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Improved Quantum Metrology Using Quantum Error Correction

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We consider quantum metrology in noisy environments, where the effect of noise and decoherence limits the achievable gain in precision by quantum entanglement. We show that by using tools from quantum error correction this limitation can be overcome. This is demonstrated in two scenarios, including a many-body Hamiltonian with single-qubit dephasing or depolarizing noise and a single-body Hamiltonian with transversal noise. In both cases, we show that Heisenberg scaling, and hence a quadratic improvement over the classical case, can be retained. Moreover, for the case of frequency estimation we find that the inclusion of error correction allows, in certain instances, for a finite optimal interrogation time even in the asymptotic limit.

Parameter estimation is a problem of fundamental importance in physics, with widespread applications in gravitational-wave detectors [1,2], frequency spectroscopy [3,4], interferometry [5,6], and atomic clocks [7,8]. Quantum metrology offers a significant advantage over classical approaches, where the usage of quantum entanglement leads to an improved scaling in the achievable precision [9,10]. However, noise and decoherence jeopardize this effect, reducing the quadratic improvement with system size to only a constant gain factor in many scenarios [10–12].

General upper bounds on the possible gain have been derived suggesting that no improvement in the scaling of precision is possible in the presence of uncorrelated, Markovian noise including local depolarizing or dephasing noise [11,12]. For non-Markovian noise [13], and noise with a preferred direction transversal to the Hamiltonian evolution [14], a scaling of $\mathcal{O}(N^{-3/4})$ and $\mathcal{O}(N^{-5/6})$ was found, respectively, where N denotes the number of probes (see also [15] for results on correlated noise). This is, however, still below the quadratic improvement attainable in the noiseless case. Moreover, for frequency estimation the optimal interrogation time, i.e., the optimal time to perform the measurement, tends to zero for large N in both these cases, making a physical realization for large N impractical.

In this Letter, we show that, by relaxing the restrictions implicit in standard quantum metrology, namely, that the only systems available are the N probes and the unitary dynamics are generated by local Hamiltonians, the no-go results for the case of uncorrelated, Markovian noise [10–12,14] can be circumvented, and Heisenberg scaling can be restored. Specifically, by encoding quantum information into several qubits, one can effectively reduce noise arbitrarily at the logical level, thereby retaining the Heisenberg limit in achievable precision. The required overhead is only logarithmic; i.e., each qubit is replaced by $m = \mathcal{O}(\log N)$ qubits. Moreover, we show that in the case of frequency estimation the optimal interrogation time in certain scenarios considered here is finite and independent of the system size, in stark contrast to all frequency estimation protocols studied to date. As the methods we employ can be readily implemented experimentally, at least for moderate system sizes, our result paves the way for the first feasible experimental realization of Heisenberg limited frequency estimation.

To be more precise, let us consider a system of Nm qubits which we imagine to be decomposed into N blocks of m qubits with m odd (see Fig. 1). First, we consider a class of many-body Hamiltonians, $H_I(m) = 1/2\sigma_z^{\otimes m}$, acting on each of the blocks, and uncorrelated, single-qubit dephasing or depolarizing noise (scenario I). Here, and in the following, $\sigma_{x,y,z}$ denote the Pauli operators. We show that, depending on the number of probe systems, N, one can choose a sufficiently large m [not exceeding $\mathcal{O}(\log N)$] such that the Heisenberg limit is achieved even in the presence of noise and that the optimal measurement time is constant. Furthermore, we generalize this model to arbitrary local noise and show that for short measurement times
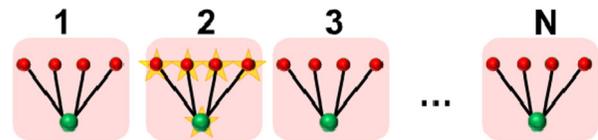
FIG. 1 (color online). Illustration of a quantum metrology scenario using error correction. We consider N blocks of size m (here $m = 5$). In scenario I, all particles in each block are affected by a Hamiltonian $H_I = 1/2\sigma_z^{\otimes m}$. In scenario II, only the lowest (green) particle of each block is affected by the Hamiltonian $H_{II} = 1/2\sigma_z^1$, and $m - 1$ ancilla particles (red) are used to generate an effective m-body Hamiltonian. In both scenarios, all particles are affected by (local) noise, and each block serves to encode one logical qubit.

the Heisenberg limit can be retrieved. Whereas this model may appear somewhat artificial, it nevertheless serves as a good example to illustrate how quantum error correction can be used to restore the Heisenberg scaling.

The second, and more physically important, scenario we consider is that of a local Hamiltonian, $H_{\mathrm{II}} = 1/2\sigma_z^{(1)}$, and local, transversal σ_x noise on all qubits. We show that this scenario can be mapped, for short times, to scenario I and hence demonstrate how quantum error correction (and other tools) can be used to arbitrarily suppress noise and restore Heisenberg scaling in precision just as in the noiseless case [16] . The key idea of our approach lies in the usage of auxiliary particles to encode and protect quantum information against the influence of noise and decoherence as done in quantum error correction. In addition, the encoding needs to be chosen in such a way that the Hamiltonian acts nontrivially onto the encoded states, such that the information on the unknown parameter is still imprinted onto the system. As long as H is many-body and the noise is local (scenario I), or the Hamiltonian is local and the noise is transversal (scenario II), both conditions can be met simultaneously.

Background.—We begin by describing the standard scenario in quantum metrology. A probe is prepared in a possibly entangled state of N particles and subsequently undergoes an evolution that depends on some parameter λ, after which it is measured. This process is repeated ν times, and λ is estimated from the statistics of the measurement outcomes. The achievable precision $\delta\lambda$ is lower-bounded by the quantum Cramér-Rao bound [17] $\delta\lambda \geq 1/\sqrt{\nu\mathcal{F}(\rho_\lambda)}$ with $\mathcal{F}$ the quantum Fisher information (QFI). For local Hamiltonians and uncorrelated (classical) probe states, $\mathcal{F} = O(N)$, leading to the so-called standard quantum limit. Entangled probe states, such as the Greenberger-Horne-Zeilinger (GHZ) state, lead to $\mathcal{F} = O(N^2)$, i.e., a quadratic improvement in precision, the so-called Heisenberg limit. In frequency estimation, time is also a variable that can be optimized, and the quantity of interest in this case is given by $\mathcal{F}/t$. We refer the reader to Ref. [18] for details.

In the presence of noise, however, a number of no-go results show that for many uncorrelated noise models, including dephasing and depolarizing noise, the possible quantum enhancement is limited to a constant factor rather than a different scaling with N [11,12]. To be more specific, we describe the time evolution of the state by a master equation of Lindblad form

$$\dot{\rho}(t) = -i\lambda[H,\rho] + \sum_{j=1}^{N}\mathcal{L}_j(\rho), \qquad (1)$$

where the action of the single qubit map $\mathcal{L}_j$ is given by

$$\mathcal{L}_j\rho = \frac{\gamma}{2}(-\rho + \mu_x\sigma_x^{(j)}\rho\sigma_x^{(j)} + \mu_y\sigma_y^{(j)}\rho\sigma_y^{(j)} + \mu_z\sigma_z^{(j)}\rho\sigma_z^{(j)}) \qquad (2)$$

and γ denotes the strength of the noise. The choice $H = H_0 = 1/2\sum_i\sigma_z^{(i)}$ and $\mu_z = 1$, $\mu_x = \mu_y = 0$ corresponds to local unitary evolution and local, uncorrelated, and commuting dephasing noise scenario considered in Ref. [10], whereas for the same Hamiltonian the choice $\mu_x = 1$, $\mu_y = \mu_z = 0$ corresponds to transversal noise considered in Ref. [14]. The choice $\mu_x = \mu_y = \mu_z = 1/3$ corresponds to local depolarizing noise. We remark that this approach includes phase estimation for fixed $t = t_0$ and frequency estimation when t can be optimized.

For any such scenario investigated so far, the attainable precession scales worse than $\mathcal{O}(N^{-1})$, and the optimal interrogation time tends to zero whenever the noise is not vanishing (see [18] for details).

Quantum metrology with error correction.—We now demonstrate that error correction can be used to recover the Heisenberg limit in the presence of noise in the two scenarios (scenarios I and II) mentioned above. For the case of frequency estimation we show that, in certain scenarios, our technique asymptotically allows for a finite, nonzero optimal time to perform measurements in contrast to all current metrological protocols.

Scenario I.—The evolution of the Nm qubits is governed by the class of Hamiltonians (see Fig. 1) $H(m) = \frac{1}{2}\sum_{k=1}^{N}H_k$, $H_k = \sigma_z^{\otimes m}$, where H_k acts on block k. We assume locality with respect to the blocks; i.e., this situation is equivalent to having N, d-level systems with $d = 2^m$. We describe the overall dynamics by Eq. (1), where the decoherence mechanism is modeled by Eq. (2). In the noiseless case ($\gamma = 0$), the maximal attainable QFI is given by $\mathcal{F} = (\partial\theta/\partial\lambda)^2N^2$ and is obtained by a GHZ-type state $|\mathrm{GHZ}_L\rangle = (|0_L\rangle^{\otimes N} + |1_L\rangle^{\otimes N})/\sqrt{2}$, with $|0_L\rangle = |0\rangle^{\otimes m}$ and $|1_L\rangle = |1\rangle^{\otimes m}$.

Let us now consider the standard metrological scenario in the presence of local dephasing noise, acting on all qubits, where the noise operators commute with the Hamiltonian evolution. In this case, Eq. (1) can be solved analytically, and the resulting state is given by $\rho_\lambda(t) = \mathcal{E}_z(p)^{\otimes Nm}(U_\lambda|\psi\rangle\langle\psi|U_\lambda^\dagger)$, where $U_\lambda = \exp(-i\theta_\lambda H)$ and $\mathcal{E}_z(p)\rho = p\rho + (1-p)\sigma_z\rho\sigma_z$, with $p = (1 + e^{-\gamma t})/2$, are acting on all physical qubits. Phase estimation corresponds to the case where $t = t_0$, for some fixed time t_0, and the parameter to be estimated is $\theta_\lambda = \lambda$ resulting from the unitary evolution for time t_0. Note that in this case one can start directly with the equation for $\rho_\lambda(t)$, with p being time independent, and a time-independent gate $U_\lambda = \exp(-i\lambda H)$ (see [18], Sec. II). As the subsequent discussion is independent of whether p is time dependent or not, we simply write p in the following whenever it does not lead to any confusion.

We now encode each logical qubit in m physical qubits. On each block of m qubits we make use of an error-correction code, similar to the repetition code, capable of correcting up to $(m-1)/2$ phase-flip errors (recall that we chose m to be odd), with code words $|0_L\rangle = (|0_x\rangle^{\otimes m} + |1_x\rangle^{\otimes m})/\sqrt{2}$, $\quad |1_L\rangle = (|0_x\rangle^{\otimes m} - |1_x\rangle^{\otimes m})/\sqrt{2}$,

where $|0_x\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|1_x\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. The error-correction procedure consists of projecting onto subspaces $P_{\vec{k}}$, spanned by $\{\sigma_z^{\vec{k}}|0_x\rangle^{\otimes N}, \sigma_z^{\vec{k}}|1_x\rangle^{\otimes N}\}$, where $\vec{k} = (k_1, ..., k_m)$ with $k_i \in \{0, 1\}$. Here, $\sigma_z^{\vec{k}}$ denotes the m qubit local operator $\sigma_z^{k_1} \otimes \sigma_z^{k_2} ... \otimes \sigma_z^{k_m}$. After obtaining outcome $\vec{k}$ the correction operation $\sigma_z^{\vec{k}}$ is applied. As long as fewer than $(m-1)/2$ σ_z errors occur, we obtain no error at the logical level. Otherwise, a logical $\sigma_z^{(L)}$ error occurs. Hence, the noise at the logical level can again be described as logical phase-flip noise, $\mathcal{E}_z^{(L)}(p)(\rho) = p_L\rho + (1 - p_L)\sigma_z^{(L)}\rho\sigma_z^{(L)}$, with

$$p_L = \sum_{k=0}^{(m-1)/2} \binom{m}{k} p^{m-k}(1-p)^k, \qquad (3)$$

where $p_L > p$ for $p > 1/2$. For small errors, i.e., $(1 - p) \ll 1$, the Taylor expansion of p_L can be approximated by $p_L = 1 - \binom{m}{(m+1)/2}(1-p)^{(m+1)/2} + \mathcal{O}[(1-p)^{(m/2)+1}]$, to leading order in $(1 - p)$. That is, noise at the logical level is exponentially suppressed.

We now consider a logical GHZ state, $|\text{GHZ}_L\rangle = (|0_L\rangle^{\otimes N} + |1_L\rangle^{\otimes N})/\sqrt{2}$, as the input state [19]. At the logical level, H_k acts as a logical $\sigma_z^{(L)}$ operation, $H_k|0_L\rangle = |0_L\rangle$, $H_k|1_L\rangle = -|1_L\rangle$, and the (time) evolved state $|\psi_\lambda^L\rangle = U_\lambda|\text{GHZ}_L\rangle = (e^{-iN\theta_\lambda/2}|0_L\rangle^{\otimes N} + e^{iN\theta_\lambda/2}|1_L\rangle^{\otimes N})/\sqrt{2}$ remains within the logical subspace. The state is then subjected to phase noise acting on each of the qubits. After correcting errors within each block of m qubits, phase noise at the logical level is reduced (see above). The state after error correction is given by $\rho_\lambda^L = [\mathcal{E}_z^L(p_L)]^{\otimes N}(|\psi_\lambda^L\rangle\langle\psi_\lambda^L|)$. As a result, the situation is equivalent to the standard phase estimation scenario with a single-qubit, σ_z Hamiltonian and local phase noise, where the error probability is, however, exponentially suppressed.

Let us now bound the precision for both phase and frequency estimation. As ρ_λ is of rank 2, the Fisher information can be easily calculated [12] (see [18], Sec. II), and for phase estimation one finds $\mathcal{F}(\rho_\lambda) = (2p_L - 1)^{2N}N^2$. In contrast to the standard scenario, where the strength of the noise is independent of N, here p_L can be made arbitrarily close to 1. Hence, one encounters a quadratic scaling and thus recovers the Heisenberg limit. For any fixed value of p and m, we have Heisenberg scaling up to a certain, finite-system size $N_{\max}$. For example, for $p = 1 - 10^{-3}$ we find $(2p_L - 1) = 1 - \epsilon_L$ with $\epsilon_L \approx 6 \times 10^{-6}$, 2×10^{-8}, 1.3×10^{-15} for $m = 3$, 5, 11, respectively. Hence, $(2p_L - 1)^{2N} = \mathcal{O}(1)$, i.e., a constant close to 1, as long as $2N\epsilon_L \ll 1$. Thus, for N up to $N_{\max} = \mathcal{O}(1/\epsilon_L)$ our error-correction technique would yield Heisenberg scaling in precision. More importantly, if $m = \mathcal{O}(\log N)$, and by using the approximation

$\binom{m}{(m+1)/2} < 2^m$, it can be shown that $(2p_L - 1)^{2N} \to 1$ and $\mathcal{F} \approx N^2$ for $N \to \infty$ as long as $4N(2\sqrt{1-p})^m \ll 1$. Thus, the QFI can be stabilized, and the Heisenberg limit is attained, with only a logarithmic overhead [20].

If instead of phase estimation we consider frequency estimation, i.e., $\theta_\lambda = \lambda t$, we obtain (see [18], Sec. II) $\mathcal{F}(\rho_\lambda) = t^2[2p_L(t) - 1]^{2N}N^2$, where $2p_L(t) - 1 = e^{-\gamma_L(m,\gamma,t)t}$ and $\gamma_L(m, \gamma, t)$ is the noise parameter at the logical level. Assuming that $\gamma t \ll 1$, the optimization of $\mathcal{F}/t$ over t can be easily performed. Assuming that $m = \mathcal{O}(\log N)$, the optimal interrogation time and the bound on precision for an arbitrary number of m are presented in Ref. [18], Sec. II. We find that the optimal interrogation time decreases for larger system sizes N. However, t_{opt} gets larger with increasing m and can hence be much more feasible in practice. Assuming that $\gamma t \ll 1$ and $m = \mathcal{O}(\log N)$, p_L can be approximated by using Stirling's formula, and we find $t_{\text{opt}} = N^{-(2/m)}/2\gamma m^{2/m} \to 1/2\gamma e^2$. Thus the optimal measurement in our scenario can be performed at a finite time for large N. This is to be contrasted with the optimal times for previously considered frequency estimation scenarios, based on GHZ and other entangled states, where $t_{\text{opt}} \to 0$ for large N [10,14]. The maximum QFI per unit time is then given by $(\mathcal{F}/t)_{\text{opt}} = N^{2[1-(1/m)]}/2\gamma m^{2/m} \to N^2/2\gamma e^2$, and the Heisenberg limit is approached for $N \to \infty$.

In Ref. [18], Sec. III, we show that any kind of local error can be treated in this way. This is done by using an error-correction code that corrects for arbitrary single-qubit errors rather than just bit-flip errors and where the Hamiltonian still acts as a logical $\sigma_z^{(L)}$ operator on the codewords. We find that one obtains Heisenberg scaling for short measurement times $t \propto N^{-1/2}$.

Scenario II.—Let us now consider the physically more relevant scenario where the Hamiltonian is given by $H = H_0 = 1/2\sum_i\sigma_z^i$ and transversal noise [21].

We now show that the Heisenberg limit is attainable also in this case. To this aim, we attach to each of the system qubits $m - 1$ ancilla qubits, not affected by the Hamiltonian, that may also be subjected to (directed) local noise (see Fig. 1). In practice, this may be achieved by using qubits associated with different degrees of freedom (e.g., other levels in an atom) or another type of physical system. The situation is hence similar to scenario I; i.e., we have Nm qubits that are decomposed into N blocks of size m. The Hamiltonian is given by $H = \frac{1}{2}\sum_{k=1}^N H_k$, $H_k = \sigma_z^{(1)} \otimes I^{\otimes m-1}$ and we consider transversal noise acting on each of the Nm qubits; see Eqs. (1) and (2).

In the following, we show that the above situation can indeed by mapped precisely to the situation considered in scenario I. To this end, imagine that after preparing the entangled (encoded) resource state (i.e., a logical GHZ state $|\text{GHZ}_L\rangle$), we apply an entangling unitary operation $\mathcal{U}^\dagger$ to all qubits, allow them to freely evolve according to Eq. (1),

and apply $\mathcal{U}$ before the final measurement. The result is that the evolution takes place with respect to a unitarily transformed master equation $\dot{\rho} = -i\lambda[\tilde{H}, \rho] + \sum_{j=1}^{Nm} \tilde{\mathcal{L}}_j(\rho)$, where $\tilde{H} = \mathcal{U}H\mathcal{U}^\dagger$ and $\tilde{\mathcal{L}}_j\rho = (\gamma/2)[-\rho + (\tilde{\mathcal{U}}\sigma_x^{(j)}\tilde{\mathcal{U}}^\dagger)\times \rho(\tilde{\mathcal{U}}\sigma_x^{(j)}\tilde{\mathcal{U}}^\dagger)]$. Here, $\mathcal{U} = \otimes_{k=1}^N \mathcal{V}_k$ with $\mathcal{V}_k = \prod_{j=2}^m CX^{(1,j)}$, where $\mathcal{V}_k$ acts on a single block and $CX = (\text{Had} \otimes \text{Had})CP(\text{Had} \otimes \text{Had})^\dagger$ with $CP = \text{diag}(1, 1, 1, -1)$ the controlled phase gate and Had the Hadamard operation. The action of such a transformation has been studied and applied in the context of simulating many-body Hamiltonians [22]. It is straightforward to verify that [22] $\mathcal{U}H_k\mathcal{U}^\dagger = \mathcal{V}_k H_k \mathcal{V}_k^\dagger = \sigma_z^{(1)} \otimes \sigma_x^{\otimes m-1}$, $\mathcal{U}\sigma_x^{(j)}\mathcal{U}^\dagger = \mathcal{V}_k\sigma_x^{(j)}\mathcal{V}_k^\dagger = \sigma_x^{(j)}$, where the transformed Hamiltonian $\mathcal{U}H_k\mathcal{U}^\dagger$ acts within a block. Up to Hadamard operations on particles $2, ..., m$, this corresponds to the situation described in scenario I, i.e., an m-qubit Hamiltonian, $H_k = \sigma_z^{\otimes m}$, and local, single-qubit noise (X noise on particle 1 and Z noise on all ancilla particles). As shown in Ref. [18], Sec. III, one can achieve Heisenberg scaling for any local noise model by using logical GHZ states as input states. This implies that we also achieve Heisenberg scaling—at least for short measurement times $t \propto N^{-1/2}$ [23]—for transversal local noise, where the required block size is again $m = \mathcal{O}(\log N)$.

Experimental realization.—We now consider a simplified version of scenario II, where only particles that are affected by the Hamiltonian are affected by noise; i.e., noise is part of the coupling process, involving a two-qubit error-correction code which can be easily demonstrated experimentally. The error-correction code with $|0_L\rangle = |0\rangle|0_x\rangle$, $|1_L\rangle = |0\rangle|1_x\rangle$ as codewords is capable of correcting arbitrary σ_x errors occurring on the first qubit, while the Hamiltonian still acts as a logical σ_z^L after the transformation $\mathcal{U}$. This opens the way for simple proof-of-principle experiments in various setups, including trapped ions or photonic systems, where a total of $2N$ qubits prepared in a GHZ-type state suffices to obtain a precision $O(N^{-1})$.

Conclusion and outlook.—We have demonstrated that quantum error correction can be applied in the context of quantum metrology and allows one to restore Heisenberg scaling in several scenarios. This includes the estimation of the strength of a multiqubit Hamiltonian in the presence of arbitrary independent local noise, as well as a single-body Hamiltonian in the presence of transversal noise. In the latter case, an improvement in the precision from $\mathcal{O}(N^{-5/6})$, previously shown in Ref. [14], to $\mathcal{O}(N^{-1})$ is demonstrated. Furthermore, for frequency estimation we have shown that the interrogation time can be finite and independent of N in contrast to all previously known parameter estimation protocols. This demonstrates that, even though recent general bounds suggest a limitation of the possible gain in noisy quantum metrology to a constant factor for dephasing or depolarizing noise, this is actually not the case in general. It remains an open question whether tools from quantum error correction can also be applied in other metrology scenarios, most importantly in the context of estimating local Hamiltonians in the presence of parallel (phase) or depolarizing noise [24].

Note added.—After completing this work, we learned about independent work using similar approaches [25–27].

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[16] Note that the reason for obtaining Heisenberg scaling lies in the usage of error correction and not in the (logarithmic) increase of system size (which could only lead to a logarithmic improvement). In fact, the Hamiltonians we consider are such that the achievable precession in the noiseless case is independent of m. Moreover, it only depends linearly on N, which is also in contrast to the nonlinear metrology scheme studied in Ref. [28]. We show in both scenarios that we can obtain the same (optimal) precession as in the noiseless case.
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[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.112.080801 for explicit calculations of the optimal interrogation time and QFI for the various scenarios considered here, as well as how results can be extended to deal with arbitrary single qubit errors.

[19] See also Ref. [29] for studies on the stability of this state under noise.

[20] By logarithmic overhead we mean that each particle is replaced by $m = \mathcal{O}(\log N)$ particles. Note that in practice there is no need for a separate error-correction step followed by measurements to determine the parameter, but a single measurement with proper reinterpretation suffices.

[21] We remark that, in practical situations, parallel noise will often be dominant. The optimal measurement is typically transversal to the Hamiltonian, and imperfections in the measurement lead to parallel noise.

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[23] If $\delta t^2 N \ll 1$, then we have to take higher order terms in the solution of the master equation into account. This leads to parallel noise of $\mathcal{O}(\delta t^2)$ and limits the maximal N until which Heisenberg scaling can be achieved [14].

[24] Note that our results from scenario II cannot be directly applied in the case of parallel or depolarizing noise. Using a unitary transformation to obtain a many-body Hamiltonian also transforms parallel noise to correlated noise, that cannot be corrected by the error-correction code used here.

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