How Energy Conservation Limits Our Measurements

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Observations in quantum mechanics are subject to complex restrictions arising from the principle of energy conservation. Determining such restrictions, however, has been so far an elusive task, and only partial results are known. In this Letter, we discuss how constraints on the energy spectrum of a measurement device translate into limitations on the measurements which we can effect on a target system with a nonconstant energy operator. We provide efficient algorithms to characterize such limitations and, in case the target is a two-level quantum system, we quantify them exactly. Our Letter, thus, identifies the boundaries between what is possible or impossible to measure, i.e., between what we can see or not, when energy conservation is at stake.

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Our success in investigating natural phenomena relies critically on our ability to perform accurate measurements. It is no wonder therefore that improving the performance of our measuring devices is one of the main goals in our technical progress. However, it turns out that in some instances our ability to perform certain measurements is not only limited by our technological capability, but by basic physical principles. One famous such limitation is that imposed by conservation laws. Consider, for example, a closed laboratory, freely floating in space. The total energy of the lab (including the lab walls, the particles inside, and all the measuring devices) is a conserved quantity. Suppose now that we would like to measure some physical quantity that does not commute with the total energy. According to quantum mechanics, this is impossible since such a measurement would disturb the energy operator (the Hamiltonian) and hence, may change the total energy, violating the conservation law. This is roughly the content of the famous Wigner-Araki-Yanase theorem [1–3].

Obviously however, if our lab is allowed to exchange energy with some other external body, say a second lab floating next to it, more possibilities open. But what exactly are these new possibilities? Or, in other words, what are the constraints imposed by conservation laws, when some form of exchange of conserved quantities is allowed? Even though some specific limits are known [4–9], an exact characterization of the constraints induced by conservation laws is still missing.

For illustration, consider two separate parties trying to violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [10] by conducting measurements on the bipartite Fock state $|\varphi\rangle_{AB} \equiv (1/\sqrt{2})(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$. If Alice and Bob are limited to using passive optical operations and photodetectors, they will never go beyond the classical limit. This follows from the fact that, if their local measurements commute with the local photon number operator, Alice and Bob cannot distinguish their state from the locally dephased (in the energy basis) separable state $\frac{1}{2}(|0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B)$.

Suppose now that they also have access to a source of Fock states of the form $|+\rangle \equiv (1/\sqrt{2})(|0\rangle + |1\rangle)$. This time, Alice and Bob will see a CHSH violation when they use the optical setup [11] depicted in Fig. 1. However, they will not get close to the quantum limit $2\sqrt{2}$ [13], no matter what energy-conserving interactions they apply (see the Supplemental Material [14]).

An interpretation of this result is that the presence of coherent superpositions of energy eigenstates $|+\rangle$ increases up to a point, the set of local measurements which Alice and Bob c an implement over the state $|\varphi\rangle$. What measurement processes are available when Alice's measurement device uses states other than $|+\rangle$, like $(1/\sqrt{3})(|0\rangle + |1\rangle + |2\rangle)$, or the coherent state $|\alpha\rangle$? How is Alice's capacity to measure limited by the dimensionality

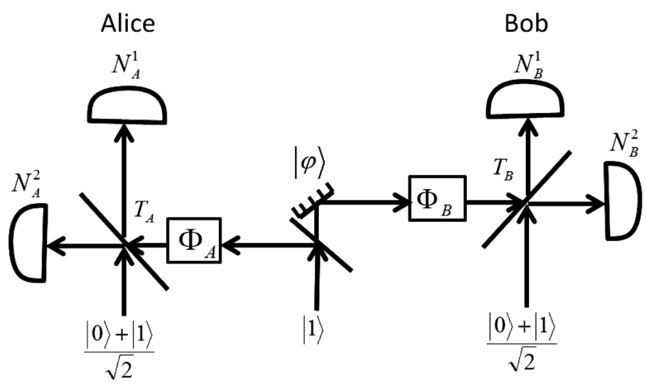

FIG. 1. Bell experiment under energy-conserving interactions. Alice and Bob can violate the CHSH Bell inequality by means of beam splitters of varying transmittivity T_A, T_B, and shifters of phase Φ_A, Φ_B, if, besides the entangled state $|\varphi\rangle$, they are distributed each a copy of the reference state $|+\rangle$. The quantity N_A^k (N_B^k) denotes the number of photons registered by Alice's (Bob's) kth detector.

of the quantum states that she can handle, or the energy of such states? These are the kinds of questions that we address in this Letter.

This Letter explores how measurements of a target system affected by a given quantum device depend on the relation between the energy spectra of one system and the other. In case the target is a two-level system, we determine the difference between the set of all conceivable measurements and the set of all measurements implementable by devices with finite energy spectrum, or with unbounded energy spectrum, but bounded average energy $\bar{E}$. Additionally, we provide an efficient algorithmic characterization of the set of accessible measurements for target systems of any dimension. Our Letter, hence, reveals to what extent energy conservation affects our capacity to see the world.

Traditional descriptions of the measurement process in quantum mechanics typically overlook the energy exchange between the system under observation and the measurement device carrying it. Our starting point will, thus, be a measurement model where the quantum nature of such energy transfer is properly accounted for.

A general measurement over a quantum system in state ρ is described by a positive operator valued measure (POVM), i.e., a set M of positive semidefinite operators $M \equiv \{M_x\}_{x=0}^n$, with $\sum_{x=0}^n M_x = \mathbb{I}_{\mathcal{H}}$, such that

$$p(x) = \mathrm{tr}(\rho M_x) \tag{1}$$

denotes the probability of obtaining outcome x. In general, such a measurement must be physically realized by attaching our *target* system S to a measuring device whose *pointer* P is in the initial quantum state $|0\rangle$ and, for simplicity, has zero Hamiltonian. The third main element is a *battery* with energy operator H_B. The battery is the auxiliary system with which our original system S may exchange energy. It is the efficiency of this battery that interests us here. Finally, we also need a clock that will switch on and off the interaction between the system S, the battery B, and the pointer P. Think of Alice's experimental setup in Fig. 1: there, the target is the laser beam carrying the state $\mathrm{tr}_A(|\varphi\rangle\langle\varphi|)$, and the ancillary state $|+\rangle$ constitutes the battery. The displacement of these two beams through optical fibers plays the role of the clock, which switches on and off an interaction with the beam splitter and the two photodetectors. The outcomes of the photodetectors can then be printed on a piece of paper (the pointer).

In general, the clock itself may also contribute to the energy exchange. This is a complication that is specific to energy conservation constraints only. Indeed, for any other conservation laws, such as momentum or angular momentum conservation, one can also consider an auxiliary system (i.e., a "battery") with which the conserved quantity can be exchanged, and the existence of such a battery will enlarge the class of possible measurements. In those cases, however, the clock that turns on and off the interaction will not interfere with the conserved quantity. In the present Letter, we will also consider that the clock switches the interaction on and off adiabatically [15], so it does not exchange energy with the SPB composite system; all the energy exchanges can, thus, be traced to the battery and accounted for. Note that this is not a loss of generality at all: we can, in fact, accommodate many other fast clocks in this picture, by considering them as part of the battery. They can then switch on and off the SPB interaction anyway we want, under the umbrella of the slow master clock whose role is only to start and end the whole process.

Suppose now that we try to measure a target system S with Hamiltonian H_S, and assume that our measurement device has a battery with energy operator $H_B = \sum_n \mu_n \Pi_n$, where $\{\Pi_i\}$ are orthogonal projector operators. If the Hamiltonians of the target and the battery do not have coincident energy gaps (i.e., if they are nonresonant), the presence of system B will not provide any advantage towards measuring or interacting with system S in a quantum way, as it can be easily seen. Resonant Hamiltonians, however, allow us to increase the set of possible measurements in a nontrivial fashion. Actually, any measurement can be approximated up to arbitrary precision by taking resonant Hamiltonians H_B with enough dimensionality d and appropriate battery states; see the Supplemental Material [14].

This observation is certainly counterintuitive: one would expect that ancillary systems with energy operator $\tilde{H}_B \approx H_B$ *nearly* resonant with H_S should induce similar effective measurements over system S (and thus, approximate the set of all possible measurements for $d \gg 1$). In the Supplemental Material [14], we provide a possible solution for this apparent paradox, by invoking an interaction with hidden continuous degrees of freedom.

The aim of the rest of this Letter is to determine exactly how the energy spectrum of our measurement device constrains the set of effective POVMs that such a device is able to implement on its target system. But, before this, we have to specify means to quantify such constraints.

Suppose that we intend to measure our target S with a battery B, that we can initially set to any *pure* quantum state (i.e., any coherent superposition of eigenstates of the energy operator) with energy distribution $f(E)dE$, among a set of distributions $\mathcal{B}$. Let $\mathcal{M}(\mathcal{B}, S)$ thus denote the set of POVMs which we can implement over a target system S with Hamiltonian H_S by preparing a suitable battery state in $\mathcal{B}$ and then applying energy conserving interactions to target, battery, and pointer. We want to quantify the difference $\epsilon(\mathcal{M}(\mathcal{B}, S), \mathcal{M}(d))$ between $\mathcal{M}(\mathcal{B}, S)$ and $\mathcal{M}(d)$, the set of all POVMs in $\mathbb{C}^d$. Notice that, if we had a notion of distance $\mathrm{dist}(M^0, M^1)$ between two arbitrary POVMs, M^0 and M^1, then we could simply define $\epsilon(\mathcal{M}(\mathcal{B}, S), \mathcal{M}(d))$ as the maximal distance between an element in $\mathcal{M}(d)$ and the smaller set $\mathcal{M}(\mathcal{B}, S)$:

$$\epsilon(\mathcal{M}(\mathcal{B},S),\mathcal{M}(d)) = \max_{M\in\mathcal{M}(d)} \mathrm{dist}(\mathcal{M}(\mathcal{B},S),M), \quad (2)$$

with $\mathrm{dist}(\mathcal{M},M) = \min_{M'\in\mathcal{M}}\mathrm{dist}(M',M)$.

Our first step is, therefore, to introduce a natural distance between POVMs. Suppose that we have a device capable of implementing either measurement $M^0 \equiv \{M_x^0\}_x$ or $M^1 \equiv \{M_x^1\}_x$ with probability 1/2, but we ignore what measurement it actually performs. We let the measuring device act on a quantum state ρ and from the outcome x obtained, we try to guess which of the two POVMs our machine is in fact implementing. Optimizing over all possible input states ρ, we obtain the maximum probability $\mathcal{P}_C$ of guessing the correct POVM. Note that $\mathcal{P}_C \geq 1/2$, since we can always make a completely random guess. $\mathcal{P}_C$ induces a *classical distance* $\mathrm{dist}_C(M^0,M^1)$ in the space of POVMs via the relation $\mathcal{P}_C = \frac{1}{2}\{1+\mathrm{dist}_C(M^0,M^1)\}$, with

$$\mathrm{dist}_C(M^0,M^1) = \frac{1}{2}\max_{\rho}\sum_x |\mathrm{tr}\{\rho(M_x^0 - M_x^1)\}|. \quad (3)$$

Analogously, we can define a *quantum distance* between POVMs, by means of a protocol where the measuring device acts on part A of an entangled state ρ_{AB}. Depending on the outcome x of the measurement, we then implement a (known) POVM N_a^x, with outcomes $a \in \{0,1\}$ on system B and the result of this second measurement will be our guess as to which of the two POVMs, M^0 or M^1, the unknown measuring device is actually implementing. As before, the probability $\mathcal{P}_Q$ of correctly guessing the POVM can be expressed as $\mathcal{P}_Q = \frac{1}{2}\{1+\mathrm{dist}_Q(M^0,M^1)\}$, with

$$\mathrm{dist}_Q(M^0,M^1) = \frac{1}{2}\max_{\rho_{DQ}}\sum_x \|\mathrm{tr}_D[\rho_{DQ}(M_x^0 - M_x^1)] \otimes \mathbb{I}_Q\|_1. \quad (4)$$

Note that the classical (quantum) distance $(\mathrm{dist}_C(M^0,M^1))$ $\mathrm{dist}_Q(M^0,M^1)$ corresponds to the norm defined in Ref. [16] (the diamond norm [17,18]) between the quantum channels $\Omega^a(\rho) = \sum_x \mathrm{tr}(\rho M_x^a)|x\rangle\langle x|$, with $a = 1, 2$. Like the classical distance, dist_Q satisfies the triangle inequality and has maximum value 1. Also, both distances can be proven different even in the qubit case, with $\mathrm{dist}_Q(M^0,M^1) \geq \mathrm{dist}_C(M^0,M^1)$, for all M^0, M^1; see the Supplemental Material [14] for details.

These two distances lead to two different ways to quantify the difference between an arbitrary set $\mathcal{M}$ of POVMs in $\mathbb{C}^d$ and the set of all possible POVMs $\mathcal{M}(d)$, namely, a classical and a quantum one, ϵ_C and ϵ_Q, as defined by Eq. (2). Intuitively, ϵ_C, ϵ_Q measure the feasibility of distinguishing a device capable of implementing any measurement in $\mathcal{M}(d)$ from another one restricted to POVMs in $\mathcal{M}$ when separable (C) or entangled (Q) inputs are allowed.

We will next apply the former notions to study the effect of energy conservation over the set of feasible measurements when the target system is a qubit.

Suppose then that S is a two-level system with $H_S = \Delta|1\rangle\langle 1|$, which we probe with a battery B that we can prepare in any state with energy distribution $f(E)dE \in \mathcal{B}$. In these conditions, we prove that

$$\epsilon_C(\mathcal{M}(\mathcal{B},S)) = \epsilon_Q(\mathcal{M}(\mathcal{B},S)) = \frac{1}{2}\{1-\tau(\mathcal{B},S)\}, \quad (5)$$

where

$$\tau(\mathcal{B},S) = \max_{f(E)dE\in\mathcal{B}} \int_0^\infty dE f^{1/2}(E) f^{1/2}(E+\Delta). \quad (6)$$

Note that, in the absence of a battery, we can take $H_B \propto \mathbb{I}$ and hence, Eqs. (5), (6) imply that $\epsilon_{C,Q}(\mathcal{M}(\emptyset,S)) = \frac{1}{2}$.

It turns out that equatorial von Neumann measurements maximize Eq. (2); i.e., they are the most difficult to simulate. That explains why we obtain identical results for ϵ_C, ϵ_Q, since both quantities coincide when used to compare two different dichotomic measurements (see the Supplemental Material [14]). From Eq. (6), it is also clear that any pure state whose energy distribution $f^\star(E)dE \in \mathcal{B}$ maximizes Eq. (6) allows us to simulate the whole set $\mathcal{M}(2)$ with accuracy $1-\tau(\mathcal{B},S)$, or, equivalently, $\epsilon_{C,Q}(\mathcal{B},S) = \epsilon_{C,Q}(\{f^\star(E)dE\},S)$. Notice that this relation was derived for qubit target systems; we should not expect it to hold in higher dimensions.

In the next lines, we will compute the value of $\tau(\mathcal{B},S)$ for two cases of practical interest: finite dimensionality and finite energy.

Picture an experimental scenario where we are allowed to prepare our measurement device's battery in any superposition of d discrete energy levels $|\Psi\rangle_B = \sum_{k=0}^{d-1} c_k|E_k\rangle_B$. From Eq. (6), it is clear that the optimal energy spectrum H_B for our battery must be resonant with H_S, i.e., $E_k = E_0 + k\Delta$ [19]. The problem of computing $\tau(\mathbb{C}^d,S)$ hence reduces to maximizing $\sum_{k=0}^{d-2}\sqrt{p_k p_{k+1}}$ over all discrete distributions $\{p_k\}_{k=0}^{d-1}$.

From the seminal papers of Aharonov and collaborators [20], one is tempted to assume that the optimal state should have an equal superposition of all energy levels, as this is expected to better fix a time frame of reference. Surprisingly, in the Supplemental Material [14], we show that such is not the case: the solution of this problem is $\tau(\mathbb{C}^d,S) = \cos(\pi/(d+1))$, and thus,

$$\epsilon_{C,Q}(\mathcal{M}(\mathbb{C}^d,S),M(2)) = \frac{1}{2}\left\{1-\cos\left(\frac{\pi}{d+1}\right)\right\}. \quad (7)$$

$\mathcal{M}(\mathbb{C}^d,S)$, therefore, tends to $\mathcal{M}(2)$ as $O(1/d^2)$.

Consider now a scenario where, in principle, we can prepare any initial battery state, but we do not wish to spend too much energy in the process. Note that, if we set the origin of energies to zero ($E_0 = 0$) in the previous example, the average energy of the d-level states maximizing Eq. (6) grows linearly with d. This makes one wonder whether that much energy is actually needed in order to reach that degree of measurement accuracy. In other words: how well can we approximate $\mathcal{M}(2)$ when our battery is infinite dimensional, but its average energy is bounded?

Call $\mathcal{M}(\bar{E}, 2)$ the set of two-level POVMs attainable via a measurement device with a battery of average energy smaller than or equal to $\bar{E} > 0$. To compute $\epsilon_{C,Q}(\mathcal{M}(\bar{E}, 2), \mathcal{M}(2))$, we must maximize Eq. (6) under the constraint $\int_0^\infty dEf(E)E \leq \bar{E}$. It is easy to see that the optimal energy density $f(E)dE$ must be discrete and resonant with the system, i.e., $f(E) = \sum_{k=0}^\infty p_k \delta(E - k\Delta)$. The corresponding probabilities p_k must then satisfy

$$\sum_{k=0}^{\infty} p_k k\Delta \leq \bar{E}. \quad (8)$$

Maximizing Eq. (6) with respect to this constraint leads to $\tau(\bar{E}) = \varphi(\bar{E}/\Delta)$, with

$$\varphi(z) = \min_{\lambda \geq 0} \frac{z + \{\mu : j_{\mu-1,1} = 2\lambda\}}{2\lambda}, \quad (9)$$

where $j_{n,1}$ denotes the first zero of $J_n(y)$, the Bessel function of the first kind. This follows from the theory of continuous fractions [21] (see the Supplemental Material [14] for a proof). For $z \gg 1$, $\varphi(z)$ behaves as $\varphi(z) \approx 1 - (0.9468/z^2)$ [22]. Consequently,

$$\epsilon_{C,Q}(\mathcal{M}(\bar{E}, 2), \mathcal{M}(2)) = \frac{1}{2}\{1 - \varphi(\bar{E}/\Delta)\} \approx \frac{0.4734\Delta^2}{\bar{E}^2}, \quad (10)$$

for $\bar{E} \gg \Delta$. Let $\lambda^\star > 0$ denote the minimizer in Eq. (9). Then, the quantum state whose energy density maximizes Eq. (6) can be expressed in the number basis as $|\Psi^\star_{\bar{E}}\rangle = \sum_{k=0}^\infty c_k |k\rangle$, where $H_B|k\rangle = \Delta k|k\rangle$ and the coefficients $\{c_k\}$ are given by the recurrence formula:

$$c_{k+1} = \frac{k + \{\mu : j_{\mu-1,1} = 2\lambda^\star\}}{\lambda^\star} c_k - c_{k-1}. \quad (11)$$

We will refer to $\{|\Psi^\star_{\bar{E}}\rangle : \bar{E} \in \mathbb{R}^+\}$ as *power states*.

Figure 2 shows a comparison between the performance of power states against coherent states $|\alpha\rangle = e^{-(|\alpha|^2/2)} \times \sum_{n=0}^\infty (\alpha^k/\sqrt{k!})|k\rangle$, with energy $\langle\alpha|H_B|\alpha\rangle = |\alpha|^2\Delta = \bar{E}$. The plot shows that, as soon as $\bar{E}/\Delta \gtrsim 10$, the newly defined states become considerably advantageous. This was expected, given that $\tau(|\alpha\rangle) \approx 1 - (\Delta/8\bar{E})$, for $\alpha \gg 1$ [23]. It is, therefore, an interesting question whether current technology allows producing power states in the lab.

The results discussed above are "global"—they refer to the distance between the set of POVMs that one can perform with a constrained battery and the set of all conceivable POVMs. They leave open, though, the problem of characterizing which two-level POVMs $M \in \mathcal{M}(2)$ can be realized with a battery of finite energy spectrum or bounded average energy. More generally, given a POVM $M \in \mathcal{M}(d)$, we would like to decide if such a POVM can be implemented in a d-dimensional target with a nonconstant energy operator H_S with a battery of Hamiltonian H_B.

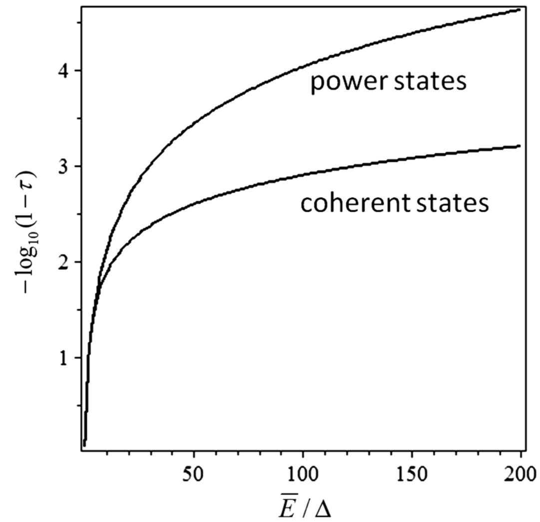

FIG. 2. Power states against coherent states. The plot shows the accuracy in reproducing $\mathcal{M}(2)$ for different values of the average energy $\bar{E}$ when our resource states are either the power states $|\Psi^\star_{\bar{E}}\rangle$ or coherent states $|\alpha\rangle$.

To answer the above question, in the Supplemental Material [14], we show how to formulate the characterization of $\mathcal{M}(\mathbb{C}^d, 2)$ as a semidefinite program [24] involving $O(d)$ 2×2 complex matrices. Due to this small scaling, using standard convex optimization packages like YALMIP and SeDuMi [25,26], we found that a normal desktop can decide if an arbitrary 3-outcome POVM M belongs to $\mathcal{M}(\mathbb{C}^d, 2)$, for $d = 4000$. As shown in the Supplemental Material [14], the algorithm can be easily adapted to characterize the set $\mathcal{M}(\mathbb{C}^d, d')$ for arbitrary (given) Hamiltonians H_B, H_S.

Characterizing $\mathcal{M}(\bar{E}, 2)$ turned out to be more difficult. The approach that we followed was to define two sequences of inner ($\{M_d^I(\bar{E}, 2)\}$) and outer ($\{M_d^O(\bar{E}, 2)\}$) approximations of $\mathcal{M}(\bar{E}, 2)$, i.e., $\mathcal{M}_d^I(\bar{E},2) \subset \mathcal{M}(\bar{E},2) \subset \mathcal{M}_d^O(\bar{E},2) \subset \mathcal{M}(2)$, with $\lim_{d\to\infty}\mathcal{M}_d^I(\bar{E}, 2) = \lim_{d\to\infty}\mathcal{M}_d^O(\bar{E}, 2) = \mathcal{M}(\bar{E}, 2)$. Each of these approximations can be computed via a semidefinite program involving $O(d)$ 2×2 positive semidefinite matrices. Moreover, the speed of convergence of the scheme is bounded by $O(\bar{E}/d\Delta)$. This algorithm can be adapted as well to describe effective measurements in multilevel quantum targets.

In this Letter, we have investigated how the energy spectrum of a quantum measurement device limits what can be observed in a quantum system with nontrivial energy operator. We have provided efficient algorithms to characterize the set of effective measurements for arbitrary target and battery systems. Moreover, we have quantified exactly the maximum efficiency of measurement devices acting over a two-level target as a function of their dimensionality and energy content. In this respect, we found that coherent states constitute a bad resource for high precision quantum measurements with bounded energy, as compared to the optimal power states. Designing a laser that produces power rather than coherent states is hence an important problem for those interested in the control of two-level systems.

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