

# Unifying Three Perspectives on Information Processing in Stochastic Thermodynamics

A. C. Barato and U. Seifert

*II. Institut für Theoretische Physik, Universität Stuttgart, 70550 Stuttgart, Germany*

(Received 19 August 2013; published 3 March 2014)

So far, feedback-driven systems have been discussed using (i) measurement and control, (ii) a tape interacting with a system, or (iii) by identifying an implicit Maxwell demon in steady-state transport. We derive the corresponding second laws from one master fluctuation theorem and discuss their relationship. In particular, we show that both the entropy production involving mutual information between system and controller and the one involving a Shannon entropy difference of an information reservoir like a tape carry an extra term different from the usual current times affinity. We, thus, generalize stochastic thermodynamics to the presence of an information reservoir.

DOI: [10.1103/PhysRevLett.112.090601](https://doi.org/10.1103/PhysRevLett.112.090601)

PACS numbers: 05.70.Ln, 05.40.-a, 89.70.Cf

A deep relation between information theory and statistical physics has been apparent from the very conception of the former in Shannon's classical formulation [1,2]. One explicit manifestation is Bennett's insight on how Landauer's result on the thermodynamic cost of erasing memory exorcises Maxwell's demon [3,4]. While, thus, the universal validity of the second law has apparently been restored, exploring the specific relationship between information theory and thermodynamics particularly in small systems has become a very active field not the least since ingenious experiments with single colloidal particles provide beautiful illustrations and a test of these concepts [5,6]. If the arguably bewildering plethora of recent theoretical work in this field [7–34] is tentatively classified into three main approaches, an important question on the uniqueness of the second law arises as follows.

In the first and most prominent approach, the classical ideas of Maxwell and Szilard are implemented in an explicit feedback scheme where immediately after a measurement some parameters of the device are altered depending on the outcome of the measurement [7–23]. The subsequent evolution of the system, thus, depends on the state after the measurement and the new control parameter. For such a setup, Sagawa and Ueda have derived an integral fluctuation theorem (FT) [11]. The corresponding inequality implies that the extracted work, which from the perspective of the first law is compensated by a corresponding heat transfer from the bath, is less than the information acquired in the measurement [10]. This inequality, thus, generalizes the second law to such feedback-driven schemes. Since the thermodynamic costs of neither the measurement nor of the erasure of the acquired information are included, the analysis is necessarily somewhat incomplete from a thermodynamic point of view.

A second approach where the system is allowed to interact explicitly with an information storage device such as a tape consisting of a sequence of bits overcomes this deficiency. In such a scheme, an inequality has been

derived that shows that the work extracted from a heat bath is necessarily less than the information theoretic entropy difference between outgoing and incoming tape [24] (see also Refs. [25,26]). How is this inequality related to the one derived in the first approach? Does it also follow from an underlying fluctuation theorem?

In the third approach, “ordinary” transport through a device such as a quantum dot controlled by a gate is considered [27,28]. The corresponding nonequilibrium steady state (NESS) complies with the well-established rules of stochastic thermodynamics including a well-defined rate of thermodynamic entropy production [35]. *A posteriori*, however, a term in the entropy production is interpreted as an “information current,” which is related to an idealized feedback procedure happening much faster than the time scales for the transitions between states. Again, the question arises whether and how the genuine thermodynamic entropy production of a NESS relates to the second laws discussed within the first two approaches. A recent work in this direction compares this genuine entropy production with the one arising from the first approach by considering two models with similar dynamics that can be either driven by an input of chemical work or by feedback [29].

In this Letter, we will show that the second laws arising from these three approaches are in fact three different inequalities involving three different quantities each bounding the maximal extractable work from such devices. We will do so by first discussing the simplest paradigmatic device based on a two-level system from all three perspectives. For a system with an arbitrary number of states, we then derive one master FT that can be specialized to yield the three second laws pertaining to the three perspectives discussed above. On the basis of these insights, we can thus generalize stochastic thermodynamics to include an information reservoir, like a tape that mediates transitions between a pair of states in a general NESS. Surprisingly, differing from the usual thermodynamic

entropy production that can be written as a sum of currents multiplied by affinities, the contribution due to the information reservoir does not involve a current.

Let us set the stage with a paradigmatic two-level system [10,29,36,37]. The upper level  $u$  has energy  $E > 0$ , and the lower level  $d$  has energy 0. The system is connected to a heat bath at temperature  $T$  so that the transition rates fulfill the detailed balance relation  $k_+/k_- = \exp(-E)$ , where we set Boltzmann's constant multiplied by the temperature to  $k_B T \equiv 1$ ,  $k_+$  is the transition rate from  $d$  to  $u$ , and  $k_-$  is the reversed one. The feedback is introduced in the following way. After every period  $t$ , a measurement gives information to a controller whether the state of the system is  $d$  or  $u$ . If the measurement is error free and if at the end of the time interval the system is at  $u$ , the energy of the upper level is lowered to 0, leading to the extraction of work  $E$ . Furthermore, the energy of the empty state is elevated to  $E$  at no cost. This instantaneous change in the energy levels of the system corresponds to an effective jump from state  $u$  to state  $d$ , because after the energy levels are switched the labels are also switched with  $d$  always representing the state with energy 0 and  $u$  the state with energy  $E$ .

More generally, we assume a probability of a measurement error given by  $\epsilon$  so that if the state at time  $t$  is  $x = u$  ( $x = d$ ) the measurement yields  $y = u$  ( $y = d$ ) with probability  $1 - \epsilon$  and  $y = d$  ( $y = u$ ) with probability  $\epsilon$ . Moreover, for  $y = u$  the energy levels are interchanged and for  $y = d$  they remain fixed. Note that whenever an error occurs the initial state in the next time interval is  $u$ . Hence, the system reaches a periodic steady state for which the probability of finishing the period at state  $u$  is

$$p_t = p + (\epsilon - p) \exp(-kt), \quad (1)$$

where  $k \equiv k_+ + k_-$  and  $p \equiv k_+/k$ . The mean extracted work per time interval  $t$  is given by

$$W_t = E[p_t(1 - \epsilon) - (1 - p_t)\epsilon] = E[p_t - \epsilon]. \quad (2)$$

The probability of the measurement outcome  $y = u$  is

$$q_t = p_t(1 - \epsilon) + (1 - p_t)\epsilon = p_t + \epsilon(1 - 2p_t), \quad (3)$$

whereas the probability for  $y = d$  is  $1 - q_t$ . Therefore, the Shannon entropy of the controller is  $H_y = H(q_t) \equiv -q_t \ln q_t - (1 - q_t) \ln(1 - q_t)$ . This Shannon entropy conditioned on the state of the system  $x$  becomes  $H_{y|x} = H(\epsilon)$  (see the Supplemental Material [38]). Using the standard definition for the mutual information  $I_t$  between the system  $x$  and the controller  $y$  [39], we obtain  $I_t \equiv H_y - H_{y|x} = H(q_t) - H(\epsilon)$ . The second law of thermodynamics for feedback controlled systems then implies [10]

$$I_t - W_t = H(q_t) - H(\epsilon) - W_t \geq 0; \quad (4)$$

i.e., the extracted work is bounded by the mutual information due to measurements.

This very model allows a second interpretation that leads to another second-law-like inequality; see Fig. 1. We now consider a system connected to a thermal bath, mediating the interaction between a work reservoir and a tape (a sequence of bits), which corresponds to a simplified version of the original model proposed in Ref. [24]. In this interpretation, each bit from the tape interacts for a time  $t$  with the system. During this time interval, the bit state 0 (1) is coupled to the system state  $d$  ( $u$ ) so that when the system jumps from  $d$  ( $u$ ) to  $u$  ( $d$ ) the bit changes from 0 (1) to 1 (0). After interacting with the system for a time  $t$ , the bit moves forward, and a new bit comes to interact with the tape. This new incoming bit generates effective transitions by determining the initial state of the system for the subsequent interaction time interval, where for an incoming 0 the system will start at  $d$  and for an incoming 1 at  $u$ . More precisely, if the system finishes in state  $u$  ( $d$ ) and the new incoming bit is 0 (1) then the energy levels are interchanged, leading to an amount of energy  $E$  extracted from (delivered to) the system. If the system finishes in state  $d$  ( $u$ ) and the new incoming bit is 0 (1), then the energy levels remain fixed and no energy is exchanged with the work reservoir. Furthermore, the probability of an incoming 1 is  $\epsilon$  and, hence, the Shannon entropy of the incoming tape is  $H(\epsilon)$ . On the other hand, the outgoing tape is a record of the interaction with the system, with the probability of a 1 being  $p_t$  and the Shannon entropy  $H(p_t)$ . Importantly, in this second interpretation, we have an autonomous system with no explicit measurement and feedback; the new incoming bit simply determines the initial state of the system for the coming interaction period.

Mandal and Jarzynski [24] showed that a second-law-like inequality bounds the work (2) delivered to the reservoir by the Shannon entropy difference between the incoming and the outgoing tapes, i.e.,

$$H(p_t) - H(\epsilon) - W_t = D_{\text{KL}}(\epsilon||p) - D_{\text{KL}}(p_t||p) \geq 0, \quad (5)$$

where  $D_{\text{KL}}(x||y) \equiv x \ln(x/y) + (1 - x) \ln$  is the Kullback-Leibler distance. Hence, as our first result, we realize that this two-level system allows for two different

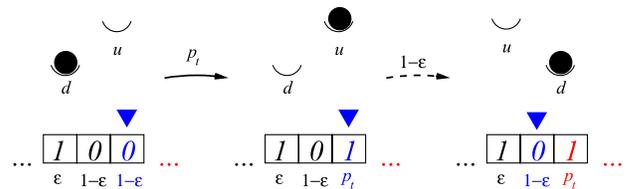


FIG. 1 (color online). Two-state model with feedback at fixed time intervals interpreted as a tape interacting with a heat bath and a work reservoir. The full arrow represents an interaction time interval  $t$ , and the dotted arrow represents an instantaneous effective transition due to the new incoming bit.

interpretations leading to two different second-law-like inequalities. Since  $H(p_t) \leq H(q_t)$  (Supplemental Material [38]), the bound on the extracted work in Eq. (5) is tighter than that in Eq. (4). Moreover, while the mutual information  $I_t$  is always non-negative,  $H(p_t) - H(\epsilon)$  can be negative. Hence, for  $p_t < \epsilon < 1/2$  the work delivered to the system  $-W_t$  is used to erase information, with the reduction of the Shannon entropy of the tape being bounded by  $-W_t$ , as given by Eq. (5) (see Refs. [24,25]). Such information erasure cannot be addressed within the second-law inequality of Eq. (4) as  $I_t \geq 0$ .

Preparing for a third perspective on this model as a NESS, we assume that the feedback procedure does not take place at constant time intervals  $t$  but that it is rather a Poisson process with rate  $\gamma$ . Consequently, the previous expressions obtained for a fixed  $t$  must be weighted with  $e^{-\gamma t}$ . The average extracted work then becomes with Eq. (2)

$$W_\tau \equiv \gamma \int_0^\infty dt \exp(-\gamma t) W_t = E(p_\tau - \epsilon) \quad (6)$$

where  $\tau \equiv k/(k + \gamma)$  and  $p_\tau \equiv \tau p + (1 - \tau)\epsilon$ . Using the inequality  $\int_0^\infty dt \gamma \exp(-\gamma t) H(p_t) \leq H(p_\tau)$  for the concave function  $H(x)$ , the second-law inequality of Eq. (5) can be written in the form

$$\dot{s}_1 \equiv \gamma(H(p_\tau) - H(\epsilon) - W_\tau) \geq 0, \quad (7)$$

where  $\dot{s}_1$  represents a rate of entropy production. Analogously, the inequality involving the mutual information of Eq. (4) becomes

$$\dot{s}_2 \equiv \gamma(H(q_\tau) - H(\epsilon) - W_\tau) \geq 0, \quad (8)$$

where  $q_\tau = p_\tau + \epsilon(1 - 2p_\tau)$ .

The NESS description of this model then follows by considering two states  $d$  and  $u$  with two links between them. One link is related to the thermal reservoir, and the corresponding transitions rates are  $k_+$  and  $k_-$  as before. The other link is related to the effective transitions mediated by the tape with transition rates  $\gamma\epsilon$  and  $\gamma(1 - \epsilon)$ . The master equation for this model is analogous to the master equation for the previous model with feedback. More precisely, the stationary state probability distribution is  $P_u = p_\tau$ . The rate at which work is delivered to the mechanical reservoir is  $\dot{w} = \gamma W_\tau$ . The usual rate of thermodynamic entropy production specialized to this NESS becomes [35]

$$\begin{aligned} \dot{s} &= k[p_\tau(1 - p) - (1 - p_\tau)p] \ln \frac{1 - p}{p} \\ &\quad + \gamma[p_\tau(1 - \epsilon) - (1 - p_\tau)\epsilon] \ln \frac{1 - \epsilon}{\epsilon} \\ &= \gamma \left[ (p_\tau - \epsilon) \ln \frac{1 - \epsilon}{\epsilon} - W_\tau \right] \geq 0. \end{aligned} \quad (9)$$

For  $\epsilon \rightarrow 0$  this thermodynamic entropy production diverges in contrast to both Eq. (8), for which  $\epsilon = 0$  implies error free measurements, and Eq. (7) for which  $\epsilon = 0$  means a fully ordered incoming tape. In the first case, the physical reason for this very different behavior comes from the fact that Eq. (8) does not contain the thermodynamic cost of acquiring or erasing information [16,21,29]. In the second case, a remarkable result is obtained if we compare Eq. (7) with Eq. (9). Let us consider  $\epsilon < p_\tau < 1/2$  so that the flow of work to the mechanical reservoir is positive. The minimal rate of work  $\dot{w}_c$  that would have to be provided by the mechanical reservoir in order to restore the original tape (with a fraction  $\epsilon$  of 1's) from the processed tape (with a fraction  $p_\tau$  of 1's) is obtained in the adiabatic limit  $k \gg \gamma$  with  $E = \ln[(1 - \epsilon)/\epsilon]$ . According to Eq. (7), it is given by  $\dot{w}_c = \gamma(p_\tau - \epsilon) \ln[(1 - \epsilon)/\epsilon] \geq \gamma(H(p_\tau) - H(\epsilon))$ . Thus, if we apply Eq. (7) twice, first for extracting work at the expense of increasing the entropy in the tape and second for restoring the original tape by applying mechanical work in the limit  $k \gg \gamma$ , we find for the total entropy production the bound of Eq. (9). Hence, the NESS description contains the full thermodynamic cost including the one for restoring the original tape. This observation shows that in a fully integrated description, an error-free or perfect tape scheme implies an infinite thermodynamic cost somewhere else, as noted previously for a particular case study in Ref. [28].

Leaving the paradigmatic two-state system, we now derive a master fluctuation theorem for a general Markov process with transition rates from generic states  $i$  to  $j$  denoted by  $W_{ij}$ , which will lead to the generalized version of the three entropy productions of Eqs. (7)–(9). In stochastic thermodynamics, the transitions rates are related to reservoirs, with the ratio  $W_{ij}/W_{ji}$  given by the local detailed balance condition [35]. For simplicity, we consider the case where there is at most one link for each pair of states except for one pair. Denoting the two states of this special pair by  $u$  and  $d$ , besides the ordinary transition rates  $W_{ud}$  and  $W_{du}$  (which can be zero), there are also rates  $R_{ud}$  and  $R_{du}$ , which will become related to an information reservoir.

In order to derive the master FT, it is convenient to formally duplicate the system; see Fig. 2. We represent the two copies of the system by the subscripts  $A$  and  $B$ . The “internal” transition rates  $W_{ij}$  are the same for both sides, and they involve states with the same subscript, i.e., from  $i_A$  to  $j_A$  or from  $i_B$  to  $j_B$ . The transition rates related to the information reservoir,  $R_{ud} = R_{dd} = \gamma(1 - \epsilon)$  and  $R_{du} = R_{uu} = \gamma\epsilon$ , must involve states with different subscripts. This description is clearly symmetric with the stationary state probability distribution fulfilling  $P_{i_A} = P_{i_B} = P_i/2$ , for all states  $i$ . Therefore, the stationary properties of the duplicated system and of the original one are the same.

We denote a stochastic trajectory from time 0 to  $T$  with  $N$  jumps visiting states  $x_n$  (with  $n = 0, 1, \dots, N$ ) by  $X_T$ .

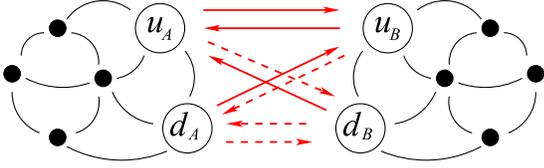


FIG. 2 (color online). Representation of the formal duplication of the system. The full circles represent generic states  $i$  different from  $u$  and  $d$ ; the curved lines represent links between states with nonzero transition rates  $W_{ij}$  and  $W_{ji}$ ; the arrows represent the transition rates between the replicas, with the full arrows representing  $\gamma\epsilon$  and the dotted arrows representing  $\gamma(1-\epsilon)$ .

The master FT is derived by considering a reversed trajectory  $\tilde{X}_T$  subjected to, in general, different transition rates, denoted by an overline, fulfilling the constraints  $\bar{W}_{ij} = W_{ij}$  and  $\bar{R}_{du} + \bar{R}_{dd} = \bar{R}_{ud} + \bar{R}_{uu} = \gamma$ . Considering the total internal current from  $i$  to  $j$

$$\mathcal{J}_{ij}[X_T] \equiv \sum_{n=0}^N \sum_{C=A,B} (\delta_{x_n, i_C} \delta_{x_{n+1}, j_C} - \delta_{x_n, j_C} \delta_{x_{n+1}, i_C}) \quad (10)$$

and the counter of jumps from  $i$  to  $j$  between the two replicas

$$\mathcal{K}_{ij}[X_T] \equiv \sum_{n=0}^N (\delta_{x_n, i_A} \delta_{x_{n+1}, j_B} + \delta_{x_n, i_B} \delta_{x_{n+1}, j_A}), \quad (11)$$

we define the functional

$$\Omega[X_T] \equiv \sum_{i<j} \mathcal{J}_{ij}[X_T] \alpha_{ij} + \sum_{ij} \mathcal{K}_{ij}[X_T] \beta_{ij}, \quad (12)$$

where  $\alpha_{ij} \equiv \ln(W_{ij}/W_{ji})$  and  $\beta_{ij} \equiv \ln(R_{ij}/\bar{R}_{ji})$ . The first sum is over all pairs  $ij$  with  $i < j$ , and the second constrained sum is over the states  $i = u, d$  and  $j = u, d$  ( $\mathcal{K}_{ij} = 0$  if  $i \neq u, d$  or  $j \neq u, d$ ). As a main result, we can show that  $\Omega$  obeys the integral FT  $\langle \exp(-\Omega) \rangle = 1$ , which implies  $\langle \Omega \rangle \geq 0$ , with the brackets representing an average over all trajectories (Supplemental Material [38]). From this inequality, the three second-law inequalities can be derived from three different choices of  $\bar{R}$  as follows (Supplemental Material [38]).

First, for  $R = \bar{R}$ , the well-known standard rate of entropy production generalizing Eq. (9) follows as

$$\dot{s} = \sum_{i<j} J_{ij} \alpha_{ij} + J'_{ud} \ln \frac{1-\epsilon}{\epsilon} \geq 0, \quad (13)$$

where  $J_{ij} \equiv P_i W_{ij} - P_j W_{ji}$  and  $J'_{ud} \equiv \gamma(P_u + P_d)(p_\tau - \epsilon)$ , with  $p_\tau \equiv P_u/(P_u + P_d)$ .

Second, choosing

$$\bar{R}_{uu} = \bar{R}_{du} = \gamma p_\tau \quad \text{and} \quad \bar{R}_{dd} = \bar{R}_{ud} = \gamma(1-p_\tau), \quad (14)$$

we obtain as the generalization of Eq. (7)

$$\dot{s}_1 = \sum_{i<j} J_{ij} \alpha_{ij} + \gamma(P_u + P_d)[H(p_\tau) - H(\epsilon)] \geq 0. \quad (15)$$

This second-law inequality generalizes the theory of stochastic thermodynamics to the presence of an information reservoir. In this entropy production, the term related to the transitions that are mediated by the tape is not the probability current  $J'_{ud}$  multiplied by the affinity  $\ln[(1-\epsilon)/\epsilon]$  that appears in the usual entropy rate (13). It is rather given by the rate at which the tape is processed  $\gamma(P_u + P_d)$ , multiplied by the Shannon entropy difference of the processed tape. Crucially, this quantity is not antisymmetric and, therefore, it is not subjected to the conservation laws of probability currents [40,41]. This observation demonstrates that a formulation of the second law containing Shannon entropy differences related to information reservoirs is fundamentally different from the ordinary thermodynamic entropy production.

The physical meaning of the choice of Eq. (14) becomes clear if we consider the two-state model again. The FT leading to the inequality of Eq. (7) is obtained by considering a reversed trajectory where the probability of a 1 in the incoming tape is  $p_\tau$ . If we go back to the initial model with feedback at fixed time intervals  $t$ , our FT is obtained by applying feedback also to the reversed trajectory [19]; however, the probability of an error for the reversed trajectory is chosen as  $p_\tau$  rather than  $\epsilon$ . This is different from the Sagawa-Ueda FT, where there is no feedback in the reversed trajectory [13].

Third and finally, the NESS version of the Sagawa-Ueda FT is obtained with transition rates  $\bar{R}$  corresponding to a ‘‘protocol’’ in the reversed trajectory determined by the measurements along the forward trajectory [13]. Therefore, with  $\bar{R}_{ud} = \bar{R}_{du} = \gamma q_\tau$  and  $\bar{R}_{uu} = \bar{R}_{dd} = \gamma(1-q_\tau)$ , where  $q_\tau = p_\tau + \epsilon(1-2p_\tau)$ , we obtain

$$\dot{s}_2 = \sum_{i<j} J_{ij} \alpha_{ij} + \gamma(P_u + P_d)[H(q_\tau) - H(\epsilon)] \geq 0, \quad (16)$$

which becomes Eq. (8) for the two-state model. The particular term  $\gamma(P_u + P_d)[H(q_\tau) - H(\epsilon)]$  in the entropy production  $\dot{s}_2$  accounts for the mutual information between the system and the information reservoir.

The three different entropy productions obey the relations

$$\dot{s} - \dot{s}_1 = \gamma(P_u + P_d) D_{\text{KL}}(p_\tau || \epsilon) \geq 0 \quad (17)$$

and

$$\dot{s}_2 - \dot{s}_1 = \gamma(P_u + P_d)[H(q_\tau) - H(p_\tau)] \geq 0, \quad (18)$$

which show that  $\dot{s}_1$  provides the tightest bound on  $\sum_{i<j} J_{ij} \alpha_{ij}$ . On the other hand, there is no general

inequality between  $\dot{s}$  and  $\dot{s}_2$ , as noted previously for the two-state model in the limit  $k \gg \gamma$  in Ref. [29].

In conclusion, our unified perspective on three different approaches to feedback-driven systems has revealed that the corresponding expressions for entropy production are genuinely different despite the fact that we could derive all of them from one master FT. Significantly, neither the one containing the Shannon entropy difference of an information reservoir like a tape of bits interacting with the system nor the one containing mutual information between a controller and the system can be written in the standard form of a current times an affinity. This result points *inter alia* to a conceptual challenge for a future comprehensive linear response theory of information processing. Apparently, an information reservoir like a tape has features that are fundamentally different from those of a heat or particle reservoir. Whether allowing the tape to reverse its direction will suffice to restore an ordinary thermodynamic behavior as found in the case study [25] remains to be seen. Finally, the second-law inequality of Eq. (15) provides a general framework to study the entropic interaction between a tape and a thermodynamic system. Two examples are the paradigmatic two-state model where this entropic interaction generates a flow of work to a mechanical reservoir (or lifts a falling mass [24]) and a refrigerator powered by it [26].

Support by the ESF through the network EPSD is gratefully acknowledged. We thank D. Hartich and D. Abreu for helpful discussions.

- 
- [1] C. E. Shannon, *Bell Syst. Tech. J.* **27**, 379 (1948).  
 [2] E. T. Jaynes, *Phys. Rev.* **106**, 620 (1957).  
 [3] C. H. Bennett, *Int. J. Theor. Phys.* **21**, 905 (1982).  
 [4] K. Maruyama, F. Nori, and V. Vedral, *Rev. Mod. Phys.* **81**, 1 (2009).  
 [5] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, *Nat. Phys.* **6**, 988 (2010).  
 [6] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, *Nature (London)* **483**, 187 (2012).  
 [7] H. Touchette and S. Lloyd, *Phys. Rev. Lett.* **84**, 1156 (2000).  
 [8] H. Touchette and S. Lloyd, *Physica (Amsterdam)* **331A**, 140 (2004).  
 [9] F. J. Cao, L. Dinis, and J. M. R. Parrondo, *Phys. Rev. Lett.* **93**, 040603 (2004).  
 [10] F. J. Cao and M. Feito, *Phys. Rev. E* **79**, 041118 (2009).  
 [11] T. Sagawa and M. Ueda, *Phys. Rev. Lett.* **104**, 090602 (2010).  
 [12] M. Ponmurugan, *Phys. Rev. E* **82**, 031129 (2010).  
 [13] J. M. Horowitz and S. Vaikuntanathan, *Phys. Rev. E* **82**, 061120 (2010).  
 [14] J. M. Horowitz and J. M. R. Parrondo, *Europhys. Lett.* **95**, 10005 (2011).  
 [15] J. M. Horowitz and J. M. R. Parrondo, *New J. Phys.* **13**, 123019 (2011).  
 [16] L. Granger and H. Kantz, *Phys. Rev. E* **84**, 061110 (2011).  
 [17] D. Abreu and U. Seifert, *Europhys. Lett.* **94**, 10001 (2011).  
 [18] D. Abreu and U. Seifert, *Phys. Rev. Lett.* **108**, 030601 (2012).  
 [19] A. Kundu, *Phys. Rev. E* **86**, 021107 (2012).  
 [20] T. Sagawa and M. Ueda, *Phys. Rev. E* **85**, 021104 (2012).  
 [21] T. Sagawa and M. Ueda, *Phys. Rev. Lett.* **109**, 180602 (2012).  
 [22] M. Bauer, D. Abreu, and U. Seifert, *J. Phys. A* **45**, 162001 (2012).  
 [23] L. B. Kish and C. G. Granqvist, *Europhys. Lett.* **98**, 68001 (2012).  
 [24] D. Mandal and C. Jarzynski, *Proc. Natl. Acad. Sci. U.S.A.* **109**, 11641 (2012).  
 [25] A. C. Barato and U. Seifert, *Europhys. Lett.* **101**, 60001 (2013).  
 [26] D. Mandal, H. T. Quan, and C. Jarzynski, *Phys. Rev. Lett.* **111**, 030602 (2013).  
 [27] M. Esposito and G. Schaller, *Europhys. Lett.* **99**, 30003 (2012).  
 [28] P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, *Phys. Rev. Lett.* **110**, 040601 (2013).  
 [29] J. M. Horowitz, T. Sagawa, and J. M. R. Parrondo, *Phys. Rev. Lett.* **111**, 010602 (2013).  
 [30] L. Granger and H. Kantz, *Europhys. Lett.* **101**, 50004 (2013).  
 [31] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, *Phys. Rev. Lett.* **98**, 080602 (2007).  
 [32] M. Esposito and C. Van den Broeck, *Europhys. Lett.* **95**, 40004 (2011).  
 [33] D. Andrieux and P. Gaspard, *Proc. Natl. Acad. Sci. U.S.A.* **105**, 9516 (2008).  
 [34] D. Andrieux and P. Gaspard, *Europhys. Lett.* **103**, 30004 (2013).  
 [35] U. Seifert, *Rep. Prog. Phys.* **75**, 126001 (2012).  
 [36] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, *Europhys. Lett.* **89**, 20003 (2010).  
 [37] N. Kumar, C. Van den Broeck, M. Esposito, and K. Lindenberg, *Phys. Rev. E* **84**, 051134 (2011).  
 [38] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.112.090601> for details.  
 [39] T. M. Cover and A. J. Thomas, *Elements of Information Theory* (Wiley-Interscience, Hoboken, NJ, 2006), 2nd ed.  
 [40] J. Schnakenberg, *Rev. Mod. Phys.* **48**, 571 (1976).  
 [41] A. C. Barato and R. Chétrite, *J. Phys. A* **45**, 485002 (2012).