

Two-Color Gauge Theory with Novel Infrared Behavior

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(Received 5 December 2013; published 18 March 2014)

Using lattice simulations, we study the infrared behavior of a particularly interesting SU(2) gauge theory, with six massless Dirac fermions in the fundamental representation. We compute the running gauge coupling derived nonperturbatively from the Schrödinger functional of the theory, finding no evidence for an infrared fixed point up through gauge couplings $\bar{g}^2$ of order 20. This implies that the theory either is governed in the infrared by a fixed point of considerable strength, unseen so far in nonsupersymmetric gauge theories, or breaks its global chiral symmetries producing a large number of composite Nambu-Goldstone bosons relative to the number of underlying degrees of freedom. Thus either of these phases exhibits novel behavior.

DOI: 10.1103/PhysRevLett.112.111601

PACS numbers: 11.10.Hi, 11.15.Ha, 11.25.Hf, 12.60.Nz

Introduction.—A new sector, described by a strongly interacting gauge theory, could play a key role in physics beyond the standard model. With the recent discovery of a 125 GeV Higgs-like scalar [1,2], SU(2) vectorlike gauge theories provide attractive candidates. Because of the pseudoreality of the fundamental representation of SU(2), two-color theories with N_f massless Dirac fermions in this representation have an enhanced chiral symmetry, a novel symmetry breaking pattern $SU(2N_f) \rightarrow Sp(2N_f)$, and, therefore, a relatively large number of Nambu-Goldstone bosons (NGBs) [3,4]. This feature has motivated SU(2)-based models of a composite Higgs boson [5,6] and of dark matter [7–9].

These models take $N_f = 2$, but new intriguing possibilities emerge for larger N_f . With N_f just below the value at which asymptotic freedom is lost, a conformal window opens up, with the theory initially governed by a weakly coupled infrared fixed point (IRFP). As N_f is decreased, the strength of the fixed point increases. Below some critical value N_f^c , chiral symmetry is broken and the theory confines. This critical value defines the lower edge of the conformal window [10,11]. Knowing the extent of the window and the behavior of theories in it and near it could be crucial for building a successful model of beyond the standard model physics.

The extent of the conformal window is also interesting from a more theoretical point of view, and this is

particularly true of the two-color theory. For example, a general notion about quantum field theories, as first applied to second-order phase transitions and critical phenomena, is that the renormalization group flow toward the infrared (IR) should result in a thinning of the degrees of freedom. This can provide an important constraint on IR behavior if it can be shown that the IR count cannot exceed the UV count. One implementation of this idea, much studied recently [12,13], defines the degree-of-freedom count through the coefficient a entering the trace of the energy momentum tensor on an appropriate space-time manifold. Although an UV-IR inequality can perhaps be proven, it does not seem to lead to useful constraints.

Another approach [14] defines the degree-of-freedom count via the thermodynamic free energy $F(T)$, using the temperature T as the renormalization group scale. The dimensionless quantity $f(T) \equiv 90F(T)/\pi^2 T^4$ is T independent for a free massless theory, leading to $f = 2N_V + (7/2)N_F + N_S$, where N_V , N_F , and N_S count the gauge, Dirac-fermion, and real-scalar fields. The conjectured inequality of Ref. [14] is that for an asymptotically free theory, $f_{\text{IR}} \equiv f(0) \leq f_{\text{UV}} \equiv f(\infty)$.

In the case of an IR phase with broken chiral symmetry and confinement, f_{IR} counts the number of NGBs. For a vectorlike SU(N) gauge theory with $N \geq 3$ and N_f Dirac fermions, this count is $N_f^2 - 1$. Also, in the UV,

$N_V = N^2 - 1$ and $N_F = NN_f$. The above inequality then demands $N_f^c < \frac{1}{4}(7N + \sqrt{81N^2 - 16})$. This is a testable constraint, and it has been satisfied by recent lattice simulations [15]. For $N = 2$ on the other hand, the enhanced chiral symmetry, the different pattern of symmetry breaking, and the resultant enhanced NGB count ($2N_f^2 - N_f - 1$) [3] lead to a significantly reduced bound on N_f for the broken phase: $N_f^c < (4 + \sqrt{30})/2 \approx 4.7$.

Crude estimates of the edge of the conformal window, based on quasiperturbative methods, also exist. Gap-equation methods [16], for example, lead to the estimate $N_f^c \approx 4N$. While this is nicely compatible with the inequality for $N \geq 3$, it clearly disagrees with it for $N = 2$. This tension suggests that the $N_f = 6$ theory could be particularly worthy of study.

Early lattice calculations attempted to explore the two-color conformal window by studying the lattice theory at strong bare coupling [17,18]. Recent efforts have primarily searched for an IRFP with nonperturbative running coupling calculations. Evidence that $N_f = 10$ ($N_f = 4$) is inside (outside) the conformal window is presented in Ref. [19]. Additionally, Ohki *et al.* argue that $N_f = 8$ is inside the conformal window [20]. The case $N_f = 6$, arguably the most interesting, while tackled by many groups [19,21–24], has remained inconclusive.

Here we study the $N_f = 6$ theory, drawing on larger computational resources than in all previous work, to determine whether $N_f = 6$ has an IRFP by calculating the Schrödinger functional (SF) [25] running coupling. We use the stout-smearing [26] Wilson fermion action, which avoids coupling the fermions to unphysical fluctuations of the gauge field on the scale of the lattice spacing. Smeared actions have also been used in SF running coupling studies of other theories [27,28].

Preliminaries.—A stout-smearing fermion action replaces “thin” gauge links by “fat” links which are averaged with nearby gauge links. It has the advantage that it is analytic and can therefore be used in conjunction with molecular dynamics updating schemes such as Ref. [29]. The formulas required to implement this smearing procedure in a molecular dynamics algorithm are derived for the case of SU(3) links in Ref. [26]. We have derived the relevant formulas for the SU(2) case. Recently, another group implemented two-color stout smearing as well [30].

We use only one level of stout smearing with an isotropic smearing parameter $\rho = 0.25$ [26]. In preliminary work [23], we show that this degree of smearing reduces lattice artifacts and allows us to reach stronger running couplings, but still not be in the vicinity of an unphysical bulk phase transition that could distort our results. As all calculations in this work are done with Dirichlet boundary conditions (BCs) in the time directions, there is some ambiguity in how to implement the smearing of the gauge field near this boundary. We choose to not smear the boundary links with

bulk links and *vice versa*. This choice results in a simpler running-coupling observable.

The Wilson fermion action contains an additional irrelevant operator that lifts the mass of the fermion doublers to the cutoff scale so they decouple from the calculation. This lattice artifact explicitly breaks chiral symmetry, and as a result the fermion mass is additively renormalized. The bare mass m_0 therefore must be carefully tuned in order to restore chiral symmetry. The critical value of the bare mass (as a function of the bare coupling) $m_c(g_0^2)$ is defined as the bare mass value that results in a zero renormalized quark mass [31]. In practice, m_c is determined, at fixed bare gauge coupling g_0^2 and lattice volume $(L/a)^3 \times 2L/a$, as the root of a fitted linear function to measurements of the renormalized quark mass vs the bare quark mass. This is done for a range of bare couplings and lattice volumes and the results are fit to a polynomial given by

$$m_c^{\text{fit}}\left(g_0^2, \frac{a}{L}\right) = \sum_{i=1}^7 g_0^{2i} \left[a_i + b_i \left(\frac{a}{L} \right) \right]. \quad (1)$$

Then, $m_c^{\text{fit}}(g_0^2, 0)$ is used in the running coupling calculations [32]. All data used to fit $m_c^{\text{fit}}(g_0^2, a/L)$ and $m_c^{\text{fit}}(g_0^2, 0)$ are shown in Fig. 1.

In order to guarantee that we can take a continuum limit, we need to obtain data only from the weak-coupling side of any spurious lattice phase transition. With this in mind, we scan through the bare parameter space and locate peaks in the plaquette susceptibility on a $L/a = 10$ lattice. This search indicates a line in the m_0 - g_0^2 plane of first order phase transitions that ends at a critical point at around $g_0^2 \approx 2.2$. For $g_0^2 \lesssim 2.2$, we see crossover behavior. In Fig. 1, we show the above transition line plotted along with $m_c^{\text{fit}}(g_0^2, 0)$. Figure 1 indicates that our action has a sensible continuum limit only for $g_0^2 \lesssim 2.175$. Therefore, we examine the running coupling only on lattices with a bare coupling within this range.

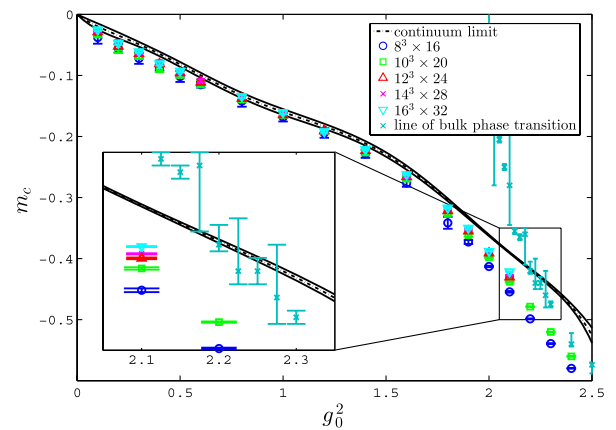


FIG. 1 (color online). Bare masses that result in zero partially conserved axial current mass at various lattice volumes. All data points fit to $m_c^{\text{fit}}(g_0^2, a/L)$ and the continuum extrapolation $m_c^{\text{fit}}(g_0^2, 0)$ (black dashed line) are shown. $m_c^{\text{fit}}(g_0^2, 0)$ determines masses used in running coupling simulations. Additionally the peak in the plaquette susceptibility (turquoise x's) is shown.

Running coupling.—To define a nonperturbative renormalized coupling, we employ the Schrödinger functional [25]. It is given by a path integral over gauge and fermion fields that reside within a four-dimensional Euclidean box of spatial extent L with periodic BCs in spatial directions and Dirichlet BCs in the time direction. We choose gauge BCs [33], $U(x, k)|_{x^0=0} = \exp[-i\eta \frac{a}{L} \tau_3]$ and $U(x, k)|_{x^0=L} = \exp[-i(\pi - \eta) \frac{a}{L} \tau_3]$, and fermion BCs [34], $P_+ \psi|_{x^0=0} = \bar{\psi} P_-|_{x^0=0} = P_- \psi|_{x^0=L} = \bar{\psi} P_+|_{x^0=L} = 0$. These BCs classically induce a constant chromoelectric background field whose strength is characterized by the dimensionless parameter η . With these BCs the SF is given by $\mathcal{Z}(\eta, L) = \int D[U, \psi, \bar{\psi}] e^{-S[U, \psi, \bar{\psi}; \eta]}$.

The running coupling is then defined by,

$$\frac{k}{\bar{g}^2(g_0^2, \frac{L}{a})} = \frac{\partial}{\partial \eta} \log \mathcal{Z} \Big|_{\eta=\pi/4} = \left\langle \frac{\partial S}{\partial \eta} \right\rangle, \quad (2)$$

with $k = -24(L/a)^2 \sin[(a/L)^2(\pi/2)]$ so that the renormalized coupling agrees with the bare coupling at tree level. The first two perturbative coefficients of the SF beta function are the universal coefficients given in Ref. [10]. This renormalization scheme has the virtue that it is fully nonperturbative and it is amenable to a lattice calculation.

We calculate the SF renormalized coupling over a range of bare couplings and lattice volumes [32]. Lattice perturbation theory gives $g_0^2/\bar{g}^2$ as an expansion in powers of g_0^2 . This motivates an interpolating fit [35],

$$\frac{1}{\bar{g}^2} - \frac{1}{\bar{g}^2(g_0^2, \frac{L}{a})} = \sum_{i=0}^{N_{L/a}} a_{i,L/a} g_0^{2i}. \quad (3)$$

We choose the lowest possible $N_{L/a}$ to give a reasonable χ^2 per d.o.f. (in practice, values in the range $\chi^2/\text{d.o.f.} \in [0.7, 1.5]$), finding $N_{L/a \leq 12} = 6$ and $N_{L/a > 12} = 5$. This procedure produces smooth functions, one for each lattice volume L/a , of the renormalized coupling vs the bare coupling. Before using this interpolation for further analysis, it is worth noting that there is no hint of an IRFP in the lattice data and therefore in the interpolating curves. At any fixed g_0^2 , the running coupling $\bar{g}^2(g_0^2, \frac{L}{a})$ is seen only to increase as a function of L/a in the range of the data.

The question is whether a careful continuum extrapolation will indicate otherwise. A step scaling [36] analysis allows us to address this issue and to study the renormalized coupling over a large range of scales in a computationally feasible manner. The continuum step scaling function $\sigma(u, s)$ is defined by

$$\int_u^{\sigma(u,s)} \frac{d\bar{g}^2}{\beta(\bar{g}^2)} = 2 \log s. \quad (4)$$

It is the renormalized coupling at a length scale sL given that the running coupling $\bar{g}^2 = u$ at a length scale L . On the lattice we calculate the discrete step scaling function,

$$\Sigma\left(u, \frac{a}{L}, s\right) \equiv \bar{g}^2\left(g_0^2, \frac{sL}{a}\right) \Big|_{\bar{g}^2(g_0^2, L/a)=u}. \quad (5)$$

We arrive back at a continuum step scaling function by taking the continuum limit:

$$\sigma(u, s) = \lim_{a/L \rightarrow 0} \Sigma\left(u, \frac{a}{L}, s\right). \quad (6)$$

From here we use $s = 2$ and drop reference to this from our notation.

To extract σ as a function of u , we first use the interpolating fits, given by Eq. (3), to evaluate Σ at each fixed value of u and integer $L/a = 5-10$ and 12. We take the continuum limit, at each u independently, by fitting $\Sigma(u, a/L)$ to a polynomial in a/L , and extrapolating to $a/L \rightarrow 0$. Our result, shown in Fig. 2, displays several plots of the quantity $[\sigma(u) - u]/u$ vs u . This quantity is a finite-difference version of the continuum beta function. In one curve (red), we fit $\Sigma(u, a/L \leq 1/6)$ to a quadratic polynomial and then extrapolate the result to $a/L \rightarrow 0$. Additionally, we show, $\Sigma(u, a/L \leq 1/5)$ extrapolated from a cubic polynomial fit (green). We see that these two curves are consistent, but the errors of the cubic extrapolation become large at $u \approx 8$. The remaining (blue) curve is obtained with a constant extrapolation to the continuum using only the three points with $a/L \leq 1/9$.

To assess the goodness of fit of any particular functional form for continuum extrapolation of Σ we examine $\chi^2/\text{d.o.f.}$ over the entire range of u . For the constant extrapolation (blue) in Fig. 2 for $L/a \geq 9$, $\chi^2/\text{d.o.f.}$ varies from 0.5–2. A quadratic extrapolation (red) for $L/a \geq 6$ and a cubic extrapolation for $L/a \geq 5$ have comparable $\chi^2/\text{d.o.f.}$ ranging from 0.5–4 throughout the range of u .

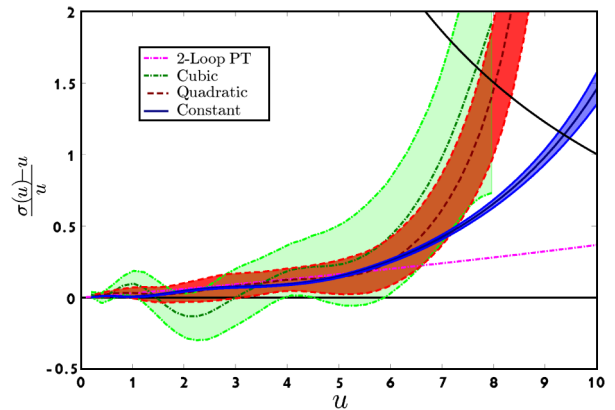


FIG. 2 (color online). $[\sigma(u) - u]/u$ vs u for three different extrapolations to the continuum. A contour at $\bar{g}^2 = 20$ is shown to provide a measure of the strength of renormalized coupling explored here. The 2-loop perturbative result is also shown here (dot-dashed magenta line).

The constant (quadratic and cubic) extrapolation relies on fits with two (three) degrees of freedom.

These various extrapolations all perform well at reproducing the perturbative two-loop curve (magenta) at small values of u . If the resulting curves were to cross zero at some larger u , this would be indicative of an IRFP. We see no indication of this; in fact we see, regardless of which extrapolation we use, the running coupling grow up to and beyond estimates of the critical coupling required to induce spontaneous chiral symmetry breaking [16]. We see no evidence even of an inflection point, which would hint at an IRFP at a stronger coupling strength. Indeed these curves are qualitatively similar to, although much more slowly running than, the $\sigma(u)$ of quantum chromodynamics.

We next compare these three continuum extrapolations more carefully and comment also on extrapolation via a linear polynomial in a/L . For each u , $\Sigma(u, a/L)$, evaluated at $L/a = 5-10$ and 12, is fit to a cubic polynomial, $p(a/L) = \sum_{i=0}^3 \alpha_i (a/L)^i$. For several values of a/L , the relative sizes of the constant, $O(a/L)$, $O(a/L)^2$, and $O(a/L)^3$ terms in the polynomial are plotted vs u . We can then assess the validity of some truncation of the polynomial continuum extrapolation within some window in a/L . We show the results of such an analysis in Fig. 3 for $L/a = 6, 9$, and 12. A number of interesting features are evident. At weak coupling the lattice artifacts are small, and a constant extrapolation adequately describes the continuum limit. But at intermediate and strong coupling ($u \gtrsim 6$), lattice artifacts become significant. Throughout the coupling range, the linear and quadratic lattice artifacts

are comparable for $a/L \geq 1/9$ and hence we cannot perform a reliable linear extrapolation to the continuum. The cubic contribution, however, is small for $a/L \leq 1/6$ and $u \lesssim 8$, indicating that a quadratic extrapolation to the continuum is reliable at least up to this input coupling strength. This indicates that the running coupling reaches a $\bar{g}^2$ of order 20 without encountering an IRFP.

Insight may also be gleaned by plotting the extrapolation to the continuum at fixed coupling strength u . We show in Fig. 4 the example of $u = 7.5$. We plot $\Sigma(u, a/L)$ vs a/L , along with a quadratic and cubic polynomial fit, as well as a constant extrapolation based on the three smallest a/L values. These correspond to the fits used in Fig. 2. Figure 4 demonstrates that a constant extrapolation to the continuum is reasonable. Taking the larger a/L points into account shows the presence of significant nonlinear lattice artifacts, in fact suggesting that the constant extrapolation significantly underestimates $\sigma(u)$ for $u \gtrsim 7$. It is also evident that the quadratic and cubic fits extrapolate to a value of σ that is well above the smallest- a/L points. It is likely that the true extrapolated value is somewhere between the constant and quadratic extrapolations.

Recently, Hayakawa *et al.* have claimed to see evidence of an IRFP in the two-color six-flavor theory [24]. They employ the SF method as we do but with the unimproved Wilson fermion action and a linear extrapolation to the continuum. It is reasonable to expect that for large enough L/a the linear term will be the dominant lattice artifact but it is difficult to quantify how large an L/a is necessary outside of perturbation theory. Other extrapolation forms, including quadratic terms can be used to fit their data with a comparable or slightly better $\chi^2/\text{d.o.f.}$ When this is done, we cannot conclude that an IRFP exists. Moreover, from *our* data set, sampling many more bare couplings and lattice volumes, we are able to study the relative contributions of different lattice artifacts. In Fig. 3, we see that in the strong coupling regime, the quadratic term becomes

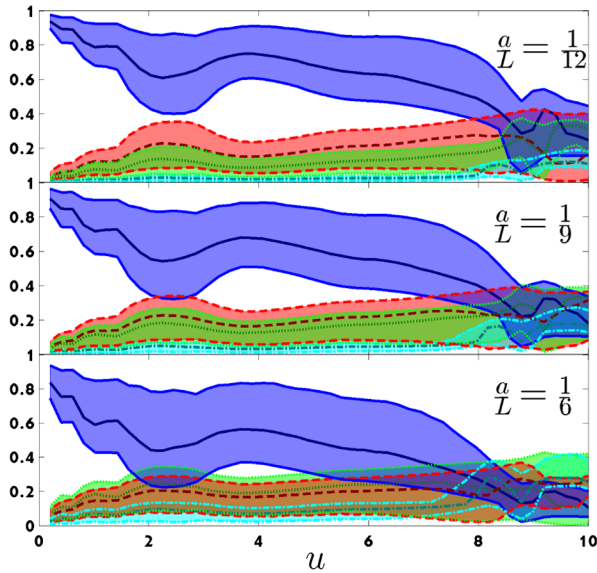


FIG. 3 (color online). Plots of relative magnitudes of low order contributions to the continuum extrapolation. We fit $s = 2$ steps at $L/a = 5-10$ and 12 to a polynomial $\sum_{i=0}^3 \alpha_i (a/L)^i$. Then $|\alpha_0|/T$ (blue line), $|\alpha_1(a/L)|/T$ (red dashed line), $|\alpha_2(a/L)^2|/T$ (green dotted line), and $|\alpha_3(a/L)^3|/T$ (cyan dot-dashed line) are plotted vs u , at various values of a/L , with $T = \sum_{i=0}^3 |\alpha_i(a/L)^i|$.

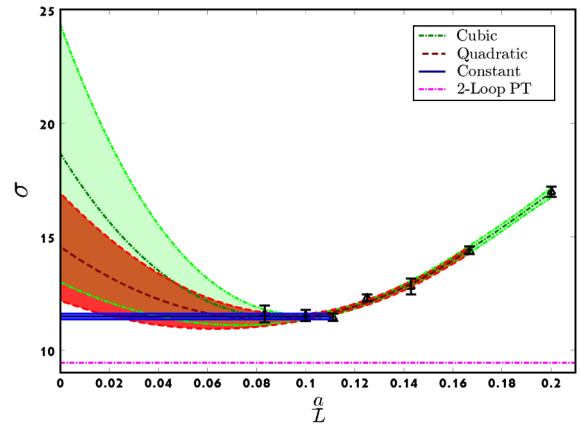


FIG. 4 (color online). Plot of $\Sigma(u = 7.5, a/L)$ vs a/L with various extrapolations to the continuum. The continuum limit of the quantity is obtained by fitting these points to a polynomial in a/L .

significant in the a/L range studied by Hayakawa *et al.* and by us. This suggests that concluding the existence of an IRFP from a linear extrapolation to the continuum is premature.

To summarize, for an SU(2) gauge theory with six massless fermions in the fundamental representation, we find no evidence of an infrared fixed point in the running gauge coupling as defined in the Schrödinger functional scheme. Our simulations reach well into a strong-coupling range, potentially capable of triggering chiral symmetry breaking and confinement. We conclude that this theory either flows to a very strong infrared fixed point, so-far unseen in nonsupersymmetric theories, or it breaks chiral symmetry and confines, producing a large number (65) of Nambu-Goldstone bosons, well above the number of underlying fermionic and gauge degrees of freedom. Thus either of these (zero-temperature) phases exhibits novel behavior. In the latter case, since the Stefan-Boltzmann value f_{UV} would lie below f_{IR} , the finite-temperature phase transition would have features distinctly different from quantum chromodynamics. We could in principle probe even larger couplings than presented here, but the computational challenges and lattice-artifact difficulties grow with coupling strength. Other approaches will be important to firmly establish the infrared nature of this theory.

We thank Robert Shrock for helpful discussions. We would like to acknowledge our use of the CHROMA [37] software package for all calculations performed here. We thank the Lawrence Livermore National Laboratory (LLNL) Institutional Computing Grand Challenge program for computing time on the LLNL Sierra, Hera, Atlas, and Zeus computing clusters. We thank LLNL for funding from LDRD10-ERD-033 and LDRD13-ERD-023. Several of us (T. A., G. T. F., R. C. B., M. C., E. T. N., M. F. L., and D. S.) thank the Aspen Center for Physics (supported by NSF Grant No. PHYS-1066293) for its hospitality while some of the research reported here was being done. This work has been supported by the U. S. Department of Energy under Grants No. DE-FG02-00ER41132 (M. I. B.), No. DE-FG02-91ER40676 (R. C. B., M. C., C. R.), No. DE-FG02-92ER-40704 (T. A.), No. DE-FC02-12ER41877 (D. S.), No. DE-FG02-85ER40231 (D. S.), and Contracts No. DE-AC52-07NA27344 (LLNL), No. DE-AC02-06CH11357 (Argonne Leadership Computing Facility), and by the National Science Foundation under Grants No. NSF PHY11-00905 (G. T. F., M. F. L., G. V.) and No. PHY11-25915 (Kavli Institute for Theoretical Physics). We thank USQCD for computer time on FNAL and JLab clusters. We thank XSEDE for computer time on Kraken under Grant No. TG-MCA08X008.

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