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Scaling Quasistationary States in Long-Range Systems with Dissipation

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Hamiltonian systems with long-range interactions give rise to long-lived out-of-equilibrium macro-scopic states, so-called quasistationary states. We show here that, in a suitably generalized form, this result remains valid for many such systems in the presence of dissipation. Using an appropriate mean-field kinetic description, we show that models with dissipation due to a viscous damping or due to inelastic collisions admit "scaling quasistationary states," i.e., states that are quasistationary in rescaled variables. A numerical study of one-dimensional self-gravitating systems confirms the relevance of these solutions and gives indications of their regime of validity in line with theoretical predictions. We underline that the velocity distributions never show any tendency to evolve towards a Maxwell-Boltzmann form.

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Physical systems characterized by long-range inter-actions (for reviews, see e.g., Refs. [1,2]) are ubiquitous, encompassing systems as diverse as self-gravitating bodies in astrophysics, plasmas [3], lasers [4], cold atoms [5] in the laboratory, and even biological systems [6]. One of the main results of recent years about such systems is that, quite generically, they relax, on times scales characterized by the mean force field, towards long-lived macroscopic states called quasistationary states (QSS) (e.g., galaxies in astrophysics [7], the red spot of Jupiter [8], steady states of a free electron laser [4]). These out-of-equilibrium states have a typical lifetime diverging with particle number, while on shorter time scales they are described within the framework of the Vlasov equation. These results apply to strictly conservative systems in a microcanonical frame-work, and the question inevitably arises of the robustness of such states beyond this idealized limit. Studies of a paradigmatic toy model—the Hamiltonian mean field (HMF) model [9]—have shown that, coupled to a canonical heat bath [10,11] or when simple energy-conserving stochastic forces are introduced [12,13], such states relax rapidly towards thermal equilibrium. We report here theoretical and numerical results of the effect of introducing dissipative forces, with or without an intrinsic stochasticity. Our main finding is that for power-law interactions such systems admit what we call "scaling QSS," i.e., solutions in which the phase space distribution remains unchanged in rescaled variables as the system evolves. Numerical study for a class of such models shows that these solutions are often realized, and in the particular cases where deviations are observed, the phase space density evolves with increas-ing correlation of velocity and position. This means in particular that these systems never show any tendency, either in the scaling QSS or when there are deviations from them, to evolve towards the space and (Maxwellian) velocity distributions of thermal equilibrium.

We consider particles interacting via a long-range central power-law pair potential $V(r) = (gm^2)/(nr^n)$ where g is the coupling, m the particle mass, and r the distance between the particles. The mean-field limit will be taken keeping the total energy E, total mass M, and system size L fixed, with $N \to \infty$ and, thus, $m \sim N^{-1}$, $g \sim N^0$. For $n > 0$, the short distance cutoff should in general be regulated. We will not explicitly do so as we will treat the mean-field dynamics that is in principle independent of the associated cut off, at least down to $n < d - 1$ where d is the dimension of space [14].

We consider in addition two different classes of dis-sipative forces: on the one hand, a viscous damping force of the form $\vec{f} = -m\eta \|v\|^{\alpha-1}\vec{v}$, where α and η are constants, which we will refer to as the viscous damping model (VDM); on the other hand, instantaneous inelastic but momentum-conserving collisions, which we will refer to as the inelastic collisional model (ICM). For the sake of simplicity, we restrict here to one-dimensional models, while the generalization to any dimension will be consid-ered elsewhere. For two colliding particles i and j of incoming velocities v_i and v_j, the postcollisional velocities are given by $v_{i,j}^* = v_{i,j} \pm [(1 + c)/2](v_i - v_j)$, where c is the coefficient of restitution. Amongst the many systems in this broad class, we note two particular ones. First, self-gravitating particles in an expanding universe are described, in certain circumstances, in so-called comoving coordinates and an appropriate time variable, by the case of $\alpha = 1$ corresponding to a simple fluid damping $\vec{f} = -\eta m\vec{v}$ (see e.g., Ref. [15] and references therein). Second, the case

of gravity with inelastic collisions corresponds to a self-gravitating granular gas. Martin and Piasecki [16] have obtained an exact solution of this model for specific regular initial conditions in the totally inelastic limit, a situation different to that we will consider below. Let us recall that, in the absence of gravity, i.e., for a simple granular gas starting from an homogeneous initial configuration, the kinetic energy of the system as well as the velocity distribution function have been shown to obey scaling laws [17], until the system reaches a collapse time where clusters appear [18]. An analogy between a self-gravitating system and granular gases was also considered for cluster formation in Refs. [19–21].

In the absence of dissipation, the time evolution is described in the mean field limit and, thus, on time scales short compared to that on which the full Hamiltonian evolution drives the system towards equilibrium by the Vlasov equation (see e.g., Refs. [2,22])

$$\partial_t f(x,v,t) + v\partial_x f(x,v,t) + \bar{a}(x,t)\partial_v f(x,v,t) = 0 \quad (1)$$

where $\bar{a}(x)$ is the mean-field acceleration given by $\bar{a}(x,t) = g \int \mathrm{sgn}(x - x')|x - x'|^{-(n+1)} f(x',v',t)dx'dv'$ where $f(x,v,t)$ is the mass density in phase space. When dissipative forces are present, the Vlasov equation is modified by the addition on the right-hand side of Eq. (1) of a term denoted $J_d[f]$, an operator accounting for the dissipation in the system [23]. For the case of a viscous damping force, this operator is [24]

$$J_{d,1} = \eta\partial_v(v^\alpha f(x,v,t)), \quad (2)$$

whereas for the case of inelastic collisions it can be expressed in terms of $f(x,v,t)$ (Stosszahl ansatz) [25] as

$$J_{d,2} = \frac{N}{M}\int dv_1|v - v_1|$$
$$\times\left[\frac{f(x,v^{**},t)f(x,v_1^{**},t)}{c^2} - f(x,v,t)f(x,v_1,t)\right] \quad (3)$$

where v_i^{**} are the precollisional velocities that are given by $v_{i,j}^{**} = v_{i,j} \pm [(1 + c^{-1})/2](v_j - v_i)$. To obtain the mean-field limit of Eq. (3), we rewrite the collision operator as a series expansion of $(1 - c)/2$ [26]. After some calculation and taking the limit $N \to \infty$ at fixed $\gamma = [(1 - c)N]/2$, we then obtain $J_{d,2} = -\partial_v(a_1(x,v,t)f(x,v,t))$, where $a_1(x,v,t) = (\gamma/M)\int du(u - v)|u - v|f(x,u,t)$ is the acceleration associated with the collisional force. This scaling $(1 - c) \sim N^{-1}$ corresponds to the so-called quasielastic limit [27,28]. As discussed below in detail, in this limit the ratio of the two essential time scales of our system, the first associated with the dissipation of the total energy and the second with the mean-field dynamics ($\tau_{\mathrm{mf}} \sim 1/\sqrt{g\rho_0}$, where ρ_0 is the mass density), is independent of N.

We now seek scaling solutions to Eq. (1), using the following ansatz:

$$f(x,v,t) = \frac{M}{\bar{x}(t)\bar{v}(t)}F\left(\frac{x}{\bar{x}(t)},\frac{v}{\bar{v}(t)}\right). \quad (4)$$

Substituting this in Eq. (1) gives

$$-\frac{\partial_t(\bar{x}(t)\bar{v}(t))}{(\bar{x}(t)\bar{v}(t))^2}F(y,z) - \frac{\partial_t\bar{x}(t)}{\bar{x}^2(t)\bar{v}(t)}y\partial_y F(y,z) - \frac{\partial_t\bar{v}(t)}{\bar{x}(t)\bar{v}^2(t)}$$
$$\times z\partial_z F(y,z) + \frac{z\partial_y F(y,z)}{\bar{x}^2(t)} + \frac{gM\bar{A}(y)}{\bar{v}^2(t)\bar{x}^{(n+2)}(t)}\partial_z F(y,z) = J_d$$

$$(5)$$

where y and z and the rescaled variables [$y = x/(\bar{x}(t))$, $z = v/(\bar{v}(t))$] and $\bar{A}(y) = \int \mathrm{sgn}(y-y')|y-y'|^{-(n+1)}$ $F(y',z)dy'dz$; furthermore, we have for VDM $J_{d,1} = [(\eta\bar{v}^{\alpha-2})/\bar{x}]\partial_z(z^\alpha F(y,z))$ and for ICM $J_{d,2} = -[\gamma/(\bar{x}^2(t))]\partial_z$ $(\bar{A}_1(y,z)F(y,z))$ where $\bar{A}_1(y,z) = \int(z'-z)|z-z'|F(y,z')dz'$. These scaling solutions are admitted if it is possible to choose functions $\bar{x}(t)$ and $\bar{v}(t)$ so that the time dependence of the coefficient of each term is the same. Comparing, first, the last two terms on the left-hand side of Eq. (5), we infer the requirement

$$\bar{v}^2(t)\bar{x}^n(t) = \text{cste}. \quad (6)$$

This means that the virial ratio, defined as $R = -2K/nU$ where K and U are the total kinetic and potential energy respectively, is constant. We now assume further that the system is in a QSS in the limit that the dissipation is absent. With this assumption, we have that

$$z\partial_y F(y,z) + \frac{gM}{\bar{v}^2\bar{x}^n}\bar{A}(y)\partial_z F(y,z) = 0, \quad (7)$$

i.e., the last two terms on the right hand side of Eq. (5) cancel. This also implies that the virial ratio R is unity. Physically, this means we assume that all time dependence of the evolution
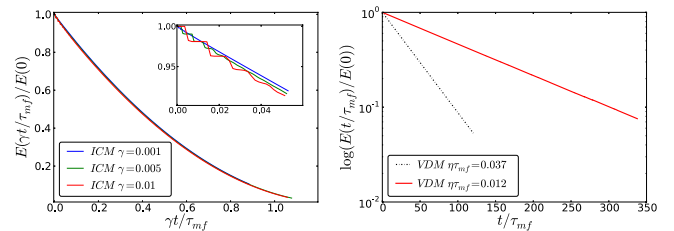

FIG. 1 (color online). Left: For the ICM, reduced total energy $E(t)/E(0)$ versus $\gamma t/\tau_{\mathrm{mf}}$, for the indicated values of γ and $R_0 = 0.01$. Inset: zoom on the shorter time evolution. Right: for the VDM, semi-log plot of $E(t)/E(0)$ versus t/τ_{mf} with the indicated values of η, and $R_0 = 0.01$.

arises solely from the dissipation. This corresponds to an adiabatic limit of weak dissipation in which the time scale on which the dissipation causes macroscopic evolution is arbitrarily long compared to the time scale associated with the mean-field dynamics. The scaling solution, thus, excludes all nontrivial time dependence due to the mean-field dynamics except its effect in virializing the system. In particular, does not describe the phase of violent relaxation to virial equilibrium. We will evaluate further below the validity of this crucial approximation.

With the use of Eq. (6) it is simple to infer that the sole additional requirement on the scaling solution is $\bar{v}(t)^{-1}\partial_t\bar{v}(t) = -A_0\eta\bar{v}^\beta(t)$ with $\beta = \alpha - 1$, for the VDM, and $\bar{v}(t)^{-1}\partial_t\bar{v}(t) = -A_1\gamma\bar{v}(t)\bar{x}(t)^{-1} = -A_1\gamma\bar{v}^\beta$ with $\beta = (n+2)/n$ for the ICM, where A_0 and A_1 are dimensionless positive constants. Integrating these equations, we obtain

$$\bar{v}(t) = v_0 \begin{cases} \left(1 + \mathrm{sgn}(\beta)\frac{t}{t_c}\right)^{-1/\beta} & \beta \neq 0 \\ e^{-t/t_c} & \beta = 0 \end{cases} \quad (8)$$

where t_c is a characteristic time scale. The solutions for $\bar{x}(t)$ follow from Eq. (6). The case $\beta = 0$ corresponds to the VDM with $\alpha = 1$ and the ICM with $n = -2$, the trivial case of a harmonic potential.

For a virialized state, the total energy $E = [1 - (2/n)]K$ and so scales as $\bar{v}^2(t)$. For attractive pair potentials with $n < 0$, which is the class of long-range potentials we are considering here (in $d = 1$), the scaling solution therefore describes, for cases with $\beta < 0$, a system that undergoes a collapse in the finite time t_c. Otherwise, the system undergoes a monotonic contraction characterized by the same time but never collapses.

We now return to the essential approximation Eq. (7) that we have made in deriving the scaling solution. This corresponds to assuming that $\tau_{\mathrm{diss}} \gg \tau_{\mathrm{mf}}$, where τ_{diss} and τ_{mf} are the characteristic times for, respectively, the dissipation of the system energy E and the mean-field (Vlasov) dynamics. For the VDM, we have that

$$\frac{dE}{dt} = -\eta\langle|v|^{\alpha+1}\rangle_t \quad (9)$$

where $\langle X(x,v)\rangle_t = \int dx\,dv\,X(x,v)f(x,v,t)$. Substituting the scaling solution, in which $E \propto \bar{v}^2(t)$, in this equation, we can then infer that

$$t_c \equiv \frac{2E_0}{\eta\langle|v|^{\beta+2}\rangle_0} = \frac{1}{\eta}\left(1 - \frac{2}{n}\right)\frac{\langle v^2\rangle_0}{\langle|v|^{\beta+2}\rangle_0}. \quad (10)$$

For the ICM, a similar relation can be written, the only difference being that η is replaced by γI_0 where I_0 is a dimensionless integral. For both cases, t_c diverges as the inverse of the strength of the dissipation; t_c represents the time scale for dissipation starting from the (arbitrary) time $t = 0$. In the scaling solution, the characteristic time for

dissipation of energy starting from an arbitrary time t, thus, scales as $\tau_{\mathrm{diss}}(t) \propto \bar{v}^{-\beta}(t)$. For a typical system size $\bar{x}(t)$, the mean-field acceleration scales as $\bar{x}^{-(n+1)}$ and τ_{mf} (mean time for a particle to cross the system) as $\bar{x}^{(n+2)/2}$. It follows that $\tau_{\mathrm{diss}}/\tau_{\mathrm{mf}} \propto \bar{v}^{[-\beta+[(n+2)/n]]}$, and therefore, if $\beta > \beta_c = (n+2)/n$ the ratio of these time scales increases as a function of time. In other words, if $\beta > \beta_c$, the scaling solution drives the system to a regime in which the approximation underlying it becomes arbitrarily well satisfied. In this case, we then expect that the scaling solution may be an attractor for the system's behavior, whereas for $\beta < \beta_c$, the opposite is the case and the scaling solution is at most expected to represent a transient behavior. The case $\beta = \beta_c$, which corresponds precisely to the ICM, is the marginal one. In this case, the ratio $\tau_{\mathrm{diss}}/\tau_{\mathrm{mf}}$ remains constant in the scaling solution, and one would expect it to be a transient that persists on a time scale dependent on this ratio.

To explore the validity of this analysis, we have performed a numerical study of two one-dimensional (1D) self-gravitating systems, i.e., the case $n = -1$ corresponding to the pair potential $\phi(x) = gm|x|$ derived from the 1D Poisson equation $\partial_x^2\phi(x) = 2gm\delta(x)$, one with the dissipation of the ICM and the other with that of the VDM for the case $\alpha = 1$. In the absence of dissipation, the equations of motion may be integrated exactly between particle collisions (equivalent to crossings), and the system can be evolved using an event-driven algorithm [15,29,30], which determines the collision times exactly (up to round-off errors). For the ICM, inelastic collisions are implemented in an existing code with the appropriate postcollisional velocities (including the coefficient of restitution c). For the VDM, an event-driven algorithm is also implemented as the collision time between particles in this case can also be computed exactly by finding the roots of a quintic equation (see Ref. [15] and references therein). We use "rectangular water-bag" initial conditions; i.e., velocities and positions are chosen randomly and uniformly in phase space in $[-v_0, v_0] \times [-L_0/2, L_0/2]$. These are fully characterized by the initial virial ratio R_0. We have performed simulations of different sizes ($N = 256, \ldots, 4096$), and no noticeable finite effects have been observed. All quantities have been averaged over 100 independent realizations for the ICM and 50 for the VDM (which is less noisy). For both models, the simulation is interrupted when the difference between two possible collision times becomes smaller than the accuracy of the computer.

We define the mean-field time as $\tau_{\mathrm{mf}} = 2\sqrt{L_0/(gN)}$ and recall the behavior of this system in the absence of dissipation from initial conditions of this kind (as detailed, e.g., in Ref. [31]): it evolves on a time scale of order 10–$100\,\tau_{\mathrm{mf}}$ towards a QSS, in which the virial ratio R is unity. Monitoring R in the present case (with dissipation), we find essentially identical behavior but, as expected, a
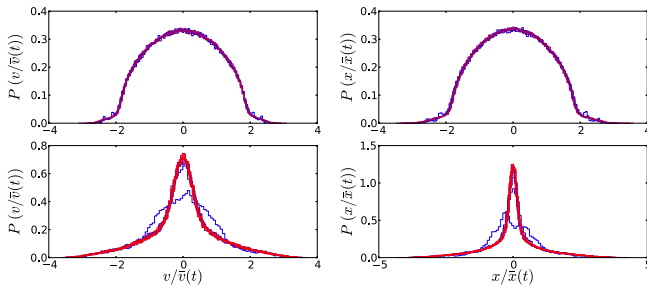
FIG. 2 (color online). Velocity (left panels) and position (right panels) distributions as a function of rescaled variables, for the ICM with $\gamma = 0.01$ at different times $t/\tau_{\mathrm{mf}} = 10, 31, 53, 74, 95$ and $R_0 = 1$ (upper panels) and for the VDM with $\eta\tau_{\mathrm{mf}} = 0.037$ and $R_0 = 0.01$ (lower panels) at $t/\tau_{\mathrm{mf}} = 10, 20, \ldots, 100$. In the lower panels, the curves which do not superpose correspond to $t/\tau_{\mathrm{mf}} = 10$, in the phase of violent relaxation.

very different behavior for the energy. The left panel of Fig. 1 shows, for the ICM, the normalized energy as a function of the dimensionless time $\gamma t/\tau_{\mathrm{mf}}$, for an initial virial ratio $R_0 = 0.01$ and the different given values of γ; in the right panel, the same quantity is plotted versus t/τ_{mf}, for the VDM with $\eta\tau_{\mathrm{mf}} = 0.012, 0.037$. We observe excellent agreement with the scaling solutions: for the ICM, the energy decay is fitted by $[1 - (t/t_c)]^{\delta}$ with $\delta = 2.00 \pm 0.01$; for the VDM, the energy decay is fitted by $E(t_s)/E(0) = \exp[-(2/3)\lambda(t/\tau_{\mathrm{mf}})]$ with $\lambda = \eta\tau_{\mathrm{mf}}$, as predicted by the scaling solution [Eqs. (8) and (10)], to 10^{-4}. The inset of Fig. 1 shows small deviations from the scaling behavior at short times, associated with the virial oscillations during the initial violent relaxation.

The velocity and position distributions versus appropriate rescaled variables are shown in Fig. 2 at different times, for the ICM with $R_0 = 1$ and for the VDM with $R_0 = 0.01$. The superposition of the curves illustrates the accurate description of the kinetics by scaling QSS (in the lower panels the blue curves correspond to $t/\tau_{\mathrm{mf}} = 10$ in the phase of the violent relaxation). Moreover, one observes significant deviations from a Gaussian shape of the distribution corresponding to the existence of a "core-halo" structure in the QSS [32,33].

The ICM is a marginal case for the validity of the approximation in which we obtained the scaling QSS, whereas for the VDM with $\alpha = 1$ and $n = -1$ (i.e., $\beta_c = -1$), we expect it to become exact asymptotically. To quantify deviations from the scaling QSS, it is convenient to monitor the dimensionless quantity [31]

$$\phi_{11} = \frac{\langle |x||v|\rangle}{\langle |x|\rangle\langle |v|\rangle} - 1 = \frac{\int |x||v|f dx dv}{\int |x|f dx dv \int |v|f dx dv} - 1, \quad (11)$$

which provides a measure of the correlation between the spatial and velocity variables. For conservative self-gravitating systems, ϕ_{11} can be interpreted as an order parameter for the QSS, which goes to 0 as the system goes to thermal
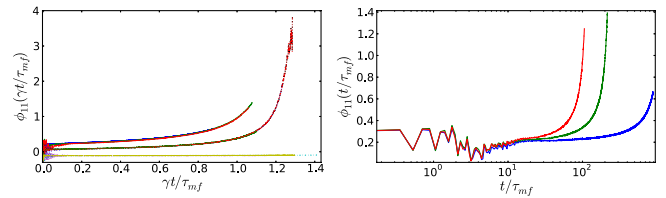

FIG. 3 (color online). Left: ϕ_{11} versus $\gamma t/\tau_{\mathrm{mf}}$ for $\gamma = 0.01$, 0.005, 0.001 and $R_0 = 0.01$ (three upper curves), $R_0 = 0.1$ (three middle curves), and $R_0 = 1$ (three lower curves). Right: ϕ_{11} versus t/τ_{mf} in a semi-log plot for $R_0 = 0.01$ and $\gamma = 0.001$ [blue (right)], 0.005 [green (middle)], 0.01 [red (left)].

equilibrium [31]. Inserting Eq. (4), in Eq. (11), we see that ϕ_{11} is constant in time also in the scaling QSS: the evolution of the system is through a sequence of QSS with identical correlations.

Figure 3, right panel, shows the evolution of ϕ_{11} for the ICM starting from $R_0 = 0.01, 0.1$, and 1 and for the three different values of γ. We have used here this rescaled time $\gamma t/\tau_{\mathrm{mf}}$ because we observe that it gives a good collapse of all curves; in the left panel, the result for the first case ($R_0 = 0.01$) is shown without this rescaling. For the VDM (not shown here) ϕ_{11} remains constant (after the initial violent relaxation). We observe that while in the case of $R_0 = 1$ ϕ_{11} is almost constant, visible deviations are evident for the two other cases, with evolution away from the scaling setting in fastest for the case $R_0 = 0.01$. Similarly, the space and velocity distributions deviate progressively from the scaling solutions for $R_0 = 0.1$, 0.01 (not shown here). In both cases, the system energy decreases as predicted by the scaling solution. Moreover, this evolution appears to depend only on the QSS attained (which is different for each R_0) and on γ through the rescaled variable. Thus, as the total energy goes to 0, inelastic collisions drive the system through a given family of ever more correlated QSS. It implies that the system never shows any tendency to drive the system towards a Maxwell-Boltzmann distribution of velocities (nor towards the spatial distribution of the thermal equilibrium of the model), despite the effective stochasticity of the inelastic collisions. This contrasts to what is observed in a stochastically perturbed HMF model in Refs. [12,13]. This tendency towards more correlated states can be interpreted as follows: for $R_0 \ll 1$, the violent relaxation drives the system to a core-halo structure (see Refs. [3,31,32]), whereas for $R_0 \simeq 1$, the QSS is rather homogeneous in phase space. For the ICM, the (kinetic) temperature of the core decreases more rapidly than that of the halo; in the VDM, on the other hand, the systems cools down uniformly. Further it may be, as observed in three-dimensional gravitating systems [34], that inefficiency of energy exchange between the core and halo impedes relaxation towards thermal equilibrium.

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