstration of a chimeralike state in model (1) for parameter set $\omega = 1$, $\alpha = 1/3$, $\beta = \pi/2 + 0.01$, $\tau = \pi - 0.02$, $\varepsilon = 0.05$, and $N = 100$. In Fig. 1(a),(b) we show this state after transients in the dynamics are over; the snapshot and the time evolution of the phases clearly depict a synchronized cluster of 64 oscillators and a cloud of 36 asynchronous ones. (Notice that throughout this example we number the oscillators in a way that units with indices $k = 1, \dots, n$ are in the cluster, whereas units with $k = n + 1, \dots, N$ belong to the cloud.) Temporal phase dynamics is further illustrated in Fig. 1(c): for the elements in the cluster it is highly regular with nearly constant instantaneous frequencies, while oscillators in the cloud are chaotic and their instantaneous frequencies strongly fluctuate. Moreover, individual frequencies in the cloud are only weakly correlated, so that the phase differences demonstrate many phase slips and are unbounded. This irregularity is also reflected in the strong fluctuations of the cloud contribution to the mean field, to be compared with nearly constant contribution from the cluster [Fig. 1(d)].

Formation of the chimera state is illustrated in Fig. 2. Here in panel (a) we show the cluster growth for different

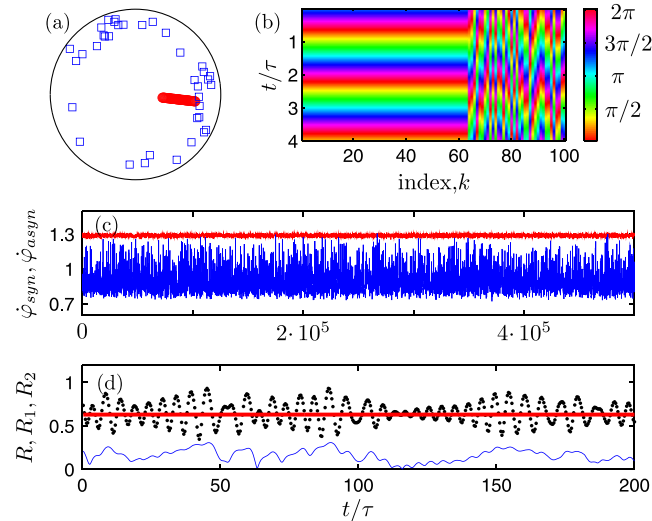


FIG. 1 (color online). Chimera state in model (1). (a) Snapshot of the phases reveals that 64 oscillators (red circles, numbered with $k = 1, \dots, 64$) are in the cluster and 36 oscillators (blue squares) belong to the cloud. For visibility, the radial coordinate is increased proportionally to the oscillator index k . (b) Temporal evolution $\varphi_k(t)$, shown by color/grey coding. (c) Instantaneous frequencies of an oscillator from the cluster (upper red curve) and of an oscillator from the cloud (lower blue curve). The average values are $\langle \dot{\varphi}_{syn} \rangle_t = 1.2897$ (cluster) and $\langle \dot{\varphi}_{asyn} \rangle_t = 0.9033$ (cloud). (d) Amplitude of the mean field component contributed by the cluster, $R_1 = |\sum_{k=1}^{64} e^{i\varphi_k}|/100$ (red bold line), and by the cloud, $R_2 = |\sum_{k=65}^{100} e^{i\varphi_k}|/100$ (blue solid line). Black dotted line shows the amplitude R of the total mean field.

initial conditions (different initial cluster size and random uniform distribution of cloud phases); we see that the cluster size saturates at a value between $n = 60$ and $n = 71$. Notice the logarithmic scale of the time axis: formation of the cluster with $q = n/N \approx 0.5$ is relatively fast, while its further growth is an extremely slow process (below we will argue that the full synchrony, i.e., the cluster with $q = 1$, cannot appear).

To show that formation of the chimeralike state is not a finite-size effect, in Fig. 2(b) we illustrate formation of the chimeralike state for ensembles of different sizes, up to $N = 1000$. In all cases the final state has cluster of size $q \approx 0.6$. As shown below, for the stability of the chimeralike state it is important, that the fluctuation of the order parameter R_2 of the cloud does not vanish in the thermodynamic limit $N \rightarrow \infty$; Fig. 2(c) demonstrates that the variance of R_2 practically does not depend on N up to values $N = 2000$. This fact indicates that the units of the cloud are not uncorrelated, but are organized in a collective chaotic mode. Finally, we emphasize that chimeras exist not only for parameters chosen above for an illustration, but in a finite parameter domain, shown in Fig. 3(a) together with domains of other types of dynamics.

Next, we present theoretical arguments explaining existence of a chimeralike state in model (1). Let us consider first the fully synchronized, uniformly rotating one-cluster state $\varphi_1 = \dots = \varphi_N = \Phi = \Omega t$, where frequency Ω is yet unknown. Substituting this expression into Eq. (1) we obtain equation

$$\Omega = \omega - \alpha \sin \Omega \tau + \varepsilon \sin \beta; \quad (2)$$

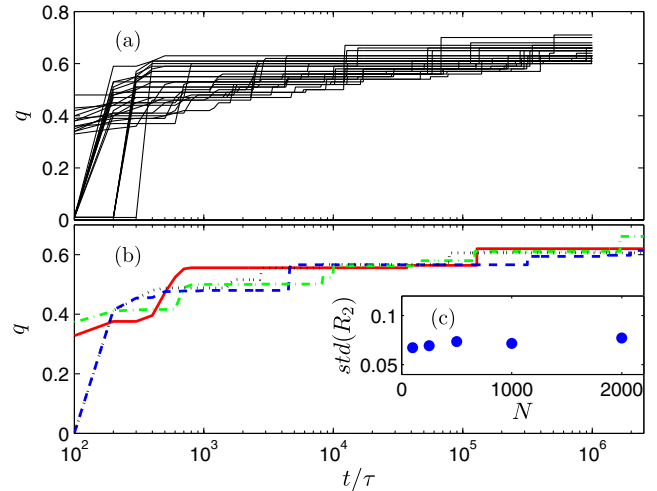


FIG. 2 (color online). Temporal evolution of the cluster and saturation of its size. (a) Growth of the relative cluster size $q = n/N$ for different initial conditions for $N = 100$ oscillators. (b) Saturation of q for different ensemble size: $N = 250$ (red solid), $N = 500$ (blue dashed), $N = 750$ (green dash-dotted), and $N = 1000$ (black dotted). (c) Standard deviation for the amplitude of the mean field component R_2 contributed by the cloud, for different ensemble size N .

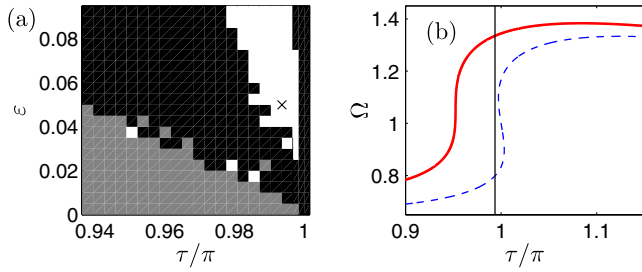


FIG. 3 (color online). (a) Approximate domain of chimera states (white region); ω , α , and β are same as above, $N = 256$. Symbol \times marks the parameters used in Figs. 1, 2. In the black domain we observed multi-cluster states, while the gray domain corresponds to the states with zero mean field and equal rotation frequencies for all units. (b) Solution of Eq. (2): frequency of the one-cluster state Ω as function of τ , for uncoupled oscillators, $\varepsilon = 0$, (blue dashed line) and for $\varepsilon = 0.05$ (red bold line). Vertical black line marks $\tau = \pi - 0.02$.

its solution $\Omega(\tau)$ is shown in Fig. 3(b) for cases $\varepsilon = 0$ (uncoupled oscillators) and $\varepsilon = 0.05$ (one-cluster state). We see that in both cases, the solution for the chosen delay τ is unique; i.e., there is no multistability. The fully synchronous cluster is, however, unstable. Indeed, consider a symmetric small perturbation to two arbitrary oscillators, $\varphi_{1,2} = \Phi \pm \delta$. Such a perturbation is transversal to the synchronization manifold and leaves the mean field unchanged; it obeys linearized equation $\dot{\delta} = \alpha \cos(\Omega\tau)(\delta_\tau - \delta) - \varepsilon \delta \cos\beta$. Most important is the eigenvalue which is close to zero; using its smallness we obtain in the first approximation $\lambda = -\varepsilon \cos\beta [1 + \tau\alpha \cos(\Omega\tau)]^{-1}$. Because for parameters used in Fig. 1 the quantity in brackets is positive, the fully synchronous state for $\varepsilon \cos\beta < 0$ is unstable. Physically, this means evaporation of the oscillators from the cluster. Numerical studies show that the fully asynchronous state with uniform distribution of phases is unstable, too. Although we cannot exclude less trivial asynchronous states, i.e., with a nonuniform distribution of phases or with several clusters and zero mean field, we have not observed them for the chosen parameters.

A natural question is why a partial cluster with $n < N$ elements (we denote its phase by Φ) is stable, while the full synchrony for $n = N$ is not. To analyze this, we again denote the perturbed phases of oscillators in the cluster as $\Phi \pm \delta$ and obtain after linearization,

$$\begin{aligned} \dot{\delta}(t) = & \alpha \cos(\Phi_\tau - \Phi)(\delta_\tau - \delta) \\ & - \left[\frac{\varepsilon n}{N} \cos\beta + \frac{\varepsilon}{N} \sum_{j=n+1}^N \cos(\varphi_j - \Phi + \beta) \right] \delta. \end{aligned} \quad (3)$$

Simultaneously we want to check, whether formation of another cluster via merging of oscillators from the cloud is possible. For this purpose we assume that two oscillators in the cloud come close to each other, so that $\Delta(t) = \varphi_k - \varphi_l$,

$l, k > n$, is small, and we can linearize the corresponding equations to obtain for the difference,

$$\begin{aligned} \dot{\Delta}(t) = & \alpha \cos(\varphi_{l,\tau} - \varphi_l)(\Delta_\tau - \Delta) \\ & - \left[\frac{\varepsilon n}{N} \cos(\Phi - \varphi_l + \beta) - \frac{\varepsilon}{N} \sum_{j=n+1}^N \cos(\varphi_j - \varphi_l + \beta) \right] \Delta. \end{aligned} \quad (4)$$

We cannot solve Eqs. (3,4) analytically, as $\varphi_j(t)$ are unknown irregular functions of time. However, we solve them numerically for large time interval T together with the full system (1) and compute the corresponding Lyapunov exponents $\lambda = \lim_{T \rightarrow \infty} (\ln \delta(T)/T) \approx -1.25 \times 10^{-2}$ and $\Lambda = \lim_{T \rightarrow \infty} (\ln \Delta(T)/T) \approx 2.38 \times 10^{-2}$. Because the Lyapunov exponent λ describing transversal stability of the cluster is negative, and the exponent Λ describing transversal stability in the cloud is positive, the cluster is stable towards evaporation of the oscillators, while merging of cloud oscillators to another minicluster is forbidden.

Stabilization of the cluster can be qualitatively explained as follows. Contrary to the fully synchronized case, in presence of a cloud, oscillators in the cluster are subject to a force which has two components, as illustrated by Fig. 1(d): a regular force from the cluster and an irregular one from the cloud [last term in Eq. (3)]. In the first approximation, the irregular component can be treated as a random force, and this effective noise is common for all elements of the cluster. It is known that common noise tends to synchronize oscillators [19,20]. Here, for sufficiently strong noise, this tendency to synchrony overcomes the internal repulsion in the cluster and stabilizes it. However, the cluster cannot absorb all elements, because for $n = N$ the noisy component vanishes; hence, $n < N$.

Considering now the system from a different viewpoint, we discuss, why the periodic forcing from the cluster does not entrain the cloud oscillators and they eventually do not join the cluster. Indeed, at initial state of chimera formation more and more oscillators join the cluster (see Fig. 2) and the more oscillators merge into the cluster, the stronger is the forcing on the cloud oscillators. Hence, one may expect the increased tendency to synchrony. However, with increase of n , the frequency of the cluster grows as described by $\Omega = \omega - \alpha \sin \Omega\tau + \varepsilon(n/N) \sin\beta$, where in the first approximation we neglect the random forcing from the cloud. For $n = 64$ the estimated frequency is $\Omega = 1.2901$, in a perfect agreement with the observed value 1.2897 [see Fig. 1(c)]. Thus, not only the amplitude $\varepsilon n/N$ of the forcing on nonsynchronized units grows with n , but also the frequency mismatch. The growth of the cluster saturates when these values drift outside of the synchronization domain for the forced oscillators in the cloud. To confirm this, we have determined this domain for chosen parameters using a periodic forcing with parameters taken from the cluster dynamics, and found

that the forcing with the cluster frequency and the corresponding amplitude lies almost exactly at the border of the domain. Thus, for $q \approx 65$ further entrainment of oscillators by the cluster is not possible.

The presented discussion explains the mechanism of the dynamically sustained bistability that underlies the chimeralike state in our globally coupled system of identical units: the ensemble splits into two parts with completely different dynamics, and these parts together create a mean field that allows such a bistability. This mechanism is nontrivial, because, as illustrated in Fig. 3(b), for the chosen parameters the uncoupled systems are monostable. However, due to interaction, the oscillators become effectively bistable: being forced by the same field, they exhibit two very different dynamical patterns. The oscillators in one group are regular and therefore easily synchronize with each other, while the others are highly irregular and remain in different asynchronous, although correlated, states. The global field that leads to the bistability is dynamically sustained in a self-consistent way.

Next we discuss less nontrivial, though more transparent, setups where already noncoupled oscillators are bistable. Here the coupling is organized in a way, that it acts repulsively on the oscillators in one state and attractively on those which are in the other state. For the first example we consider a model,

$$\dot{\varphi}_k = \omega + \alpha \sin(\varphi_{\tau,k} - \varphi_k) + \varepsilon R \sin(\Theta_T - \varphi_k + \beta), \quad (5)$$

where $R e^{i\Theta} = Z$ and $\Theta_T = \Theta(t - T)$. In contradistinction to our model (1), here not only individual oscillators possess a delayed feedback loop, but the global coupling is also delayed, with another delay time $T \neq \tau$. Parameters of oscillators are taken as $\omega = \pi$, $\tau = 0.99$, and $\alpha = 1.2$, so that uncoupled units oscillate either with the frequency $\Omega_1 = 2.0845$ or $\Omega_2 = 4.0795$, i.e., are bistable. For coupling parameters $\varepsilon = 0.1$, $\beta = \pi/2$, and $T = 0.2\tau$, we observe a chimera state (not shown, very similar to the state depicted in Fig. 4), which can be explained as follows. Suppose there is a nonzero mean field with the frequency ν .

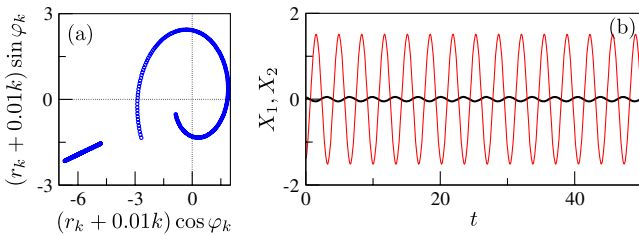


FIG. 4 (color online). Chimera state in the system of identical Stuart-Landau-type oscillators in Eq. (6). (a) Snapshot clearly demonstrates one cluster and a group of asynchronous units. Notice that for visibility, in the plot the amplitudes of all units are substituted as $r_k \rightarrow r_k + 0.01k$. (b) Mean fields of two subgroups, $X_1 = N^{-1} \sum_{j=1}^{N/2} r_k \cos \varphi_j$ (bold black line) and $X_2 = N^{-1} \sum_{j=N/2+1}^N r_k \cos \varphi_j$ (solid red line).

In the first approximation, the delay in the coupling is equivalent to the phase shift νT which sums with the constant phase shift parameter β . The coupling is attractive if the total shift obeys $|\nu T + \beta| < \pi/2$, and repulsive otherwise. Since the phase shift is frequency dependent, the effective coupling through the same global mean field is attractive for individual oscillators having frequency $\nu = \Omega_1$ and repulsive for those with $\nu = \Omega_2$. As a result, the subpopulation of oscillators which initially are in the state with Ω_1 synchronize, while the elements with Ω_2 remain asynchronous.

A similar scenario can be implemented with bistable identical oscillators without delays. Consider N Stuart-Landau-type oscillators, (here written in polar coordinates r_k, φ_k) having two stable limit cycles and let these oscillators be globally coupled via an additional linear circuit, described by variable u :

$$\begin{aligned} \dot{r}_k &= 0.1 r_k (1 - r_k^2)(4 - r_k^2)(9 - r_k^2) + \varepsilon \dot{u} \cos \varphi_k, \\ \dot{\varphi}_k &= 1 + \alpha r_k^2 - \varepsilon \frac{\dot{u}}{r_k} \sin \varphi_k, \\ \ddot{u} + \gamma \dot{u} + \eta^2 u &= N^{-1} \sum_j^N r_j \cos \varphi_j. \end{aligned} \quad (6)$$

Parameters are $\alpha = 0.1$, $\varepsilon = 0.1$, $\gamma = 0.01$, $\eta = 1.5$, $N = 400$. In the simulation, initially $N/2$ units were close to the limit cycle with the amplitude ≈ 1 whereas the others were close to the second limit cycle, with the amplitude ≈ 3 . The observed chimera state is shown in Fig. 4. Indeed, the frequencies of the limit cycle oscillations are $\Omega_1 = 1.1$ and $\Omega_2 = 1.9$. Since the resonant frequency of the circuit η lies between them, $\Omega_1 < \eta < \Omega_2$, the phase shift in the global coupling introduced by the harmonic circuit is attractive for the state with Ω_2 and repulsive for that with Ω_1 .

In summary, we have demonstrated numerically and explained semi-quantitatively the emergence of chimera states in ensembles of identical globally coupled oscillators. We have outlined a mechanism of dynamically sustained bistability which results in symmetry-breaking of the initially homogeneous system. Here, a remarkable constructive role is played by collective chaos of non-synchronized units: the irregular forcing from the cloud counteracts the instability of the fully synchronous state, thus stabilizing the cluster of synchronized $n < N$ elements. We have also demonstrated that chimeralike states are possible without this mechanism, if the individual units are naturally bistable, like in setups described by Eqs. (5,6). We stress that the chimeralike regimes here are conceptually much simpler than in the model (1): the asynchronous oscillators are not chaotic; moreover, here the partition into synchronous and asynchronous states is fully determined by initial conditions, while in Eq. (1) the partition appears self-consistently. In this Letter we analyzed only ensembles of identical oscillators, as here the

effect is mostly striking. However, we expect that the main features survive for small heterogeneity and/or noise; this issue remains a subject for a future study.

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