Random Fields, Topology, and the Imry-Ma Argument

Thomas C. Proctor, Dmitry A. Garanin, and Eugene M. Chudnovsky
Physics Department, Lehman College, City University of New York,
250 Bedford Park Boulevard West, Bronx, New York 10468-1589, USA
(Received 26 December 2013; revised manuscript received 30 January 2014; published 3 March 2014)

We consider an n-component fixed-length order parameter interacting with a weak random field in $d = 1, 2, 3$ dimensions. Relaxation from the initially ordered state and spin-spin correlation functions are studied on lattices containing hundreds of millions of sites. At $n \leq d$ the presence of topological defects leads to strong metastability and glassy behavior, with the final state depending on the initial condition. At $n = d + 1$, when topological structures are nonsingular, the system possesses a weak metastability. At $n > d + 1$, when topological objects are absent, the final, lowest-energy state is independent of the initial condition. It is characterized by the exponential decay of correlations that agrees quantitatively with the theory based upon the Imry-Ma argument.

DOI: 10.1103/PhysRevLett.112.097201 PACS numbers: 75.10.Nr, 05.50.+q, 64.60.De

More than 40 years ago, Larkin argued that randomly positioned pinning centers destroy translational order in a flux-line lattice [1]. A more general theorem was suggested by Imry and Ma [2]. It states that a static random field, regardless of strength, destroys the long-range order associated with a continuous-symmetry order parameter below $d = 4$ spatial dimensions [3]. The prototype Hamiltonian is

$$\mathcal{H} = \int d^d r \left[\frac{\alpha_e}{2}(\nabla \mathbf{S})^2 - \mathbf{h}\cdot\mathbf{S}\right] \tag{1}$$

with $\mathbf{S}$ and $\mathbf{h}$ being n-component fixed-length vector field and static random field, respectively. In the Imry-Ma (IM) state, the directions of $\mathbf{S}$ are correlated within distance $R_f \propto (1/h)^{2/(4-d)}$. The IM argument has been applied to random magnets [4], arrays of magnetic bubbles [5], superconductors [6], charge-density waves [7], liquid crystals [8], superfluid ^{3}He-A [9], etc. Its validity for distances beyond R_f has been questioned by analytical work based upon renormalization group and replica-symmetry breaking methods [10]. Numerical work [11–16] revealed glassy properties of the random-field model. It was suggested in Ref. [16] that the energy cost of vortices prevents the spins in the xy random-field model from relaxing to a disordered state from the initially ordered state.

In this Letter, we show more generally that the long-range behavior of random-field systems is controlled by topology. Our emphasis is on glassy vs reversible behavior. The condition $\mathbf{S}^2 = S_0^2 = \text{const}$ leaves $n-1$ components of the field independent. At $n \leq d$, mapping of $n-1$ independent parameters describing the field $\mathbf{S}$ onto spatial coordinates provides topological defects with singularities. They are vortices in the xy model ($n = 2$) in $2d$, vortex loops in the xy model in $3d$, and hedgehogs in the Heisenberg model ($n = 3$) in $3d$. As we shall see, the IM state necessarily contains such topological defects. Energy barriers associated with their creation or annihilation and their pinning by the random field lead to strong metastability. In the case of $n > d + 1$, the mapping of the $\mathbf{S}$ space onto the $\mathbf{r}$ space that generates topological objects is impossible. They are absent together with the energy barriers and pinning. The stable state of the system is unique and independent of the initial condition. In this case, the long-range order is destroyed in a manner that agrees quantitatively with the IM picture. This applies to the Heisenberg model with $n = 3$ (and greater) in one dimension, $n = 4$ (and greater) in two dimensions, and $n = 5$ (and greater) in $3d$. The case of $n = d + 1$ is the borderline between the above two cases. It corresponds to nonsingular topological objects: kinks in the xy model in $1d$, Skyrmions in the Heisenberg model with $n = 3$ in $2d$, and similar nonsingular solutions for $n = 4$ in $3d$. They are characterized by the topological charge $Q = \pm 1, \pm 2, \ldots$. Its conservation is important as it is only weakly violated by the discreteness of the lattice and by a weak random field. Possession of a pinned topological charge by the IM state prevents the system from relaxing to this state from any initial state that has a different topological charge.

We have numerically studied the discrete version of the Hamiltonian (1) with the nearest-neighbor exchange interaction in the presence of the external field $\mathbf{H}$,

$$\mathcal{H} = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{s}_i\cdot\mathbf{s}_j - \sum_i \mathbf{h}_i\cdot\mathbf{s}_i - \mathbf{H}\cdot\sum_i \mathbf{s}_i, \tag{2}$$

on lattices containing hundreds of millions of spins $\mathbf{s}_i$ of length s. The relation between parameters of the continuous and discrete models is $\alpha_e = Ja^{d+2}$, $S_0 = s/a^d$, where a is the lattice parameter. We consider hypercubic lattices with periodic boundary conditions containing L^d spins, L being the linear size of the system. In computations, we use $J = s = a = 1$ and $h = H_R$. Our numerical method combines sequential rotations of spins towards the direction of the local effective field $\mathbf{H}_{i,\mathrm{eff}} = \sum_i J_{ij}\mathbf{s}_j + \mathbf{h}_j + \mathbf{H}$, with energy-conserving spin flips: $\mathbf{s}_i \to 2(\mathbf{s}_i \cdot \mathbf{H}_{i,\mathrm{eff}}) \times \mathbf{H}_{i,\mathrm{eff}}/H^2_{i,\mathrm{eff}} - \mathbf{s}_i$, applied with probabilities α and $1-\alpha$, respectively, α playing the role of the relaxation constant. High efficiency of this method for glassy systems under the condition $\alpha \ll 1$, which is physically equivalent to slow cooling, has been demonstrated in Refs. [15,16].

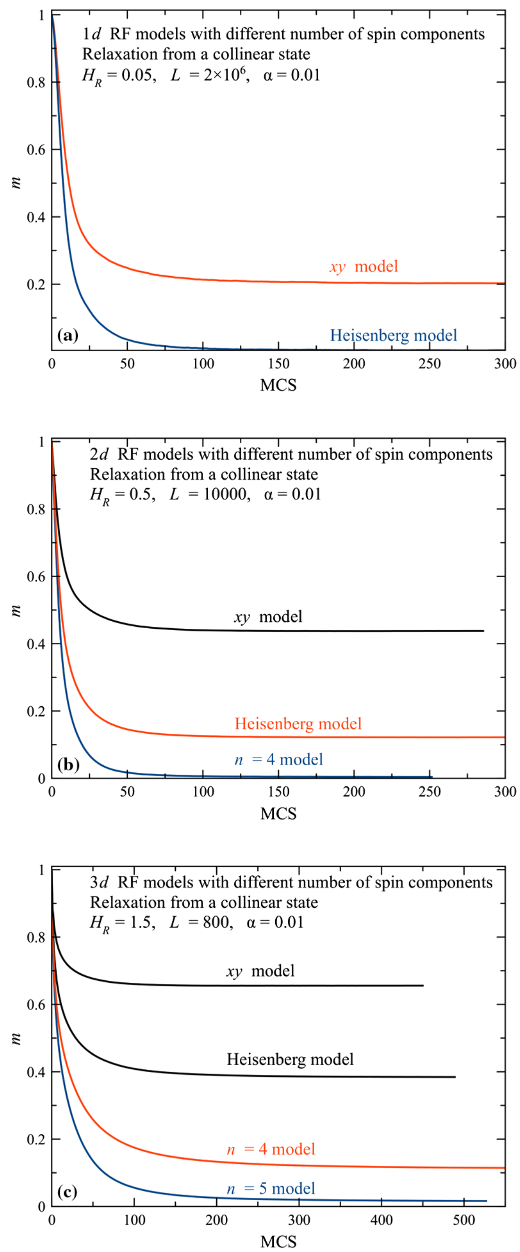

FIG. 1 (color online). Relaxation of the magnetization of the random-field spin system from fully ordered initial state for different d and n: (a) $d = 1$, $n = 2, 3$; (b) $d = 2$, $n = 2$ 3, 4; (c) $d = 3$, $n = 2$ 3, 4, 5. MCS means a full spin update, as in Monte Carlo simulations.

Relaxation of the per-site magnetization $m = \sqrt{\mathbf{m}\cdot\mathbf{m}}$, where $\mathbf{m} = (sN)^{-1}\sum_i \mathbf{S}_i$, out of a collinear state is shown in Fig. 1. For each process, it was checked that the running time was sufficient to have no further relaxation in the final state.

In one dimension, numerical analysis of different spin configurations shows that for $n = d + 1 = 2$ the IM-like state with $m = 0$ has the lowest energy. This state, however, cannot be achieved through relaxation from the initially ordered state without forming nonsingular kinks or antikinks associated with the full clockwise or counterclockwise rotations of the spin as one moves along the spin chain. While the system tends to disorder, it cannot do so completely because it requires changing the topological charge that equals the difference in the number of kinks and antikinks pinned by the random field. However, for three-component spins in one dimension, topologically stable objects are absent and the system disorders completely as is illustrated by Fig. 1(a).

Two-component spins in two dimensions form well-known topological singularities—vortices in the xy model [11,16,17]. Here again, the system wants to relax to the IM-like state with $m = 0$ but cannot do it without forming vortices that cost energy, which explains the curve in Fig. 1(b) for the xy model in $2d$. In the marginal case of $d = 2$, $n = 3$, the model possesses nonsingular topological objects: Skyrmions [17]. In the absence of the random field, the difference in the number of Skyrmions and anti-Skyrmions is a conserved topological charge. Skyrmions on the lattice tend to collapse [18]. However, pinning by the random field stabilizes them. We have checked numerically that for $d = 2$, $n = 3$ the IM state with $m = 0$ has the lowest energy. However, conservation of the topological charge prevents the system from relaxing to this state from almost any initial condition. This effect is responsible for a small but finite magnetization obtained by the relaxation from the initially ordered state, Fig. 1(b). On the contrary, for a four-component spin in two dimensions, topological objects are absent and the system relaxes to the state with $m = 0$; see Fig. 1(b).

Relaxation in a three-dimensional case is illustrated by Fig. 1(c). For $n = 2$, the system possesses vortex lines or loops that in the lattice model are singular pancake vortices in $2d$ planes stuck together; see Fig. 2(a). Similarly, the model with three-component spins in $3d$ has singular hedgehogs; see Fig. 2(b). The energy cost of vortex loops and hedgehogs prevents the $3d$ system of spins from relaxing to the $m = 0$ state, as is shown in Fig. 1(c). In the marginal case of $n = 4$, the $3d$ random-field model has nonsingular topological structures pinned by the random field that are similar to Skyrmions in $2d$. In this case, the final magnetic moment is still nonzero but small; see Fig. 1(c). We again find that the energy of the IM-like $m = 0$ state for $d = 3$, $n = 4$ is lower than that of the $m \neq 0$ state. However, the difference in the topological charge

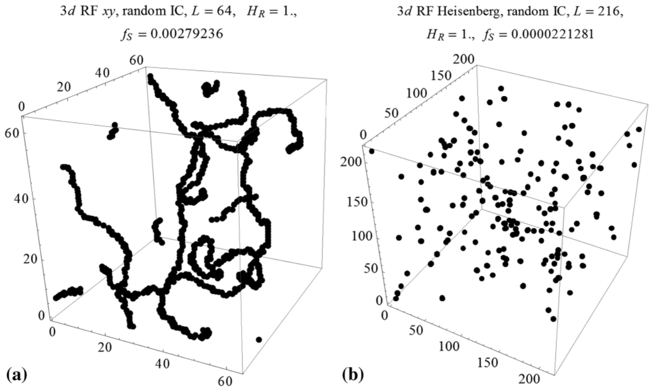

FIG. 2. Topological singularities in the random-field spin model in three dimensions obtained by relaxation from random initial orientation of spins: (a) pinned vortex loops of the xy ($n = 2$) model; (b) pinned hedgehogs of the Heisenberg ($n = 3$) model. f_S is the fraction of the lattice interstitial (body centered) sites occupied by singularities.

prevents the system from relaxing to the IM state from almost any initial state. The model with five-component spins in $3d$ does not possess any topologically stable structures. The relaxation of the system from the ordered state is unobstructed by any topological arguments, and the system ends up with $m = 0$, Fig. 1(c).

Starting from a random orientation of spins, one obtains states of vortex or hedgehog glasses with $m = 0$ and energies higher than those of the ordered states. The relation between topology and metastability in, e.g., two spatial dimensions, is further illustrated by the hysteresis curves in Fig. 3. The model with $n = 2$ that possesses xy vortices with singularities is characterized by a sizable hysteresis loop, which is indicative of strong metastability. The loop becomes thin for $n = 3$ when nonsingular Skyrmions are present. It disappears completely, resulting in a reversible magnetic behavior, at $n = 4$ when topological objects are absent. Similar behavior for different n values has been observed in $3d$.

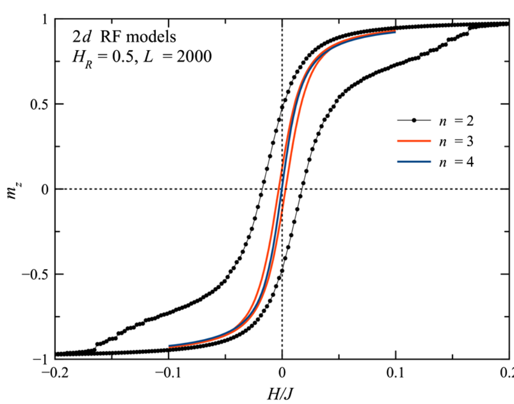

FIG. 3 (color online). Hysteresis curves of the random-field spin model in two dimensions for $n = 2, 3, 4$.

To get a better understanding of how topology modifies the IM argument, let us recall that in that argument the order parameter $\mathbf{S}$ follows the direction of the average random field $\bar{\mathbf{h}}$ on a scale R_f. The energy of the random field in Eq. (1) is proportional to $-\bar{h}S_0 \sim -hS_0/R_f^{d/2}$, whereas the nonuniformity energy is proportional to $\alpha_e S_0^2/R_f^2$. Minimization of the total energy with respect to R_f then leads to the rotation of $\mathbf{S}$ by a significant angle on a scale $R_f \propto (Js/h)^{2/(4-d)}$. For $R \gtrsim R_f$, correlations should be completely destroyed; thus, the state of the system should be disordered. This famous argument, however, does not account for the energy associated with unavoidable singularities at $n \leq d$. To show their existence in the IM state, consider components of the averaged random field $\bar{h}_\beta$, $\beta = 1, \ldots, n$. Since $\bar{h}_\beta$ are sums of many random numbers, they are statistically independent and have a Gaussian distribution. In about one half of the space $\bar{h}_\beta > 0$, and in the other half $\bar{h}_\beta < 0$. Boundaries between these regions are subspaces of dimension $d - 1$, where $\bar{h}_\beta = 0$. Their intersection, that is, $\bar{\mathbf{h}} = 0$, is unavoidable and forms a subspace of dimension $d - n$ if $n \leq d$. It is easy to see that subspaces with $\bar{\mathbf{h}} = 0$ are singularities in the spin field $\mathbf{S}$. Since $\mathbf{S}^2 = \text{const}$, crossing subspaces $\bar{\mathbf{h}} = 0$ makes all components of $\mathbf{S}$ change direction. For $n = 2$ in $2d$ subspaces $\bar{\mathbf{h}} = 0$ are points and the corresponding singularities are vortices or antivortices. A spin field in the $2d$ xy model generated in accordance with the IM prescription is shown in Fig. 4. The red line corresponds to $\bar{h}_x = 0$ and, thus, spins directed along the y axis. The blue line corresponds to $\bar{h}_y = 0$ and, thus, spins directed along the x axis. At the intersections of red and blue lines, the spins can look neither in the x nor in the y direction. This generates topological defects—vortices or antivortices. For

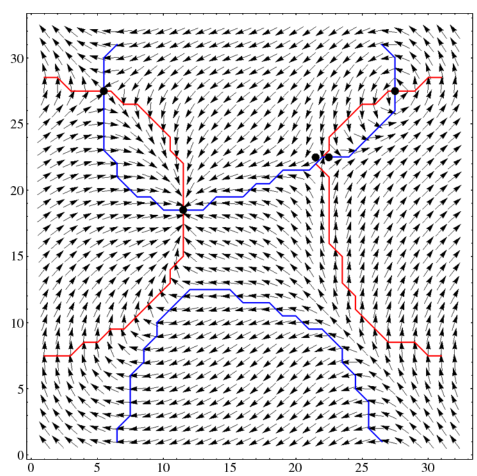

FIG. 4 (color online). Emergence of vortices and antivortices at the intersections of lines corresponding to $\bar{h}_x = 0$ (left and right curves shown in red) and $\bar{h}_y = 0$ (upper and lower curves shown in blue) in the random-field $2d$ xy model. Arrows show spins on lattice sites.

$n = 2$ in $3d$ subspaces $\bar{\mathbf{h}} = 0$ are lines and the singularities are vortex lines or loops. For $n = 3$ in $3d$ subspaces $\bar{\mathbf{h}} = 0$ are points and the singularities are hedgehogs. They emerge at the intersection of surfaces corresponding to $\bar{h}_x = 0$, $\bar{h}_y = 0$, and $\bar{h}_z = 0$.

By order of magnitude the number of singularities equals the number of IM domains $(L/R_f)^d$. The lowest energy of an xy vortex in a $2d$ IM state would be $2\pi J s^2 \ln(R_f/a)$. The energy of the vortex loop in $3d$ contains an additional factor R_f/a. Consequently, the exchange energy per spin goes up by $\ln(R_f/a)$ as compared to the IM argument that neglects vortices. The energy of a hedgehog would be $4\pi J s^2 (R_f/a)$. It changes the exchange energy by a numerical factor of order unity. Thus, topological defects only modify the IM argument by making R_f go up logarithmically in the xy model and by a factor of order unity in the Heisenberg model. However, the energies of topological defects that are needed to form the IM state as the system disorders are high compared to the Curie temperature. This slows down further disordering that requires the creation of singularities. On the contrary, for $n > d$, the averaged random field is nonzero everywhere and the spin field $\mathbf{S}$ is nonsingular. Still, at $n = d + 1$ the presence of nonsingular topological objects and conservation of topological charge prevents the ordered state from relaxing to the IM state. Only at $n > d + 1$, when topological objects are absent, does the system relax to the $m = 0$ IM state.

We have computed spin-spin correlation functions in the final state obtained through relaxation from the initially ordered state. At $n \leq d + 1$ and $R_f \ll L$, the ferromagnetic order persists: The correlation function at large distances falls to a plateau that coincides with m^2 of the plateau in Fig. 1. On the contrary, at $n > d + 1$ the order is fully destroyed in accordance with the IM picture. The $3d$ correlation function for $n = 5$ is shown in Fig. 5. We have analytically derived from Eq. (1) the short-distance form of the spin-spin correlation function in $3d$, $\langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle \cong 1 - |\mathbf{r}_1 - \mathbf{r}_2|/R_f$, where $R_f/a = 8\pi(1 - 1/n)^{-1}(Js/h)^2$. In fact, this short-range form of the correlation function agrees with our numerical results for all n in $3d$. For $n \geq 5$, the spin-spin correlation function at all distances can be very well fitted by $\langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle = \exp(-|\mathbf{r}_1 - \mathbf{r}_2|/R_f)$. A good agreement with this formula is illustrated by Fig. 5. So far we have been able to prove analytically the numerically confirmed exponential decay of the correlation function in $3d$ only for the mean-spherical model that corresponds to $n = \infty$ [19]. However, the observed exponential behavior of $\langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle$ and the observed $1/h^2$ dependence of R_f on the strength of the random field for small h at $n = 5$, $d = 3$ is clear evidence of the onset of the IM state in the absence of topological objects.

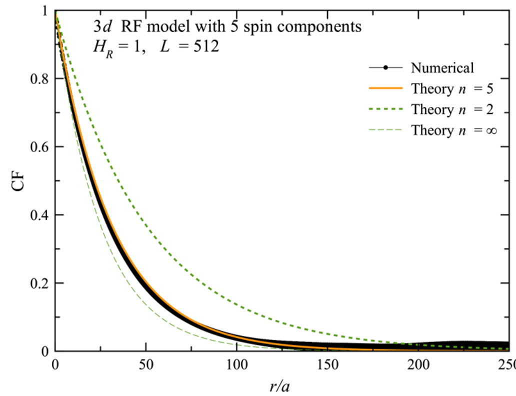

FIG. 5 (color online). Theoretical (see text) and numerical spin-spin correlation functions of the $3d$ random-field model at $n = 5$.

In conclusion, we have demonstrated that the topology of the order parameter controls whether the random-field system exhibits reversible or irreversible behavior. For the n-component spin in d dimensions, the presence of topological structures at $n \leq d + 1$ gives rise to vortex, hedgehog, and Skyrmion glasses. For $n > d + 1$, when topological structures are absent, the behavior of the system is reversible and spin-spin correlations agree quantitatively with the Imry-Ma picture. These findings provide the guiding principle for assessing the long-range behavior of various systems with quenched randomness and continuous-symmetry order parameter.

This work was supported by the Department of Energy through Grant No. DE-FG02-93ER45487.

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