

Phononic Frequency Combs through Nonlinear Resonances

L. S. Cao,^{1,2,3,*} D. X. Qi,¹ R. W. Peng,^{1,†} Mu Wang,¹ and P. Schmelcher^{2,3}

¹National Laboratory of Solid State Microstructures and Department of Physics, Nanjing University, Nanjing 210093, China

²Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany

³The Hamburg Centre for Ultrafast Imaging, Luruper Chaussee 149, D-22761 Hamburg, Germany

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We explore an analogue of optical frequency combs in driven nonlinear phononic systems, and present a mechanism for generating phononic frequency combs through nonlinear resonances. In the underlying process, a set of phonon modes is simultaneously excited by the external driving which yields frequency combs with an array of discrete and equidistant spectral lines of each nonlinearly excited phonon mode. Frequency combs through nonlinear resonance of different orders are investigated, and in particular the possibility of correlation tailoring in higher-order cases is revealed. We suggest that our results can be applied in various nonlinear acoustic processes, such as phonon harvesting, and can also be generalized to other nonlinear systems.

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An optical frequency comb (FC) is a light source whose spectrum consists of a series of discrete, equally spaced elements. It has become an important coherent optical source with diverse applications, ranging from optical frequency metrology to ultracold gases [1,2]. The nonlinear parametric frequency conversion in microresonators of high resonance quality factor [3], manifests itself as a promising generating principle of FCs. In the underlying generating process, a continuous-wave (cw) laser is introduced to a microresonator, and is subsequently converted into the eigenmodes of the resonator with discrete, equidistant spectral lines. Despite the fact that phonons can be analogized to photons in many aspects, nonlinear dispersion relations of phonons prevent a direct analogue of optical FCs in phononic systems. In this Letter we provide a novel mechanism for the generation of FCs independent of the dispersion relation, and demonstrate its existence in a phononic nonlinear system, i.e., the Fermi-Pasta-Ulam α (FPU- α) chain.

The FPU- α (β) chains [4], containing triplet (quadruplet) nonlinear terms, manifest themselves as an ideal test bed of nonlinear phenomena, and the response of FPU- α (β) chains to external influence has been widely studied. The thermal equilibration and heat transport of chains coupled to a thermal reservoir [5–7], various nonlinear excitations, such as solitons [8,9], and the supertransmission [10,11] of FPU- α (β) chains under monochromatic driving are the representative phenomena of driven FPU systems. Nonlinear resonances (NRs) can take place between external driving and intrinsic phonon modes in driven FPU systems, and the corresponding phonons are simultaneously excited by external driving, when the driving frequency matches the sum of the eigenfrequencies of a set of phonons. The nonlinear resonance serves as the source of FCs being proposed in this Letter, and we will discuss the

generation of FCs through NR of different orders. The FPU- α chain is a nonlinear three-wave-mixing system, and the first-order NR refers to a pair of phonon modes simultaneously excited, which hereafter is called a direct nonlinear resonance (DNR). The second-order NR, in which a triple pair of phonon modes is excited by a cluster of nonlinear terms, is called a cluster nonlinear resonance (CNR). Once a phonon interacts strongly with atoms in a one-dimensional nonlinear system, its energy is converted into two or more parts, and two or more additional phonons are excited. This process is well known for photons, but it remains uncommon for phonons [12,13]. Harvesting phonons may have potential applications on acoustic amplification by stimulated radiation (analogous to lasers), thermal information transfer, nonlinear acoustics, and even for the entanglement of phonons.

In this Letter we consider an N -site FPU- α chain with a monochromatic driving force of strength f and frequency ω_d applied to the first site, which manifests as a driving for all phonon modes. With fixed boundary conditions on both ends of the chain, the Hamiltonian can be written in the phonon mode picture [14] as

$$H = \sum_{i=1}^N \left[\frac{P_i^2}{2} + \frac{\omega_i^2 Q_i^2}{2} + f_i \cos(\omega_d t) Q_i \right] + \sum_{i,j,k} \frac{\alpha B_{ijk}}{2(N+1)} Q_i Q_j Q_k. \quad (1)$$

The first term describes the linear phonon under monochromatic driving, and $Q_i(P_i)$ is the canonical coordinate (momentum) of the phonon, with $f_i = \sqrt{2/(N+1)} f \sin[\pi i/(N+1)]$. The second term is the nonlinear phonon interaction term, with $B_{ijk} = (\delta_{i\pm j\pm k,0} - \delta_{i\pm j\pm k,2(N+1)})$, where all possible sign combinations are allowed. Dimensionless units are used, so the

mass, the linear coupling constant, and the spacing of atoms in the chain are set to unity. The phonon mode frequency is given as $\omega_i = 2 \sin[\pi i/2(N+1)]$ for the fixed boundary conditions, while the effects discussed here are expected not to depend significantly on the choice of boundary conditions. We demonstrate here the existence and provide the physical properties of phononic FCs both analytically and numerically via the Poincaré-Lindstedt (PL) perturbation method [15] and time-integration of the classical equations of motion based on a Runge-Kutta method, respectively. In the numerical simulations FPU- α chains of 10 and 200 sites are considered.

Frequency combs through DNR.—The DNR of a driven FPU- α chain refers to the excitation of a pair of phonon modes (PMs) (Q_i, Q_j), when the frequency matching condition $\omega_d \approx \omega_i + \omega_j$ is fulfilled. Hereafter we denote PMs (Q_i, \dots, Q_l) as $Q_{i,\dots,l}$. ω_d is set beyond the dispersion band of the chain, in order to avoid the linear excitation of phonon modes close to ω_d . In addition to the nonlinear dispersion relation, we also assume $Q_{i,j}$ to form the only pair of modes, which is in nonlinear resonance with the external driving.

Let us consider a general DNR of $Q_{i,j}$ and assume this pair is coupled by the nonlinear term $\alpha/2(N+1)Q_i Q_j Q_k$. Now the DNR can be understood within the truncated phase space spanned by the three $Q_{i,j,k}$. Applying the PL method in the truncated space [15], $Q_{i(j)}$ is obtained as

$$Q_{i(j)} = A_0^{i(j)} \cos(\tilde{\omega}_{i(j)} t) + \sum_{p \neq 0} A_p^{i(j)} \cos(\tilde{\omega}_{i(j)} + p\Delta\omega)t, \quad (2)$$

with the amplitude $A_p^{i(j)} \sim [\tilde{\omega}_{i(j)}^2 - (\tilde{\omega}_{i(j)} + p\Delta\omega)^2]^{-1}$ and $\tilde{\omega}_{i(j)}$ being the renormalized intrinsic frequency of $Q_{i(j)}$. The solution (2) demonstrates that an array of equidistant spectral lines arises around the intrinsic frequency in the solutions of Q_i and Q_j with spacing $\Delta\omega = \omega_d - \tilde{\omega}_i - \tilde{\omega}_j$. These equidistant spectral lines manifest themselves as a new type of FC structure.

The generating process of FCs through DNR takes place when the summation mode of the external driving is suppressed by choosing the driving frequency beyond the dispersion band, and the driving can only excite eigenmodes Q_i and Q_j with $\omega_d \approx \omega_i + \omega_j$. On top of such a NR process, the frequency difference $\Delta\omega$ gives rise to a cascadelike generation of subspectral lines around $\tilde{\omega}_{i(j)}$ with equal distance $\Delta\omega$, i.e., the FCs. Comparing this to the existing generating mechanisms for FCs, such as parametric conversion in microcavities and higher harmonic generation in surface acoustic wave systems [16], FCs through DNR can greatly reduce the frequency separation between the spectral lines. This opens the possibility to improve precision in frequency metrology.

The FCs predicted by the PL method are confirmed by numerical simulations of a FPU- α chain of 10 sites, as shown in Fig. 1. All numerical simulations in this Letter are based on the complete Hamiltonian (1), and no truncation is applied in the simulations. We tune the external driving to let PMs Q_1 and Q_{10} in nonlinear resonance. The DNR of these two modes is quantified by their effective phonon number $N_{\text{eff}} = (P_i^2 + \omega_i^2 Q_i^2)/(2\omega_i)$, which is essentially the energy of the modes divided by the intrinsic frequency. The temporal

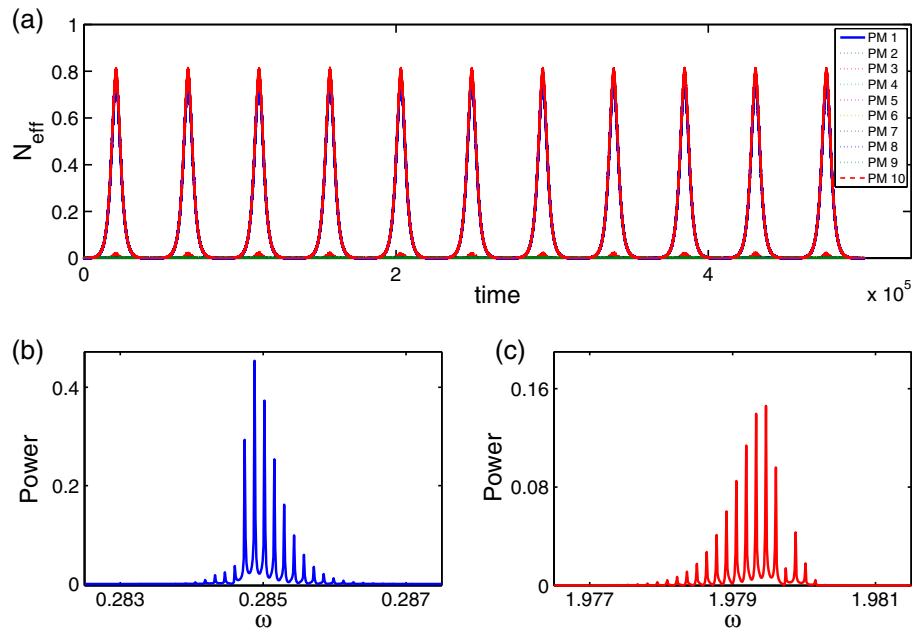


FIG. 1 (color online). Illustrations of DNR FCs of $Q_{1,10}$ in a ten-site FPU- α chain, with N_{eff} oscillations of all PMs shown in (a), and the spectra of $Q_{1,10}$ in (b) and (c), respectively. N_{eff} oscillation of $Q_{1,10}$ overlap, and the oscillation amplitude of other PMs is at least one order of magnitude smaller than those of $Q_{1,10}$. Parameters are $(f, \alpha, \omega_d) = (0.1, 0.1, 2.264)$, and the frequency spectra are obtained by fast Fourier transformation of $Q_{1(10)}(t)$.

evolutions of N_{eff} of $Q_{1,10}$ are shown in Fig. 1(a), and identical trains of temporal pulses are observed for the two modes. Such pulse trains arise exclusively for $Q_{1,10}$ and manifest themselves as the temporal fingerprint of FCs, of which the pulse period is given by $T = 2\pi/\Delta\omega$. The FCs can be more directly seen in the spectra of $Q_{1,10}$, as shown in Figs. 1(b) and 1(c), respectively, where an array of equidistant spectral lines (the FC) is observed in each figure. The numerical simulation, in this way, provides a clear temporal and spectral signature of the FCs. As also depicted in Fig. 1, the DNR FCs of $Q_{1,10}$ represent a dominating process: the amplitudes of other PMs are at least one order of magnitude smaller, thereby justifying the truncation of the small amplitude nonresonant modes, as applied in PL treatment.

It is important to study the dependence of $\tilde{\omega}_{i(j)}$ on ω_d . Though it is difficult to analytically derive $\tilde{\omega}_{i(j)}$ in PL method, as this involves solving highly nonlinear equations, a qualitative understanding can be obtained by the equations for $\tilde{\omega}_{i(j)}$ with the lowest order renormalization,

$$\tilde{\omega}_{i(j)}^2 - \omega_{i(j)}^2 = -\frac{F_k^2 \tilde{\omega}_{j(i)}^2}{2\omega_{j(i)}^2 [\tilde{\omega}_{j(i)}^2 - (\omega_d - \tilde{\omega}_{i(j)})^2]}, \quad (3)$$

with $F_k = f_k/(\omega_k^2 - \omega_d^2)$. Without explicitly solving the equations, we can see that $\tilde{\omega}_{i(j)}$ strongly depends on ω_d , and particularly the renormalization of the intrinsic frequency becomes larger when ω_d approaches the resonance, which effectively shifts the resonant point from ω_d . There are various mechanisms of frequency normalization in nonlinear phononic systems, such as that induced by the external-field [17,18] and that induced by thermal equilibrium [19]. Different from these mechanisms, here the renormalization in equation (3) is induced by the NR foldover effect [20].

Figure 2 shows numerical simulations on the generation of FCs of $Q_{1,10}$. Since the dynamical behavior of the two modes is almost identical, here we show the results for Q_1 only. Figures 2(a) and 2(b) illustrate the dependences of FCs of Q_1 on driving frequency ω_d and driving force f , respectively. In Fig. 2(a), the evolution of the spectrum can be divided into three different regimes with respect to ω_d , with two off-resonant and one resonant regime. In the off-resonant regimes of Fig. 2(a), two frequency branches, $\tilde{\omega}_1$ and $\omega_d - \tilde{\omega}_{10}$, can be identified, which resemble the PL solution up to the first-order perturbation theory. As approaching the resonant regime, a sharp transition to the comb structure occurs, and all the spectral lines become strongly dependent on ω_d , indicating the renormalization effect of equation (3). Fig. 2(b) shows that the separation of the combs can be tuned by the strength of the external driving, and a linearlike dependence is observed, which indicates a high tunability of the phononic FCs. The phononic FCs via DNR require a weak external driving, and Fig. 2 shows that FCs via DNR arise in a narrow resonant frequency window over a relatively broad regime of the driving strength f .

Frequency combs in CNR.—In an FPU- α chain, higher-order nonlinear resonances can occur with simultaneous excitation of more than two phonon modes. Such NRs occur via a cluster coupling of multiple nonlinear terms, and is termed as cluster nonlinear resonance (CNR). The CNR resembles the resonance clustering in the evolutionary dispersive wave systems [21,22] and also the effective many-body interactions intermediated by virtual excitations [23]. We demonstrate here the generation of FCs via CNR of a triplet pair of phonons.

The CNR of a triplet pair $Q_{i,j,k}$ and the consequent generation of FCs can also be interpreted by the PL method. The nonlinear resonance channel for a triplet pair with external driving is built by a series of nonlinear terms; for instance, the triplets $Q_{1,2,8}$ and $Q_{1,3,5}$ are

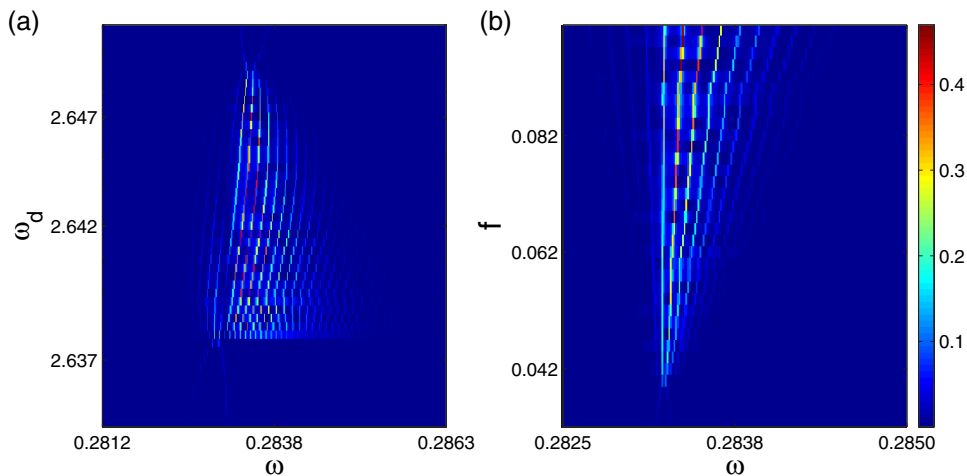


FIG. 2 (color online). The frequency spectrum of Q_1 versus the external driving frequency ω_d (a) and the driving strength f (b) in the vicinity of the DNR of $Q_{1,10}$, with the encoded color indicating the power of the spectrum.

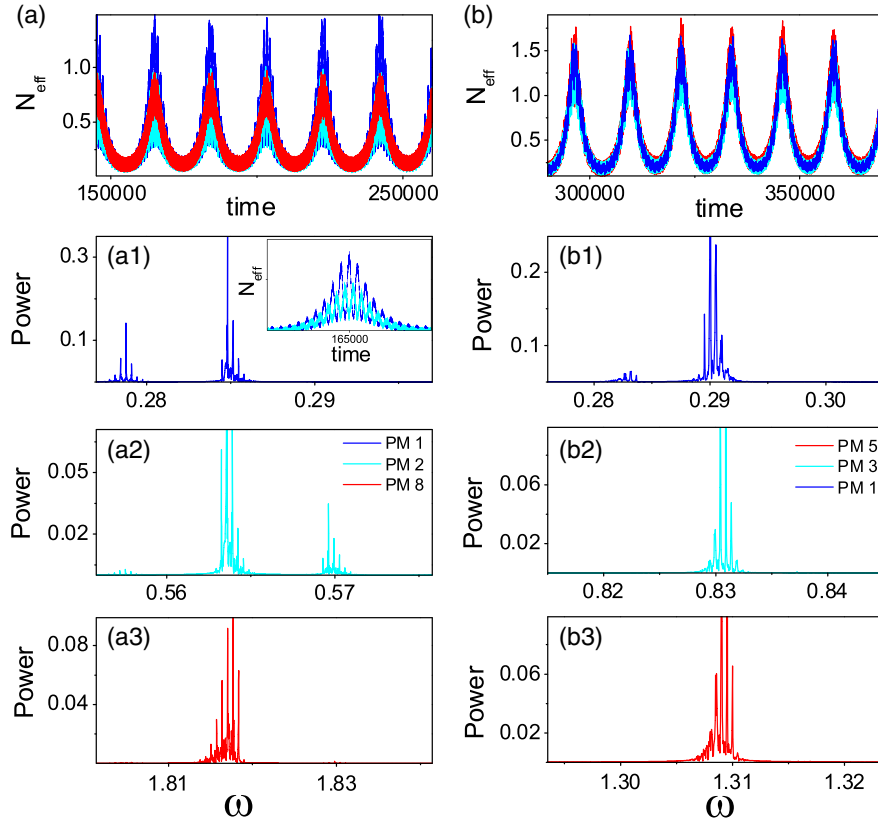


FIG. 3 (color online). CNR FCs of $Q_{1,2,8}$ and $Q_{1,3,5}$. (a) and (a1)–(a3) show N_{eff} oscillations of $Q_{1,2,8}$ and the spectra of corresponding PMs, respectively. (b) and (b1)–(b3) present N_{eff} oscillations and corresponding spectra of $Q_{1,3,5}$, respectively. The inset of (a1) shows the detailed structure of Q_1 and Q_2 within one broad pulse.

coupled to the external driving by the cluster channels ($Q_1Q_1Q_2$, $Q_1Q_2Q_3$, $Q_2Q_3Q_5$, $Q_3Q_5Q_8$) and ($Q_1Q_2Q_3$, $Q_2Q_3Q_5$), respectively. Applying the PL method in the truncated space spanned by triplet and intermediate phonons leads to a FC solution for the triplet, as $Q_\alpha^0 = \sum_n A_{\alpha,n}^0 \cos(\tilde{\omega}_\alpha - n\Delta\omega_{ijk})t$, $\Delta\omega_{ijk} = \omega_d - \tilde{\omega}_i - \tilde{\omega}_j - \tilde{\omega}_k$, with the amplitudes $A_{\alpha,n}^0 \sim [\tilde{\omega}_\alpha^2 - (\tilde{\omega}_\alpha - n\Delta\omega)^2]^{-1}$. At resonance $\Delta\omega_{ijk} \approx 0$, the amplitudes diverge and the CNR of $Q_{i,j,k}$ takes place with a series of spectral lines in the spectra of $Q_{i,j,k}$, and FCs arise in the spectra of the triplet phonons.

More importantly, the detailed properties of CNR channels give rise to additional terms in even higher-order expansions, and modify the comb structure. Taking the CNR of $Q_{1,2,8}$, for instance, higher-order expansions of the PL method predict a satellite comb $Q_\alpha^1 = \sum_n A_{\alpha,n}^1 \cos(\tilde{\omega}_\alpha \mp \Delta\omega' - n\Delta\omega)t$ ($\Delta\omega' = \tilde{\omega}_2 - 2\tilde{\omega}_1$, $\alpha \in \{1, 2\}$), which is attributed to the nonlinear term $Q_1Q_1Q_2$ and provides additional correlation of Q_1 and Q_2 . As this additional correlation is exclusively between Q_1 and Q_2 , such satellite combs appear in the spectra of Q_1 and Q_2 only, but not Q_8 . This example demonstrates the possibility of correlation tailoring in the CNR frequency comb generation by simply choosing triplet PM pairs with particular correlations in the CNR channels.

Figure 3 shows numerical results for FCs of $Q_{1,2,8}$ and $Q_{1,3,5}$ generated via related CNRs. In Fig. 3(a), oscillations of N_{eff} of $Q_{1,2,8}$ present a twofold structure on different time scales. First, on the long time scale, a train of broad pulses appears for each mode, whereas a fast oscillation of large amplitude can be observed within each broad pulse of Q_1 and Q_2 that measures a short time scale. The inset of Fig. 3(a1) illustrates fast oscillations of Q_1 and Q_2 in a broad pulse, with a π phase shift between the fast oscillations of the two PMs. Correspondingly, in the spectrum of each mode, there arises a main comb around the intrinsic frequency, and an additional satellite comb also appears in the spectra of Q_1 and Q_2 . The temporal broad pulses of the three PMs are related to the main combs of these modes, and the fast oscillations of Q_1 and Q_2 within the broad pulse are attributed to the appearance of the satellite combs. The π phase shift of the fast oscillations of Q_1 and Q_2 is given by the opposite location of the satellite combs with respect to the main combs. The above temporal and spectral properties are in agreement with the prediction of the PL method.

In Fig. 3(b) we observe a train of broad pulses with relatively small fluctuation for the N_{eff} oscillations of $Q_{1,3,5}$, and there is only one main comb in the spectrum of each PM, as shown in Figs. 3(b1)–3(b3). The small

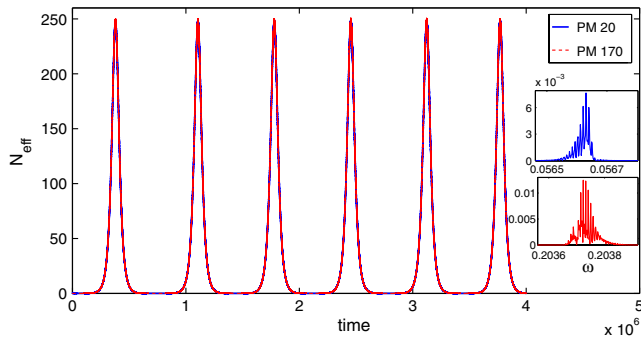


FIG. 4 (color online). DNR FCs of $Q_{20,170}$ of an FPU- α chain of 200 sites. The main graph shows the oscillation $N_{\text{eff}}(t)$ of both PMs, and oscillation of the two modes lies on top of each other. The insets present the frequency spectra of $Q_{20,170}$, respectively.

fluctuation is induced by multiple intermediated couplings in the resonance channels. The difference between the CNR of $Q_{1,3,5}$ and that of $Q_{1,2,8}$ is due to the absence of additional correlations in the CNR channel of $Q_{1,3,5}$. Such differences in the CNR FCs demonstrate that the generating principle of FCs via nonlinear resonances can quantitatively tune the spacing and location, and qualitatively tailor the correlation as well.

The FCs generated through NR are robust with respect to intrinsic decoherent channels induced by nonlinear couplings between the NR PMs and the rest. To demonstrate this feature, we numerically investigate the DNR FC in an FPU- α chain with 200 sites, which possesses more decoherent channels than the 10-site model. The results are shown in Fig. 4, where a temporal and spectral fingerprint of FCs can be easily identified.

To conclude, we introduce here a new mechanism in generating FCs via nonlinear resonances. The universality of NRs in various nonlinear systems, ranging from mechanical cantilevers and chains of vibrators to nonlinear optics, suggests that the FCs via NRs occur in many nonlinear systems. In particular, the controllability of FCs via NRs provides a new approach in optimizing energy conversion rate in phonon harvesting. Moreover, NRs also exist in quantum nonlinear systems, e.g., quantum FPU chains, where $Q_i(P_i)$ become noncommuting operators, suggesting that it is possible to generalize FCs to quantum nonlinear systems and possess applications in the entanglement of phonons.

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*lcao@physnet.uni-hamburg.de

†rwpeng@nju.edu.cn

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