Superradiance of Degenerate Fermi Gases in a Cavity

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In this Letter we consider spinless Fermi gases placed inside a cavity and study the critical strength of a pumping field for driving a superradiance transition. We emphasize that the Fermi surface nesting effect can strongly enhance the superradiance tendency. Around certain fillings, when the Fermi surface is nearly nested with a relevant nesting momentum, the susceptibility of the system toward a checkboard density-wave ordered state is greatly enhanced in comparison with a Bose gas with the same density, because of which a much smaller (sometime even vanishingly small) critical pumping field strength can give rise to superradiance. This effect leads to interesting reentrance behavior and a topologically distinct structure in the phase diagram. Away from these fillings, the Pauli exclusion principle brings about the dominant effect for which the critical pumping strength is lowered in the low-density regime and increased in the high-density regime. These results open the prospect of studying the rich phenomena of degenerate Fermi gases in a cavity.

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Recently, a series of experiments have studied a weakly interacting degenerate Bose gas in a cavity [1,2] in which a superradiance induced density-ordered superfluid phase [1] and the softening of roton excitations in the vicinity of a superradiance phase transition have been observed [2]. Studying degenerate quantum gases inside a cavity offers new insights to many-body systems [3]. First, the cavity field is a dynamical photon field rather than a classical laser configuration; cavity photon modes affect the many-body system as dynamical variables. For example, cavity photons can mediate effective long-range interactions between atoms [4,5]; a multimode cavity can introduce frustration to atoms that enhance quantum fluctuations [6]. Second, the inevitable decay of cavity photons makes the system interesting for studying nonequilibrium phenomena.

In free space without a cavity, superradiance has been proposed theoretically for fermions [7,8] and subsequently demonstrated experimentally [9]. So far, limited attention has been paid to degenerate Fermi gases inside cavities [10–12]. However, there is no fundamental difficulty in realizing such a system experimentally. To stimulate experimental efforts along this direction, it is therefore desirable to theoretically investigate interesting physics in this setup. In this work we shall start from the simplest case, i.e., spinless fermions, and show that nontrivial effects already exist.

In contrast to bosons, due to the Pauli exclusion principle, a degenerate Fermi gas forms a Fermi sea, occupying a collection of single particle states of the lowest energies. Moreover, the system exhibits a Fermi surface (FS) where "Fermi surface nesting" is the crucial feature responsible for many collective phenomena in fermionic systems, such as charge-density wave and spin-density wave [13], as well as some strongly correlated unconventional superconductivity [14]. FS nesting means that when a FS is shifted by a certain momentum, a sizable portion of the shifted FS will overlap with the original one. If a FS is perfectly nested, particle-hole excitations of the nesting momentum cost infinitesimally small energies and the FS becomes unstable in the presence of infinitesimally small local repulsive (attractive) interactions and reconstructions to be gapped by spin-density (charge-density) wave order.

The purpose of this Letter is to point out that FS nesting and the Pauli exclusion principle both have strong effects on superradiance in a degenerate Fermi gas. Explicitly, we show (i) For the one-dimensional case, perfectly nested FS leads to a dramatic result that an infinitesimal pumping field can induce superradiance when the nesting momentum matches the wave-vector magnitude of the cavity field. For the two-dimensional case, superradiance is greatly enhanced close to certain fillings when the nesting momentum matches the momentum transfer $\mathbf{Q}$ between the pumping laser and the cavity field photons. In the nesting regime, the phase diagram exhibits several interesting behaviors. (ii) In the low-density regime, the occupation of different single particle states due to the Pauli exclusion principle enhances superradiance, while in the high-density regime, superradiance is suppressed by the Pauli exclusion principle.

Model.—We consider spinless fermionic atoms trapped inside a high-Q cavity. Two linearly polarized pumping laser beams counterpropagate along $\hat{y}$ perpendicular to the main axis $\hat{x}$ of the cavity, as schematically shown in Fig. 1. The gas can be either one, two, or three dimensional. For the one-dimensional case, we consider a strong confinement potential in the yz plane that can prohibit the momentum transfer along the $\hat{y}$ direction during scattering between atoms and pumping light; fermions can only move along the direction of the cavity mode $\hat{x}$, as shown in Fig. 1a. For the two-dimensional case, the atoms' motion

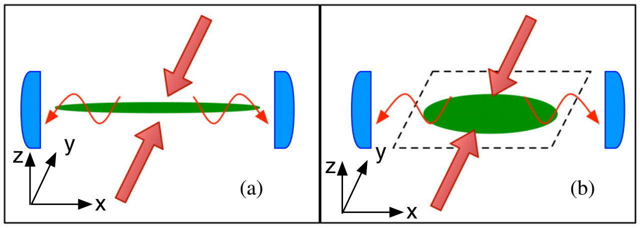

FIG. 1 (color online). Schematic of the experimental setup. Arrows are pumping lasers along $\hat{y}$. Wiggly lines with arrows represent cavity field along $\hat{x}$. Panel (a) presents a one-dimensional gas along $\hat{x}$, and panel (b) presents a two-dimensional gas in the xy plane with strong confinement along $\hat{z}$.

along $\hat{z}$ is frozen by tight confinement and fermions can only move in the xy plane, as shown in Fig. 1b. The cavity is fine-tuned such that only one mode has a frequency ω_c that is close to the frequency of pumping lasers ω_p. Both ω_p and ω_c are far off the resonance with respect to the electronic transitions of the atoms so that we can adiabatically eliminate the electronic excited states of the atoms, and obtain the Hamiltonian ($\hbar = 1$ throughout) [1,15]

$$\hat{H} = \int d^d\mathbf{r}(\hat{\psi}^\dagger(\mathbf{r})\hat{H}_0\hat{\psi}(\mathbf{r})) - \Delta_c \hat{a}^\dagger \hat{a}, \tag{1}$$

$$\hat{H}_0 = \hat{H}_{\rm at} + \eta(r)(\hat{a}^\dagger + \hat{a}) + U(r)\hat{a}^\dagger\hat{a}, \tag{2}$$

$$\hat{H}_{\rm at} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{3}$$

where $\hat{\psi}$ is the field operators for spinless fermion atoms and $\hat{a}$ is the field operators for the cavity mode. $V(\mathbf{r})$ and $U(\mathbf{r})$ are the optical potentials generated by the pumping lasers and the cavity field, respectively, and $V(\mathbf{r}) = V_0\cos^2(k_0 y)$, $U(\mathbf{r}) = U_0\cos^2(k_0 x)$ with $V_0 = \Omega_p^2/\Delta_a$ and $U_0 = g^2/\Delta_a$. The interference between the pumping lasers and the cavity field gives rise to $\eta(\mathbf{r}) = \eta_0 \cos(k_0 x)\cos(k_0 y)$ with $\eta_0 = g\Omega_p/\Delta_a$. Here Δ_a is the laser detuning with respect to the electronic exciting energies of the atoms, $\Delta_c = \omega_p - \omega_c$ is the cavity mode detuning, Ω_p is the strength of the pumping lasers, g is the single-photon Rabi frequency of the cavity mode, and k_0 is the wave-vector magnitude of the pumping lasers and the cavity mode. We define the recoil energy $E_r = \hbar^2 k_0^2/2m$ for later use. In the following discussion, g, Δ_a, Δ_c, and U_0 are kept fixed, and superradiance is driven by increasing the pumping field strength Ω_p, which simultaneously increases η_0 via $\eta_0 = \sqrt{U_0 V_0}$.

Method.—The weak leakage of electromagnetic fields from the high-Q cavity leads to a small decay rate κ for the cavity mode [1]. The mean field of the cavity mode $\alpha = \langle \hat{a} \rangle$ satisfies the equation of motion in a similar fashion as the boson case [1]

$$i\frac{\partial \alpha}{\partial t} = (-\tilde{\Delta}_c - i\kappa)\alpha + \eta_0 \Theta, \tag{4}$$

with the effective detuning $\tilde{\Delta}_c = \Delta_c - \int d^d\mathbf{r} U(\mathbf{r}) n(\mathbf{r})$ and the fermion density order $\Theta = \int d^d\mathbf{r} n(\mathbf{r})\eta(\mathbf{r})/\eta_0$. The fermion density function is $n(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$. Because of the presence of the cavity decay term κ, the system is generally in a nonequilibrium situation [1]. We seek a steady state in which $\partial\alpha/\partial t = 0$ and find

$$\alpha = \frac{\eta_0 \Theta}{\tilde{\Delta}_c + i\kappa}. \tag{5}$$

This steady state requirement fixes the relation between α and Θ.

To determine the critical pumping strength for the superradiance transition, we calculate the free energy by the second-order perturbation theory. With the above mean-field treatment for the cavity field and by integrating out the rest fermion fields, we obtain the free energy to the second order of α as

$$F_\alpha = -\beta^{-1} \ln \mathcal{Z}_\alpha = -\tilde{\Delta}_c \alpha^* \alpha - \chi_\eta (\alpha + \alpha^*)^2, \tag{6}$$

and the susceptibility χ_η is given by

$$\chi_\eta = -\frac{1}{2\beta}\mathrm{Tr}[G_0 \eta(\mathbf{r}') G_0 \eta(\mathbf{r})] \equiv \eta_0^2 N_{\rm at} \frac{f}{E_{\rm r}}, \tag{7}$$

where f is the dimensionless susceptibility, N_{at} is the total atom number, Tr includes the frequency summation, and $G_0^{-1} = i\partial_t - H_{\rm at}$. Substituting Eq. (5) into the free energy expression, Eq. (6), in the vicinity of the superradiance transition, we obtain

$$F_\alpha = -\left(\frac{\tilde{\Delta}_c}{\tilde{\Delta}_c^2 + \kappa^2} + \chi_\eta \frac{4\tilde{\Delta}_c^2}{(\tilde{\Delta}_c^2 + \kappa^2)^2}\right)(\eta_0 \Theta)^2. \tag{8}$$

Across a superradiance transition, Θ spontaneously evolves from zero to a finite value. Therefore, the transition is determined by the sign change of the coefficient of Θ^2 in Eq. (8), which yields the critical value of η_0:

$$\eta_0^{cr}\sqrt{N_{\rm at}} = \frac{1}{2}\sqrt{\frac{\tilde{\Delta}_c^2 + \kappa^2}{-\tilde{\Delta}_c}}\sqrt{\frac{E_{\rm r}}{f}}. \tag{9}$$

It is straightforward to show that in terms of the eigenfunctions $\phi_\mathbf{k}$ and the eigenenergies $\epsilon_\mathbf{k}$ of $\hat{H}_{\rm at}$,

$$f = \frac{E_r}{\eta_0^2 N_{\rm at}} \sum_{\mathbf{k}\mathbf{k}'} \left| \int d^d r \phi_\mathbf{k}^*(\mathbf{r}) \phi_{\mathbf{k}'}(\mathbf{r}) \eta(\mathbf{r}) \right|^2 \frac{n_F(\epsilon_\mathbf{k})}{\epsilon_{\mathbf{k}'} - \epsilon_\mathbf{k}}, \tag{10}$$

with n_F the Fermi distribution function. For the fermion case, f depends on the atom density, the pumping field strength, and the dimensionality of the atomic gas. For the boson case, the critical value is also determined by Eq. (9), and the difference is that in the expression for f, $n_F(\epsilon_{\mathbf{k}})$ in Eq. (10) should be replaced by $N_{at}\delta_{\mathbf{k},0}$ at zero temperature, where $\mathbf{k}=0$ corresponds to the Bose condensed single particle ground state. Thus, for the noninteracting boson case, f is independent of atom density and $f \approx 1/2$ for the weak pumping field [1]. The magnitude of f determines the easiness of inducing superradiance. The larger f is, the smaller the critical pumping strength. We present the numerical results for f at zero temperature based on Eq. (10) in Fig. 2 for Fermi gases at different dimensions and compare them with noninteracting Bose gases.

Low-density and high-density limits.—If the filling factor $\nu \equiv n/(\zeta k_0^d)$, with $\zeta = 2$ (for $d=1$) and $\zeta = 4$ (for $d = 2$, 3) for a d-dimensional Fermi gas of average density n, is much smaller than one, the degenerate fermions mainly occupy the lowest lying single particle states. At zero temperature, in the limit $\nu \to 0$, one finds that f approaches the same value for bosons and fermions [7,8,16]. And this value increases as the lattice depth increases, which means the lattice effect enhances the superradiance tendency, as shown in Figs. 2(b) and (c). In fact, a similar effect has also been found in resonance physics where the lattice effect enhances the tendency of molecule formation [17,18]. When ν increases from zero, f for fermions becomes larger than that for bosons, while the later remains unchanged due to its independence of the atoms' density. The increment of f for fermions comes from the population of finite momentum states, since some of the finite momentum states have a smaller energy denominator in Eq. (10).

We also find that as ν increases to the high-density regime with $2k_F > |\mathbf{Q}|$ ($\mathbf{Q} = (\pm k_0, \pm k_0)$ for the two-dimensional case and $\mathbf{Q} = (\pm k_0, \pm k_0, 0)$ for the three-dimensional case) f for fermions will finally drop below that for bosons. This is because when the Fermi surface is large enough, for a certain number of occupied states with momentum $\mathbf{k}$, the states with momentum $\mathbf{k}+\mathbf{Q}$ will also be occupied and the Pauli exclusion principle blocks the scattering between these states. The superradiance tendency is suppressed accordingly.

Nesting effect.—The nesting effect can be best illustrated in the one-dimensional case, where f can be calculated analytically as (up to a constant [19])

$$f = \frac{1}{8}\frac{k_0}{k_F} \ln \left| \frac{2k_F + k_0}{2k_F - k_0} \right|, \tag{11}$$

at zero temperature. As shown in Fig. 2(a), f diverges when $k_0 = 2k_F$, which means that an infinitesimally small pumping field can lead to superradiance. The divergence is due to the fact that in one dimension, FS contains only two points and is generically nested with the nesting momentum $2k_F$. The interaction mediated by cavity photons $\sim \cos(k_0 x)$ can only transfer a fixed momentum k_0. Thus, only when $2k_F$ matches k_0, can infinitesimal cavity-mediated attraction between fermions induce a density-wave order of fermions. Finite temperature is expected to smear out the divergence of f and result in a finite critical strength for the pumping field.

Generally a FS is not perfectly nested in dimensions higher than one; it is more difficult to find a nested FS when the dimensionality becomes higher. In the two-dimensional case, there are still cases in which a sizable portion of FS is nearly nested. When the nesting momentum roughly matches $\mathbf{Q}$, f will be largely increased although remains finite. For the two-dimensional case, as shown in Fig. 2(c), f as a function of ν displays two peaks in the regime $\nu \approx 0.5$, though the exact locations of these peaks depend on the pumping field strength. In Fig. 3, we plot the FS for $\hat{H}_{\rm at}$ at these peak positions. Indeed, we find that part of the FS is well nested with the relevant momentum $\mathbf{Q}$, which proves that the nested FS is responsible for the significant increasing of f in the two-dimensional case. Similarly, a peak around $\nu \approx 1/2$ is found in the three-dimensional case, as shown in Fig. 2(b).

Determining the phase diagram.—For bosons or fermions of a given density in one dimension, f is

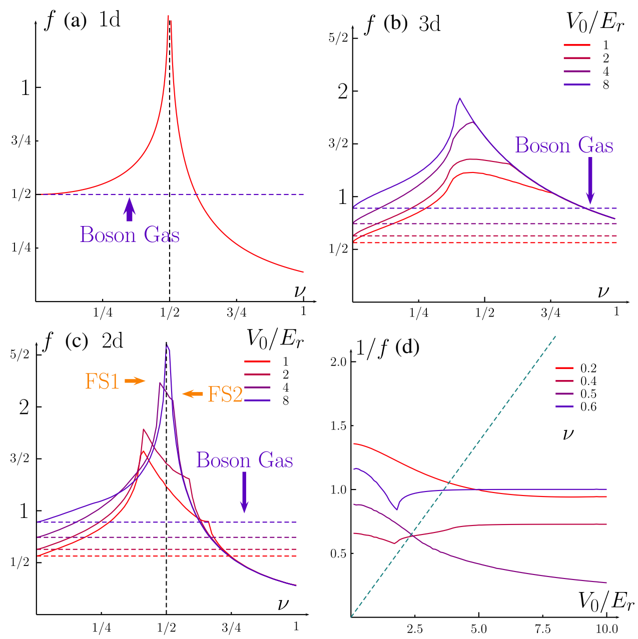

FIG. 2 (color online). (a)–(c) Dimensionless charge-density-wave susceptibility f is plotted as a function of filling ν, for the one-dimensional (a), three-dimensional (b), and two-dimensional cases (c). f is defined in Eq. (7) and is related to the critical pumping strength via Eq. (9). In (b) and (c), different lines represent different pumping field strengths V_0/E_r. FS1 and FS2 in (c) mark the place where corresponding FSs are shown in Figs. 3(a) and (b), respectively. For comparison, horizontal dashed lines represent f for the noninteracting boson case with different pumping field strengths. (d) $1/f$ as a function of V_0/E_r for various fillings ν. The dashed line represents $(1/\mathcal{C})V_0/E_r$ defined in Eq. (12).

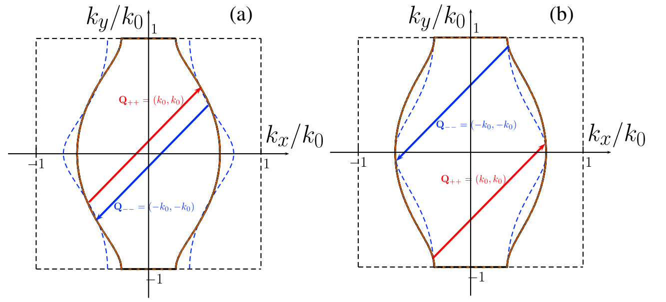

FIG. 3 (color online). Nesting of FS for Hamiltonian $\hat{H}_{at}$ before superradiance takes place, with two different fillings as marked in Fig. 2. The solid line is the original FS, and dashed lines are FSs shifted by $\mathbf{Q} = (\pm k_0, \pm k_0)$. Arrows indicate the momentum transfer $\mathbf{Q}$.

independent of V_0/E_r. The phase boundary V_0^{cr}/E_r as a function of $\tilde{\Delta}_c/E_r$ can be derived directly from Eq. 9 if κ/E_r and $\sqrt{U_0 N_{at}/E_r}$ are given. For fermions in higher dimensions, f is also a function of V_0/E_r. To determine the phase boundary one needs to solve the equation

$$\frac{V_0^{cr}}{E_r} = \frac{C(\tilde{\Delta}_c/E_r)}{f(V_0^{cr}/E_r)}, \tag{12}$$

where

$$C(x) = \frac{1}{4}\left(\frac{x^2 + (\kappa/E_r)^2}{-x}\right)\left(\frac{1}{U_0 N_{at}/E_r}\right). \tag{13}$$

In Fig. 2(d), we plot $1/f$ as a function of V_0/E_r and a straight line representing $V_0/E_r C(\tilde{\Delta}_c/E_r)$, whose crossing marks the superradiance transition point.

In Fig. 4 we plot the phase diagram for different densities; the curves are the boundary separating the normal

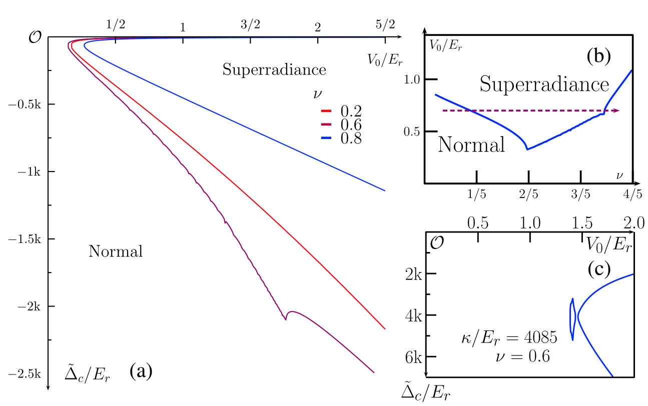

FIG. 4 (color online). (a) and (c) The phase diagram for the two-dimensional case, in terms of the effective detuning $\tilde{\Delta}_c/E_r$ and pumping lattice depth V_0/E_r. Different lines in (a) represent the phase boundary with different fillings. (b) Critical V_0/E_r as a function of filling ν for $\tilde{\Delta}/E_r$ fixed at 2×10^3. $\kappa/E_r = 250$ for (a) and (b), $\kappa/E_r = 4085$ for (c), and $U_0 N_{at}/E_r = 1 \times 10^3$ for (a)–(c).

and the superradiance phases. For a fixed effective detuning $\tilde{\Delta}_c/E_r$, the critical pumping strength V_0/E_r is shown to reach its minimum in the nesting regime $\nu \approx 1/2$. In other words, there is a density-driven superradiance transition and a reentrance behavior as shown in Fig. 4(b): The system starting in the normal phase undergoes a transition to the superradiance phase and comes back to the normal phase as the density further increases. In addition, due to the nonmonotonic behavior of $1/f$ for filling $\nu \approx 1/2$, for certain fine-tuned κ/E_r, the phase diagram can exhibit topologically distinct behavior as shown in Fig. 4(c), where an additional isolated island of the superradiance regime exists in the phase diagram.

Final remark.—In this work we have revealed that many-body effects have a much stronger impact on the superradiance of degenerate Fermi gases in a cavity, even for spinless fermions and a single mode cavity. Though the quantitative results we have shown are for zero temperature, the enhancement of superradiance for fermions compared to bosons is expected to maintain at finite temperatures. Our results lay the base for further efforts to understand more intriguing phenomena in this system, for instance, by including the fluctuations of cavity modes, and considering multiple cavity modes or interactions between fermions of different spin degrees of freedom.

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Note added.—Recently, two other works addressing a similar problem appeared [20,21].

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