Certifying Separability in Symmetric Mixed States of N Qubits, and Superradiance

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Separability criteria are typically of the necessary, but not sufficient, variety, in that satisfying some separability criterion, such as positivity of eigenvalues under partial transpose, does not strictly imply separability. Certifying separability amounts to proving the existence of a decomposition of a target mixed state into some convex combination of separable states; determining the existence of such a decomposition is "hard." We show that it is effective to ask, instead, if the target mixed state "fits" some preconstructed separable form, in that one can generate a sufficient separability criterion relevant to all target states in some family by ensuring enough degrees of freedom in the preconstructed separable form. We demonstrate this technique by inducing a sufficient criterion for "diagonally symmetric" states of N qubits. A sufficient separability criterion opens the door to study precisely how entanglement is (not) formed; we use ours to prove that, counterintuitively, entanglement is not generated in idealized Dicke model superradiance despite its exemplification of many-body effects. We introduce a quantification of the extent to which a given preconstructed parametrization comprises the set of all separable states; for "diagonally symmetric" states our preconstruction is shown to be fully complete. This implies that our criterion is necessary in addition to sufficient, among other ramifications which we explore.

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Despite extensive interest in many-body entanglement [1–4], the long-standing question of how, exactly, entanglement is generated at all remains open. To establish the minimal requisite common features of entanglement generation, we must seek counter-intuitive instances to challenge our preconceptions. To that end, this Letter was motivated by initial indications which—inconclusively—suggested that entanglement may not be a feature of Dicke model superradiance. Superradiance is a coherent radiative phenomenon resulting from collective and cooperative atomic effects [5–8], and thus, it possesses the typical hallmark of an entangling process; see, for example [9]. Various necessary criteria for separability [10–12], nevertheless, failed to find signatures of entanglement. The extraordinary claim "superradiance occurs without entanglement," demands the highest standard of evidence; to prove that superradiance need not be entangling, we must certify its separability by employing some sufficient separability criterion.

For pure states, various methods can be employed to quantify entanglement [2–4]. Mixed states, however, lack a general solution [13,14]. Inspired in part by the generalization of Glauber-Sudarshan P invoked in Eq. (28) of Ref. [13], we derived a separable decomposition applicable to superradiating systems. Whereas Ref. [13] is an existence proof, our decomposition explicitly solves a separability ansatz. Indeed, the bulk of our Letter effort was dedicated to identifying this sufficient separability criterion. Rewardingly, we subsequently realized that the technique we developed is applicable to far more than just superradiating systems; our approach for certifying separability is remarkably efficient throughout a broad class of states.

Our procedure amounts to explicitly parametrizing both the general family of states of interest, as well as some set of preconstructed separable states. Testing if the general-family parameters can be mapped to the separable-set parameters ("Does it fit?") is, therefore, a sufficient determination of separability. We demonstrate this method in detail on the "general diagonal symmetric" states, within which Dicke model superradiance evolves, and we successfully certify the perpetual separability of that model. This scenario is further exemplary in that our parametrization of separable states surprisingly appears to encompass all separable diagonally symmetric states; thus, the separability criterion developed in this Letter is apparently not only sufficient, but also necessary.

We define the general diagonal symmetric (GDS) mixed states as those which are diagonal in the symmetric eigenbasis of N-partite 2-level Dicke states. Each Dicke-basis pure state is a superposition of equal-energy states; it is the normalized sum over all permutations of a (separable) computational-basis state. Using bold font to indicate sets, such as $\mathbf{n} = \{n_0, n_1\}$, we have

$$|D_{\mathbf{n}}\rangle = w_{\mathbf{n}} \sum_{\substack{\text{perms.} \\ \{|0\rangle, |1\rangle\}}} |\underbrace{0 \ldots 0}_{n_0}, \underbrace{1 \ldots 1}_{n_1}\rangle, \tag{1}$$

where $n_0 + n_1 = N$ and $w_{\mathbf{n}} = \sqrt{n_0! n_1! / N!}$.

So, for example,

$$|D_{3,1}\rangle = \frac{|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle}{\sqrt{4}}. \quad (2)$$

The state $|D_{\mathbf{n}}\rangle$ is entangled for all $0 < n_0 < N$; Dicke states are natural generalizations of the W state [15], and can also be described as the simultaneous eigenstates of total spin and spin-z operators with $J = N/2$ and $M = (n_1 - n_0)/2$.

The most general mixed state which is diagonal in this basis can be parametrized as

$$\rho_{\mathrm{GDS}} = \sum_{\mathbf{n}} \chi_{\mathbf{n}} |D_{\mathbf{n}}\rangle\langle D_{\mathbf{n}}|, \quad (3)$$

where the $\chi_{\mathbf{n}}$ represent the eigenvalues in the eigendecomposition of ρ_{GDS}, which, in the convention of quantum optics, we refer to as the populations of ρ_{GDS}.

Next, we preconstruct a set of separable states to serve as targets for our decomposition. We start with a completely generic single qubit pure state $|\psi\rangle = \sqrt{y}|0\rangle + \sqrt{1-y}e^{i\phi}|1\rangle$, defined as $\rho^1[y,\phi] \equiv |\psi\rangle\langle\psi|$ in operator form, where we take an N-fold tensor product of the single qubit state with itself, and mix uniformly over all phases, but discretely over arbitrary amplitudes y_j with weights x_j,

$$\rho_{\mathrm{SDS}} \equiv \int_0^{2\pi} (2\pi)^{-1} \sum_{j=1}^{j_{\max}} x_j (\rho^1[y_j,\phi])^{\otimes N} \mathrm{d}\phi. \quad (4)$$

We call such parametrized states separable diagonally symmetric (SDS) states, and the value of $j_{\max}$ depends on N. Note that, by definition, all the variables x_j, y_j appearing in Eq. (4) must be real numbers between 0 and 1. Note, also, that our mixing protocol differs markedly from the spherical harmonics basis suggested in Ref. [13], and furthermore, the SDS states cannot be resolved by the partial-separability method of Ref. [16], as that protocol is incompatible with continuous mixtures.

As proven in the Supplemental Material [36], Eq. (4) can be expressed equivalently as

$$\rho_{\mathrm{SDS}} = N! \sum_{\mathbf{n}} \sum_{j=1}^{j_{\max}} \frac{x_j {y_j}^{n_0} (1-y_j)^{n_1}}{n_0! n_1!} |D_{\mathbf{n}}\rangle\langle D_{\mathbf{n}}|, \quad (5)$$

which more clearly parallels the form of Eq. (3). Orthogonality of the Dicke states allows us to match up terms inside the sums of Eq. (3) and Eq. (5), implying $N+1$ polynomial equations [17] which define a decomposition the populations $\boldsymbol{\chi}$ of ρ_{GDS} into the parameters $\mathbf{x}$, $\mathbf{y}$ of a ρ_{SDS}. Explicitly, if we can successfully identify a mapping

$$\forall_{\mathbf{n}} \chi_{\mathbf{n}} = N! \sum_{j=1}^{j_{\max}} \frac{x_j {y_j}^{n_0} (1-y_j)^{n_1}}{n_0! n_1!}, \quad (6)$$

then, we will have demonstrated that our particular ρ_{GDS} exists in the subspace defined by all possible ρ_{SDS}, $\rho_{\mathrm{GDS}} \in \boldsymbol{\varrho}_{\mathrm{SDS}}$, and, thus, that ρ_{GDS} is necessarily separable.

$j_{\max}$ is chosen in order for the system of equations (6) to be well behaved, i.e., that there should be exactly $N+1$ variables $\mathbf{x}$, $\mathbf{y}$ appearing in the $N+1$ equations. Considering that x_j and y_j always come in pairs, then plainly when $N+1$ is even we should set $j_{\max} = (N+1)/2$. When $N+1$ is odd the situation requires a manual adjustment; we take $j_{\max} = \lceil (N+1)/2 \rceil$ and fix the extraneous variable by forcing $y_{(N+2)/2} = 0$ [18]. To demonstrate, here is the system of polynomial equations for $N = 4$ qubits,

$$\begin{aligned}
\chi_{4,0} &= x_1(y_1)^4 + x_2(y_2)^4, \\
\chi_{3,1} &= 4(x_1(y_1)^3(1-y_1) + x_2(y_2)^3(1-y_2)), \\
\chi_{2,2} &= 6(x_1(y_1)^2(1-y_1)^2 + x_2(y_2)^2(1-y_2)^2), \\
\chi_{1,3} &= 4(x_1(y_1)(1-y_1)^3 + x_2(y_2)(1-y_2)^3), \\
\chi_{0,4} &= x_1(1-y_1)^4 + x_2(1-y_2)^4 + x_3.
\end{aligned} \quad (7)$$

Importantly, although the system of equations mapping $\boldsymbol{\chi} \Leftrightarrow \mathbf{x}, \mathbf{y}$ can always be solved, the decomposition is valid only if it passes a "sanity check" [19]. Explicitly, this decomposition certifies that ρ_{GDS} is separable if and only if convexity conditions on the coefficients parametrizing ρ_{SDS} are satisfied [20],

$$\begin{aligned}
&\rho_{\mathrm{GDS}} \in \boldsymbol{\varrho}_{\mathrm{SDS}} \quad \text{iff} \;\; \exists\, \mathbf{x}, \mathbf{y} \text{ satisfying Eq.(6)} \\
&\quad \text{such that} \quad \forall_j\!: \; 0 \le x_j, y_j \le 1 \; .
\end{aligned} \quad (8)$$

To be clear, conditions (8) are cumulatively a sufficient criterion for certifying separability, since

$$\begin{aligned}
&\boldsymbol{\varrho}_{\mathrm{SDS}} \subseteq \boldsymbol{\varrho}_{\mathrm{SEP}\cap\mathrm{GDS}} \subset \boldsymbol{\varrho}_{\mathrm{GDS}} \\
&\text{where } \boldsymbol{\varrho}_{\mathrm{SEP}\cap\mathrm{GDS}} \equiv \boldsymbol{\varrho}_{\mathrm{SEP}} \cap \boldsymbol{\varrho}_{\mathrm{GDS}},
\end{aligned} \quad (9)$$

and where $\subseteq$ and $\subset$ are analogous to $\le$ and $<$, respectively; $\subset$ indicates a proper subset, categorically rejecting the possibility of equivalence. So, even though we have not yet ruled out the existence of a separable ρ_{GDS} incompatible with the SDS format, the criterion developed is already a sufficient one.

The ability to certify full separability is highly desired, as: (1) The necessary separability criterion of positivity under all partial transpositions [10,11] does not imply biseparability along all bipartitions [21,22]. (2) A state can be partially separable, e.g., separable along all bipartitions, but still be entangled [23], even to the extent of serving as a resource for Bell inequality violations [24].

We emphasize that this method of generating sufficient (full) separability criteria is generic and adaptable: developing criteria for different states means parametrizing some separable states of similar form, so as to allow for parameter matching.

To demonstrate the utility of possessing a sufficient separability criterion we assess the candidacy of superradiance for entanglement generation, per the original motivation for this Letter. A system initially in a pure Dicke state is said to evolve according to idealized pure Dicke model superradiance [5] if it decays to the ground state according to the first-order differential equations

$$\frac{\partial\chi_{n_0,n_1}[\tau]}{\partial\tau} = -(n_0+1)n_1\chi_{n_0,n_1}[\tau] + n_0(n_1+1)\chi_{n_0-1,n_1+1}[\tau], \tag{10}$$

where τ is a dimensionless time parameter, $\tau = \Gamma t$ [25]. The idealization is that of perfect indistinguishability of the particles; experimentally, it corresponds to the small-volume limit without dipole-dipole induced dephasing. Our question is whether such idealized superradiance can generate entanglement.

Intuitively, this indistinguishable-particles idealization should yield the strongest entanglement possible, such that if less-idealized superradiance were to generate entanglement, then presumably, entanglement would also be evident in this extremal model; see for example the discussion of volume-dependent many-body effects in Refs. [6,26]. To consider entanglement generation, we utilize an unentangled initial state; the only nonground, separable, pure, Dicke state, is the maximally excited state [27]; i.e., we use initial conditions

$$\chi_{\mathbf{n}}[\tau\to 0] = \begin{cases} 1 & n_1 = N, n_0 = 0 \\ 0 & n_1 < N, n_0 > 0 \end{cases}. \tag{11}$$

Solving the differential equations yields populations χ as functions of τ; one may then test the system for separability at any time τ. Consider the Peres-Horodecki criterion [10,11], which notes that genuinely separable states remain positive semidefinite under partial transpositions (PPT). The property of PPT is necessary but insufficient for separability [21–24], although for symmetric states it is sufficient for $N = 2$, 3, but still insufficient for $N \geq 4$ [28–30]. We find that the PPT criterion is satisfied for all $\tau > 0$ for all $N \leq 10$ [31]. This consistency with separability per the PPT criterion underscores the need for an unambiguous, i.e., sufficient, criterion, a challenge which conditions (8) rise to fulfill.

To certify separability, one merely inspects the decomposition parameters $\{\vec{x}, \vec{y}\}$ obtained by substituting the solved-for populations $\chi_{\mathbf{n}}[\tau]$ into the system of polynomial equations given by Eq. (6). Certification amounts to verification that $\{\vec{x}, \vec{y}\}$ satisfy conditions (8). Indeed, we numerically verified that for pure Dicke model superradiance, conditions (8) are satisfied for all $\tau > 0$, thereby certifying full separability throughout the time evolution, for $N \leq 8$. This is demonstrated graphically in the Supplemental Material [36] for both $N = 4$ and $N = 8$.

We now conjecture that whenever a state ρ_{GDS} is entangled, conditions (8) must be violated, making conditions (8) a necessary and sufficient separability criterion. The sufficiency is by construction, the necessity we can demonstrate by comparison to a known necessary criterion, namely PPT [10,11]. We evidence that, upon restricting to GDS states, the PPT criterion coincides with conditions (8). We claim

$$\text{Lemma}: \varrho_{\mathrm{SDS}} = \varrho_{\mathrm{SEP}\cap\mathrm{GDS}} = \varrho_{\mathrm{PPT}\cap\mathrm{GDS}}, \tag{12}$$

where we prove Lemma (12) for $N = 4$ and conjecture that it continues to hold for all N [32]. Demonstrating Lemma (12) may seem rather daunting; proving equivalence between separability criteria with formal logic is, indeed, an intimidating task. However, we can skip the logical proof and, instead, use integration to directly establish that volume of both ϱ_{SDS} and $\varrho_{\mathrm{PPT}\cap\mathrm{GDS}}$ are identical. To do so, we establish a metric on the spaces of density matrices, the metric can be arbitrary but must be consistent: we choose the populations of ρ_{GDS} as our integration coordinates [33,34]. Thus,

$$\mathrm{PPTGDSVol}_{N=4} = \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \mathbf{1}_{\mathrm{PPT}}(\boldsymbol{\chi})\delta_{(1-|\boldsymbol{\chi}|_1)}\,\mathrm{d}\boldsymbol{\chi}, \tag{13}$$

where $|\boldsymbol{\chi}|_1 = \sum_{\mathbf{n}}\chi_{\mathbf{n}}$ and

$$\mathbf{1}_{\mathrm{PPT}}(\boldsymbol{\chi}) = \begin{cases} 1 & \boldsymbol{\chi}\in\varrho_{\mathrm{PPT}} \\ 0 & \boldsymbol{\chi}\in\varrho_{\mathrm{PPT}} \end{cases}$$

is an indicator function which cuts off the integration whenever the populations violate the PPT conditions. Here, the PPT conditions mean that all eigenvalues are nonnegative for all bipartitions of the qubits for partial transposition [35]. We find numerically that $\mathrm{PPTGDSVol}_{N=4} = (3808 \pm 2)\times 10^{-6}$. In contrast, the volume of all GDS states, including entangled, follows from Eq. (13) absent the indicator function; $\mathrm{GDSVol}_N = 1/N!$. For four qubits $\mathrm{GDSVol}_{N=4} = 41\,666.\bar{6}\times 10^{-6}$.

In principle, one could calculate the volume of ϱ_{SDS} along the same lines as Eq. (13) with a different indicator function based on conditions (8), but there is a much easier way to do it: perform the integration for SDSVol using $\mathbf{x}$ and $\mathbf{y}$ as the integration coordinates, thus, eliminating the need for any indicator function whatsoever. To stay consistent with the originally established metric of the populations $\boldsymbol{\chi}$, we must insert a volume element in the integrand, namely the absolute value of the determinant of Jacobian matrix for the change of variable. For $N = 4$, there are five $\chi_{\mathbf{n}}$ expressible in terms of $\mathbf{x}$, $\mathbf{y}$ via Eq. (6), which correspond to the columns of the Jacobian matrix. The five rows of the Jacobian matrix are given by taking the derivative of the χ list with respect to each of x_1, x_2, x_3, y_1, y_2. The Jacobian's determinant, happily *a priori* nonnegative, is $\mathrm{jac} = 96x_1x_2(1-y_1)^2(1-y_2)^2(y_1-y_2)^4$. Last, we must ensure a one-to-one mapping between χ and $\mathbf{x}$, $\mathbf{y}$. To avoid the problematic interchangeability between the variable pairs x_1, y_1 and x_2, y_2 we impose the ordering $x_1 \geq x_2$.

Therefore,

$$\mathrm{SDSVol}_{N=4} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbf{1}_{x_1 \geq x_2} \times \mathrm{jac} \times \delta_{(1-|\mathbf{x}|_1)}\, d\mathbf{x} d\mathbf{y},$$

where $|\mathbf{x}|_1 = \sum_{k=1}^3 x_k$, and unlike the $\mathbf{x}$, the $\mathbf{y}$ variables have no further restrictions placed upon them due to the normalization of ρ_{SDS}. We find that $\mathrm{SDSVol}_{N=4} = 2/525 \approx (3809.5) \times 10^{-6}$. Because we must have $\varrho_{\mathrm{SDS}} \subseteq \varrho_{\mathrm{PPT} \cap \mathrm{GDS}}$, we are forced to revise $\mathrm{PPTGDSVol}_{N=4}$ to the upper limit of its uncertainty, which indicates convincingly that Lemma (12) is true for $N = 4$.

The authors suspect that Lemma (12) is true for all N for reasons as follows: As previously mentioned, we found that Dicke model superradiance time evolution, per Eq. (10), is PPT for any $\tau \geq 0$ for at least $N \leq 10$. Thus, superradiance serves as a sort of representative sample of PPT $\cap$ GDS states, or formally $\varrho_{\mathrm{SUP\text{-}RAD}} \subset \varrho_{\mathrm{PPT} \cap \mathrm{GDS}}$. But also, as mentioned earlier, we found that such systems apparently always fit the SDS form, in that they satisfy conditions (8) for any $\tau \geq 0$ for at least $N \leq 8$. If Lemma (12) were false, then the unflappable fitting of superradiant states into the SDS form would be surprising, as we would have expected $\varrho_{\mathrm{SUP-RAD}} \not\subset \varrho_{\mathrm{SDS}}$. Thus, we have accumulated evidence by contraposition to support Lemma (12) for $N > 4$.

If Lemma (12) is true for all N, as evidence suggests, then the ramifications are numerous. First, it implies that conditions (8) amount to a necessary and sufficient criterion for separability. Second, it implies that the basic PPT criterion is a sufficient separability test for diagonally symmetric states. Third, we can generate novel practical necessary (but not sufficient) separability criteria by simply considering weaker extensions of conditions (8). For example, presuming that all separable diagonally symmetric states fit the form of Eq. (6) allows us to identify "separable maxima" for the populations such that if even a single population exceeds its "maximum separable value," then entanglement is incontrovertible. We find that, for ρ_{GDS} to be separable, it is necessary (but not sufficient) to satisfy this weaker form of Eq. (6) expressed as

$$\forall_{\mathbf{n}} \chi_{n_0,n_1} \leq \left(\frac{n_0! n_1!}{N!}\right)^{-1} \max_{0<y<1}[y^{n_0}(1-y)^{n_1}]$$
$$\therefore \chi_{n_0,n_1} \leq \left(\frac{n_0^{n_0}}{n_0!}\right)\left(\frac{n_1^{n_1}}{n_1!}\right)\left(\frac{N!}{N^N}\right), \tag{14}$$

which is computationally optimal as a first-pass test to detect entanglement.

The symmetric basis of Dicke states can be extended to general qudits. We desire a generalization of Eq. (6) for qudits, and we wonder if said generalization would also be necessary in addition to sufficient, à la Lemma (12). We hope to consider this in a future work.

In conclusion, what was originally an analysis of superradiance has led to a broad approach for studying multipartite entanglement. We found that a guess and check technique can be surprisingly efficient, as evidenced by the derivation of conditions (8) which apply for all states diagonal in the symmetric basis. Moreover, the derived criterion is a completely tight characterization of separability properties, since we found that it maps out a volume of states no smaller than that defined by the PPT criterion. Additionally, our motivating question has been firmly answered in the negative; pure Dicke model superradiance cannot generate entanglement, begging the question "What is, then, the essential prerequisite of entanglement?" We hope that our techniques for generating sufficient separability criteria, and for certifying the sufficiency of known necessary separability criteria, may prove useful in furthering the understanding of entanglement.

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[17] There are $N+1$ ways to choose $\mathbf{n}$, since $n_0, n_1 \in \mathbb{Z}^+$ and $n_0 + n_1 = N$.

[18] Setting $y_{(N+2)/2} = 0$ when N is an even number actually doesn't induce loss of generality, evidenced in that $\mathrm{PPTGDSVol} = \mathrm{SDSVol}$ still holds when N is an even number.

[19] Note that x_3 does not appear until the final equation, this is a consequence of having set $y_3 = 0$ to ensure that only five free variables exist in the five equations (7).

[20] $\sum x_j = \mathrm{Tr}[\rho_{\mathrm{GDS}}] = 1$ is guaranteed by normalization.

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[26] Note that this presumption does not constitute proof; we cannot confidently infer an absence of entanglement in the realistic cases of dephasing and lower symmetry from our null finding of entanglement in the pure Dick model. Proof of inference is desirable for future research.

[27] Alternative separable initial states include the SDS states (which are not pure), pure superpositions of Dicke states, and even mixed states outside of the GDS manifold. The authors consider such variants of initial conditions, along with other generalizations of superradiance, in another paper now in preparation.

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[32] States diagonal in the symmetric basis are a subset of general permutation-symmetric states, $\varrho_{\mathrm{GDS}}\varrho_{\mathrm{SYM}}$, thus, Lemma (12) is both trivially true for $N = 2, 3$ [28,29] and consistent with the existence of permutation-symmetric PPT-entangled states [29,30].

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[36] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.112.140402 for extended proofs and graphical examples of separability certification.