Universal Quantum Graphs

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For time-reversal invariant graphs we prove the Bohigas-Giannoni-Schmit conjecture in its most general form: For graphs that are mixing in the classical limit, all spectral correlation functions coincide with those of the Gaussian orthogonal ensemble of random matrices. For open graphs, we derive the analogous identities for all S-matrix correlation functions.

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Introduction.—The distribution of eigenvalues of a classically chaotic Hamiltonian is one of the central themes of quantum chaos. In 1984, Bohigas, Giannoni, and Schmit (BGS) [1] formulated the celebrated "BGS conjecture" (see also Refs. [2–4]): *The spectral fluctuation properties of a Hamiltonian quantum system that is classically chaotic (mixing) coincide with those of the random-matrix ensemble in the same symmetry class.* Here, "spectral fluctuation properties" refers to the totality of spectral fluctuation measures. The symmetry class (orthogonal, unitary, or symplectic) is determined [5] by the properties of the system under time reversal and under rotation.

In addition to substantial numerical evidence [6], the BGS conjecture has received analytical support along two lines. (i) With the help of the semiclassical approximation and periodic-orbit theory, the level-level correlator ("two-point function") for chaotic systems was shown to coincide with that of random-matrix theory [7–10]. (ii) The two-point function for quantum graphs [11] was shown [12,13] to obey the BGS conjecture (even though graphs are not strictly Hamiltonian systems). That result was extended to the S-matrix correlation function for open graphs [14,15]; see also Ref. [16].

In this Letter, we prove the BGS conjecture for time-reversal invariant graphs in its most general form. Generalizing the approach of Refs. [12–15], we show that for graphs with incommensurate bond lengths that are mixing in the classical limit, all spectral correlation functions coincide with those of the Gaussian orthogonal ensemble (GOE) of random matrices. For open graphs, we derive the analogous identities for all S-matrix correlation functions.

Graphs.—We need to define the correlation functions for levels and for S-matrix elements. To make the Letter self-contained, we first collect the relevant definitions and properties of graphs. A closed graph [11,17] is a system of V vertices labeled $\alpha, \beta, \ldots$ connected by B bonds labeled $(\alpha\beta), \ldots$ or simply by $b = 1, 2, \ldots, B$. We consider simple, completely connected graphs (every pair of vertices is connected by a single bond). Then $B = V(V-1)/2$. We eventually take the limit $B \to \infty$. The lengths L_b of the bonds are assumed to be incommensurate (there is no set $\{i_b\}$ of positive, negative, or zero integers for which $\sum_b i_b L_b$ vanishes). For $B \to \infty$, the lengths are assumed to remain bounded, $L_{\min} \le L_b \le L_{\max}$ for all b. On each bond b, the Schrödinger wave is written as $s_{b1} \exp\{ikx_b\} + s_{b2} \exp\{-ikx_b\}$ with the same real wave number k for all bonds. The variable x_b denotes the distance to one of the two vertices connected by the bond. The set of coefficients $\{s_{b1}, s_{b2}\}$ is determined by boundary conditions defined on each vertex α and written as $\mathcal{O}^{(\alpha)} = \sigma^{(\alpha)} \mathcal{I}^{(\alpha)}$. Here, $\mathcal{I}^{(\alpha)}$ ($\mathcal{O}^{(\alpha)}$) is the vector of incoming (outgoing) wave amplitudes on the bonds attached to vertex α, respectively. The matrices $\sigma^{(\alpha)}$ have dimension $V-1$ and are symmetric (time-reversal invariance) and unitary (flux conservation). Open graphs are defined by attaching to each of the vertices labeled $\alpha = 1, 2, \ldots, \Lambda$ an additional bond (a "channel") labeled α that extends to infinity. For these vertices, the boundary conditions $\mathcal{O}^{(\alpha)} = \Gamma^{(\alpha)} \mathcal{I}^{(\alpha)}$ involve the symmetric and unitary boundary condition matrices $\Gamma^{(\alpha)}$ of dimension V given by

$$\Gamma^{(\alpha)} = \begin{pmatrix} \rho^{(\alpha)} & \tau^{(\alpha)}_\beta \\ \tau^{(\alpha)}_\gamma & \sigma^{(\alpha)}_{\gamma\beta} \end{pmatrix}. \quad (1)$$

Here, $\rho^{(\alpha)}$ is the amplitude for backscattering into channel α, and $\tau^{(\alpha)}_\beta$ is the amplitude for scattering from channel α to vertex β or vice versa. The matrices $\sigma^{(\alpha)}$ in Eq. (1) are subunitary. For $B \to \infty$, the number Λ of channels is held fixed.

To introduce the spectral determinant $\xi(k)$ for closed graphs [11–13,17] and the scattering matrix (S matrix) $S(k)$ for open graphs [11,14,15], we define in both cases the block-diagonal symmetric "vertex scattering matrix" $\Sigma^{(V)}$. That matrix contains the matrices $\sigma^{(\alpha)}$, $\alpha = 1, 2, \ldots, V$ in its diagonal blocks. It has dimension $V(V-1)$ and is unitary (subunitary) for closed (open) graphs, respectively. Since $V(V-1) = 2B$, all relevant expressions can most easily be written in matrix form by doubling the number of bonds.

The bonds $(\alpha\beta)$ are arranged in lexicographical order (so that $\alpha < \beta$). The resulting sequence is mapped onto the sequence $b = 1, \ldots, B$. These bonds carry the labels $(b+)$. To every such "directed bond" $(\alpha\beta)$ with $\alpha < \beta$ the bond $(b-)$ is defined by $(\beta\alpha)$. The number of directed bonds (bd) with $d = \pm$ is $2B$. In directed-bond representation, the vertex scattering matrix is denoted by $\Sigma^{(B)}$ ("bond scattering matrix"). That matrix is also symmetric and unitary (subunitary, respectively).

Zeros of the spectral determinant $\xi(k) = \det(1 - \exp\{ik\mathcal{L}\}\sigma_1^d\Sigma^{(B)})$ define the bound states of a closed graph while scattering on an open graph is described by the symmetric unitary scattering matrix $S_{\alpha\beta}(k)$ of dimension Λ,

$$S_{\alpha\beta}(k) = \rho^{(\alpha)}\delta_{\alpha\beta} + (\mathcal{T}\mathcal{W}^{-1}\mathcal{T}^T)_{\alpha\beta}. \tag{2}$$

Here, $\mathcal{W} = \exp\{-ik\mathcal{L}\}\sigma_1^d - \Sigma^{(B)}$ while $\mathcal{T}$ is a rectangular matrix of dimension $\Lambda \times 2B$ containing the amplitudes $\tau_\beta^{(\alpha)}$ in directed-bond representation as nonzero elements. The symbol T denotes the transpose. The matrix $\exp\{ik\mathcal{L}\}$ with $\mathcal{L} = \{\delta_{bb'}\delta_{dd'}L_b\}$ describes propagation on the directed bonds, with the bond propagator $\exp\{ikL_b\}$ independent of the direction of the bond. The matrix σ_1^d is the first Pauli spin matrix in directional space multiplied by the unit matrix in non-directed-bond space. That matrix is needed to write $\xi(k)$ and $S_{\alpha\beta}(k)$ in matrix form.

The probability distributions for levels and S-matrix elements are specified in terms of average values and correlation functions. All averages (indicated by angular brackets) are taken over the wave number k. The average level density is [11] $\langle d_R\rangle = (1/\pi)\sum_b L_b$, and the average S matrix is [11] $\langle S_{\alpha\beta}\rangle = \rho^{(\alpha)}\delta_{\alpha\beta}$. The fluctuating part of the level density is [12,13] $[1/(i\pi)]\Im[d/(dk)]\ln\xi(k^+)$ where the plus sign indicates an infinitesimal positive imaginary increment. The fluctuating part of the scattering matrix is $S^{\rm fl} = \mathcal{T}\mathcal{W}^{-1}\mathcal{T}^T$. In terms of these quantities, the (P, Q) correlation functions for levels (closed graphs) and S-matrix elements (open graphs) are

$$\left\langle \prod_{p=1}^{P} \frac{d}{dk}\ln\xi(k^+ + \kappa_p) \prod_{q=1}^{Q} \frac{d}{dk}\ln\xi(k^- - \tilde{\kappa}_q) \right\rangle,$$
$$\left\langle \prod_{p=1}^{P} S^{\rm fl}_{\alpha_p\beta_p}(k + \kappa_p) \prod_{q=1}^{Q} S^{\rm fl *}_{\gamma_q\delta_q}(k - \tilde{\kappa}_q) \right\rangle. \tag{3}$$

Here P and Q are positive integers. Interest centers on fluctuations on the scale of the average level spacing $1/\langle d_R\rangle$. Therefore, the incremental wave numbers κ_p and $\tilde{\kappa}_q$ obey $\kappa_p\langle d_R\rangle \ll B$ and $\tilde{\kappa}_q\langle d_R\rangle \ll B$.

Classical limit.—In the classical limit [18,19], the time evolution of the probability density r, a vector with component $r_{bd} \geq 0$ on the directed bond (bd), is defined in terms of the discrete map $r \to \mathcal{F}r$. Here, $\mathcal{F}$ is the Perron-Frobenius (PF) operator [17], a nonsymmetric matrix with elements $\mathcal{F}_{bd,b'd'} = |(\sigma_1^d\Sigma^{(B)})_{bd,b'd'}|^2$. All elements of $\mathcal{F}$ are positive or zero. Moreover, $\mathcal{F}$ is bistochastic, $\sum_{b'd'}\mathcal{F}_{bd,b'd'} = 1 = \sum_{bd}\mathcal{F}_{bd,b'd'}$ and, since the graphs considered are completely connected, irreducible. According to the Perron-Frobenius theorem, the irreducible bistochastic matrix $\mathcal{F}$ possesses a nondegenerate maximal eigenvalue $\lambda_1 = 1$ with associated right (left) eigenvector $u_1 = (1, 1, \ldots, 1)^T$ $[w_1 = (1, 1, \ldots, 1)]$, respectively. A closed graph is mixing [17] if λ_1 is the only eigenvalue of $\mathcal{F}$ on the unit circle in the complex plane, with all other eigenvalues located within or on the surface of a disc within the unit circle. For an m-fold repeated map $r \to \mathcal{F}^m r$, we then have $r \to u_1(w_1|r)$ exponentially for $m \to \infty$. For the graph to remain mixing in the limit $B \to \infty$, we require that the minimum distance between the disc of eigenvalues λ_j with $j \geq 2$ and the unit circle remains finite, $|\lambda_j| \leq 1 - a$ with $a > 0$. We postulate that same condition for open graphs. Since Λ is held fixed, it seems plausible that for $B \to \infty$ that condition is met (perhaps with a different value of a) for any open graph the closed counterpart of which is mixing.

Averages. supersymmetry.—To average over k, the content of the angular brackets in every (P, Q) correlation function is written as a suitable derivative of a generating function $\mathcal{G}_G$ (a superintegral) [13,15]. The average is carried out using the incommensurability of the bond lengths by replacing [12] the integration over k by an integration over the independent phases $\phi_b = kL_b$ and using the color-flavor transformation [20]. The P (Q) factors in Eqs. (3) generate the retarded block (the advanced block, respectively). The result is [13,15] $\langle\mathcal{G}_G\rangle = \int d(Z, \tilde{Z})\exp\{-\mathcal{A}\}$ where the effective action is

$$\mathcal{A}(Z, \tilde{Z}) = -{\rm STL}(1 - Z\tilde{Z}) + \frac{1}{2}{\rm STL}(1 - z_+ Z z_- Z^\tau)$$
$$+ \frac{1}{2}{\rm STL}(1 - z_+ S_+ \tilde{Z}^\tau S_-^\dagger z_- \tilde{Z}). \tag{4}$$

Here, STL stands for the combined operations (STrln) where STr denotes the supertrace. Moreover, $S_\pm = (\sigma_1^d\Sigma^{(B)} - \mathcal{J}_\pm)$ while $z_+ = \exp\{i\kappa\mathcal{L}\}$ and $z_- = \exp\{i\tilde{\kappa}\mathcal{L}\}$ in obvious notation. Differentiation of $\mathcal{G}_G$ with respect to the source terms $\mathcal{J}_+$ $(\mathcal{J}_-)$ in the retarded (advanced) blocks yields the (P, Q) correlation functions. The source terms $\mathcal{J}_\pm$ differ for open and for closed graphs and are given in Refs. [12–15]. With $s = 1, 2, 3, 4$ the index for the supervariables, the matrices Z $(\tilde{Z})$ have elements $Z_{pbds,qb'd's'}$ $(\tilde{Z}_{qbds,pb'd's'})$, dimension $8BP \times 8BQ$ $(8BQ \times 8BP$, respectively), and are both diagonal $(\propto \delta_{bb'})$ in bond space. The integration measure is the flat Berezinian. As in Ref. [13] Z^τ is a transform of Z.

Saddle-point manifold.—Variation of $\mathcal{A}(Z, \tilde{Z})$ with respect to Z and $\tilde{Z}$ yields two saddle-point equations [12,15]. The first one yields $Z^\tau = \tilde{Z}$. The second is met if (i) $[\sigma_1^d\Sigma^{(B)}, Z] = 0$ and if (ii) $\sigma_1^d\Sigma^{(B)}(\sigma_1^d\Sigma^{(B)})^\dagger = 1$.

Condition (i) reduces the matrices Z, $\tilde{Z}$ to the saddle-point solution $Y = \{\delta_{bb'}\delta_{dd'}Y_{ps,qs'}\}$, $\tilde{Y} = \{\delta_{bb'}\delta_{dd'}\tilde{Y}_{qs,ps'}\}$. In saddle-point approximation we have $\langle \mathcal{G}_G \rangle_{sp} = \int d(Y, \tilde{Y}) (\ldots) \exp\{SB_G + CC_G\}$ where the integration measure is the flat Berezinian. The dots indicate the source terms. The "symmetry-breaking term" is

$$SB_G = i\pi\langle d_R\rangle\left\{\sum_p \kappa_p \mathrm{STr}_s\left(\frac{1}{1-Y\tilde{Y}}\right)_{pp} + \cdots\right\}. \quad (5)$$

The dots indicate a second term obtained from the first by $p \to q$, $\kappa_p \to \tilde{\kappa}_q$, $Y \leftrightarrow \tilde{Y}$. Condition (ii) is violated for open graphs. The unitarity deficit of the matrices $\sigma^{(\alpha)}$ for $\alpha = 1, \ldots, \Lambda$ and the ensuing unitarity deficit of the average S matrix are accounted for by the "channel-coupling term"

$$CC_G = -\frac{1}{2}\sum_{\alpha=1}^{\Lambda} \mathrm{STr}_{ps} \ln\left(1 + T^{(\alpha)}\frac{Y\tilde{Y}}{1-Y\tilde{Y}}\right), \quad (6)$$

where $T^{(\alpha)} = 1 - |\langle S_{\alpha\alpha}\rangle|^2$.

Massive modes.—The degrees of freedom in the superintegral for $\langle \mathcal{G}_G \rangle$ that do not belong to the saddle-point manifold are orthogonal to Y, $\tilde{Y}$ and are taken into account in Gaussian approximation [13]. We expand the effective action (4) up to second order in Z, $\tilde{Z}$, dropping the source terms and the incremental wave numbers κ_p, $\tilde{\kappa}_q$. We use $Z = \tilde{Z}^\tau$ since fluctuations away from that condition are strongly suppressed [13]. That yields two terms. One contains $(1-\mathcal{F})_{bd,b'd'}$ sandwiched between $Z_{pbds,qbds'}$ and $\tilde{Z}_{qb'd's,pb'd's'}$. It can be written as

$$\frac{1}{2}\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{j\geq 2}^{2B} \mathrm{STr}_s\{z_{j;pq}(1-\lambda_j)\tilde{z}_{j;qp}\}. \quad (7)$$

The supermatrices $z_{j;pq}$ ($\tilde{z}_{j;qp}$) are obtained by multiplying $Z_{pbds;qbds'}$ ($\tilde{Z}_{qb'd's;pb'd's'}$) with the left (right) eigenvectors of $\mathcal{F}$, respectively, that belong to eigenvalue λ_j with $j \geq 2$. Since $\Re\lambda_j < 1$ for all $j \geq 2$, Eq. (7) defines bona fide Gaussian integrals with masses $m_j = 1 - \lambda_j$ for $j \geq 2$, both for closed and for open graphs. The second term is the supertrace of $[1 - (\sigma_1^d\Sigma^{(B)})_{bd,bd}(\sigma_1^d\Sigma^{(B)})^\dagger_{bd',bd'}]$ sandwiched between $Z_{pbds,qbd's'}$ and $\tilde{Z}_{qbd's,pbds'}$ and summed over all b and all $d \neq d'$. The fluctuations due to that term are negligible because the matrices $\sigma^{(\alpha)}$ are unitary or subunitary so that for $V \gg 1$ all elements of $\sigma^{(\alpha)}$ are generically small (of order $V^{-1/2}$). We focus attention on Eq. (7). We expand the source terms and the remaining terms in the effective action (4) in Taylor series in $Z_{pbds,qbds'}$ and $\tilde{Z}_{qb'd's,pb'd's'}$, dropping all other terms. Using the right and left eigenfunctions of $\mathcal{F}$, we transform $Z_{pbds,qbds'} \to z_{j;ps,qs'}$ ($\tilde{Z}_{qbds,pbds'} \to z_{j;qs,ps'}$, respectively). We carry out the resulting Gaussian integrals. For closed graphs, the resulting expressions are bounded from above by terms of the form

$$\frac{C}{B}\prod_{l=1}^{P+Q-1}\frac{1}{B}\sum_{j_l=2}^{2B}\frac{1}{|m_{j_l}|^{k_l}}. \quad (8)$$

Here, C is some positive constant and k_l are non-negative integers. For $|m_j| > a$ (all j), the term of Eq. (8) vanishes for $B \to \infty$. The factors B^{-1} in Eq. (8) are due to the source terms for closed graphs. Detailed analysis shows that reduction factors equivalent to B^{-1} arise also for open graphs because here the source terms are matrices in directed-bond space that have a single nonvanishing element only. Hence, the expressions analogous to Eq. (8) for open graphs also vanish.

We conclude that both for closed and for open graphs, the contribution of massive modes is negligible for $B \to \infty$. Therefore, all (P, Q) correlation functions are obtained by differentiating $\langle \mathcal{G}_G \rangle_{sp}$ with respect to the source terms.

Random-matrix approach.—We turn to the GOE [21] and generalize the supersymmetry approach of Refs. [22,23] to the general (P, Q) correlation function. The real matrix elements $H_{\mu\nu}$ of the symmetric N-dimensional GOE Hamiltonian H are Gaussian-distributed random variables with zero mean values and second moments $\langle H_{\mu\nu}H_{\mu'\nu'}\rangle = (\lambda^2/N)(\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'})$. The indices run from 1 to N while $\lambda = Nd/\pi$ where d is the mean level spacing at the center of the GOE spectrum. The angular brackets denote the ensemble average. With E the energy, the (P, Q) level correlation function for the closed system is defined as

$$\left\langle \prod_{p=1}^{P} \mathrm{Tr}(E^+ + \varepsilon_p - H)^{-1} \prod_{q=1}^{Q} \mathrm{Tr}(E^- - \tilde{\varepsilon}_q - H)^{-1}\right\rangle. \quad (9)$$

The plus (minus) sign indicates an infinitesimal positive (negative) imaginary increment. The open system is obtained [23] by coupling Λ channels $a, b, \ldots$ to the states labeled μ by real channel-coupling matrix elements $W_{a\mu} = W_{\mu a}$. These obey $\sum_\mu W_{a\mu}W_{\mu b} = Nv_a^2\delta_{ab}$. The scattering matrix is $S_{ab} = \delta_{ab} - 2\pi i[W(E - H + i\pi W^\dagger W)^{-1}W^\dagger]_{ab}$. The S-matrix correlation function is defined in analogy to the second term of Eq. (3), with the replacements $S^{\mathrm{fl}}_{\alpha_p\beta_p}(k+\kappa_p) \to S_{a_pb_p}(E+\varepsilon_p)$, $S^{\mathrm{fl}\,*}_{\gamma_q\delta_q}(k-\tilde{\kappa}_q) \to S^*_{c_qd_q}(E-\tilde{\varepsilon}_q)$. In contrast to Eq. (3) the correlator now also contains the average S-matrix elements. That must be borne in mind when we later compare the source terms. The incremental energies obey $\varepsilon_p, \tilde{\varepsilon}_q \ll dN$.

The contents of the angular brackets in the (P, Q) correlation functions are written as suitable derivatives [23,24] with respect to source terms $\mathcal{J}_\pm$ of a generating function $\mathcal{G}_R$ (a superintegral). The ensemble average is calculated by straightforward generalization of the steps in Ref. [23]. The ensemble average over H is followed by the Hubbard-Stratonovich transformation and by the saddle-point approximation. At the center of the GOE spectrum, the saddle-point manifold is parametrized as

$\sigma_R = -iT_0^{-1}LT_0$. In retarded-advanced block notation, L is equal to the third Pauli spin matrix whereas T_0 is given by

$$T_0 = \begin{pmatrix} (1+t_{12}t_{21})^{1/2} & it_{12} \\ -it_{21} & (1+t_{21}t_{12})^{1/2} \end{pmatrix}. \quad (10)$$

The matrix t_{12} (t_{21}) has elements $(t_{12})_{ps,qs'}$ [$(t_{21})_{qs,ps'}$, respectively]. The elements of (t_{12}, t_{21}) span the saddle-point manifold for the (P, Q) correlation function. That gives $\langle \mathcal{G}_R \rangle_{sp} = \int d\mu(t)(\ldots)\exp\{SB_R + CC_R\}$ where the dots indicate the source terms. We suppress the definition of the invariant measure $d\mu(t)$. In analogy to Eqs. (5) and (6) the symmetry-breaking term is

$$SB_R = \frac{i\pi}{d}\left\{\sum_p \varepsilon_p \mathrm{STr}_s((t_{12}t_{21})_{pp}) + \cdots\right\}, \quad (11)$$

where the dots indicate a second term obtained from the first by the replacements $\sum_p \to \sum_q$, $\varepsilon_p \to \tilde{\varepsilon}_q$, $(t_{12}t_{21})_{pp} \to (t_{21}t_{12})_{qq}$. The channel-coupling term (present only for the open system) is

$$CC_R = -\frac{1}{2}\sum_c \mathrm{STr}_{ps}\ln(1 + T^{(c)}t_{12}t_{21}). \quad (12)$$

The transmission coefficient $T^{(c)}$ in channel c is defined as $T^{(c)} = 1 - |\langle S_{cc}\rangle|^2$.

The contribution of the massive modes to the (P, Q) correlation functions for the GOE can be shown to vanish with some inverse power of N as $N \to \infty$. Therefore, these functions are obtained by differentiation of $\langle \mathcal{G}_R \rangle_{sp}$ with respect to the sources.

Equivalence.—For $B \to \infty$ and $N \to \infty$, massive modes contribute neither to $\langle \mathcal{G}_G \rangle$ nor to $\langle \mathcal{G}_R \rangle$. The identity of all ($P$, Q) correlation functions of both approaches is, therefore, proved by showing that $\langle \mathcal{G}_G \rangle_{sp} = \langle \mathcal{G}_R \rangle_{sp}$. We equate ε_p/d with $\kappa_p\langle d_R\rangle$, $\tilde{\varepsilon}_q/d$ with $\tilde{\kappa}_q\langle d_R\rangle$, and $T^{(a)}$ with $T^{(\alpha)}$ for both a and $\alpha = 1, \ldots, \Lambda$. We define

$$\tau = -it_{12}\frac{1}{\sqrt{1+t_{21}t_{12}}}, \qquad \tilde{\tau} = it_{21}\frac{1}{\sqrt{1+t_{12}t_{21}}}. \quad (13)$$

With these substitutions and upon the identification $\tau = Y$, $\tilde{\tau} = \tilde{Y}$, the terms SB_R and CC_R in Eqs. (11) and (12) become equal to SB_G and CC_G in Eqs. (5) and (6), respectively. For the source terms (not given here) the identity is easily proved for the closed systems. For the open systems, the identity is established on the level of the transmission coefficients as the coupling matrix elements $W_{a\mu}$ of the GOE approach bear no direct analogy to the elements of the matrix $\Sigma^{(B)}$ for graphs.

With the substitutions (13) the saddle-point manifold $\sigma_R = -iT_0^{-1}LT_0$ takes the form

$$\sigma_R = -i\begin{pmatrix} 1 & \tau \\ \tilde{\tau} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & \tau \\ \tilde{\tau} & 1 \end{pmatrix}^{-1}. \quad (14)$$

For this parametrization of σ_R, the integration measure is [20] the flat Berezinian $\prod_{pq} d(\tau_{pq}, \tilde{\tau}_{qp})$, as is the case for ($Y$, $\tilde{Y}$). Complete identity of the two saddle-point manifolds is then guaranteed if for each set of block indices (p, q) there exists a one-to-one map of the two sets of matrices (Y_{pq}, $\tilde{Y}_{qp}$) and (τ_{pq}, $\tilde{\tau}_{qp}$) onto each other. That follows from the facts that all these supermatrices have dimension four, possess the same symmetries including a compact parametrization of the fermion-fermion block, and together parametrize the same supermanifold (the extension of Efetov's coset space [22] from the two-point function to the (P, Q) correlation function). It then follows that all (P, Q) correlation functions for time-reversal invariant graphs and for the GOE pairwise coincide, both for closed and for open systems.

Discussion.—We have proved the BGS conjecture for quantum graphs in its most general form both for closed and for open graphs in the limit of infinite bond number B. Our result shows how universal symmetries dominate in that limit. The proof involves a number of assumptions. (i) We have limited ourselves to graphs that are time-reversal invariant (orthogonal symmetry). We expect that the extension to unitary symmetry will be completely straightforward. (ii) Graphs must have incommensurate bond lengths. That assumption is essential as it allows the average over the wave number k to be replaced by averages over the phases $\phi_b = kL_b$ and enables the use of the color-flavor transformation. (iii) Graphs are completely connected. The removal of a finite number of bonds probably does not affect our results for $B \to \infty$. Otherwise, we expect qualitative changes that might be caused, for instance, by Anderson localization. The relation between the connectivity of the graph and the spectrum of the PF operator poses an important open problem. (iv) Graphs are classically mixing. The ensuing condition on the spectrum of the PF operator (existence of a gap separating the eigenvalue +1 from the rest of the spectrum) guarantees that the contribution of the massive modes to all (P, Q) correlation functions vanishes for closed graphs and analogously for open graphs. In Refs. [12,13,25–27], it is shown that weaker conditions on the spectrum of the PF operator suffice to guarantee certain fluctuation properties of the GOE type. It is not clear how such conditions relate to conditions on the time evolution of the classical probability density in directed-bond space and, thus, to classical chaos.

In Refs. [14,15] the complete set of (P, Q) S-matrix correlation functions for graphs was calculated explicitly in the Ericson regime, i.e., for $\sum_\alpha T^{(\alpha)} \gg 1$. It was conjectured that these results are generic. The present Letter confirms that conjecture. Beyond that regime, our results are only implicit. We prove the identity of all (P, Q)

correlation functions for graphs and for the GOE without being able to work out these functions explicitly (except for $P = 1 = Q$).

In Refs. [28–30], a field-theoretical approach to quantum chaos based upon the PF operator and on the nonlinear sigma model was advocated. Our work shows that the PF operator does indeed determine essential features of the problem. Knowledge of that operator is not sufficient, however. As shown below Eq. (7), the masses of the modes $Z_{pbds,qbd's'}$ with $d \neq d'$ are determined by quantum amplitudes that go beyond the classical PF operator.

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