Wave Systems with an Infinite Number of Localized Traveling Waves

Georgy L. Alfimov,[1] Elina V. Medvedeva,[1] and Dmitry E. Pelinovsky[2,3]
[1]Moscow Institute of Electronic Engineering, Zelenograd, Moscow 124498, Russia
[2]Department of Mathematics, McMaster University, Hamilton, Ontario L8S 4M1, Canada
[3]Department of Applied Mathematics, Nizhny Novgorod State Technical University, 603950 Nizhny Novgorod, Russia
(Received 29 August 2013; published 6 February 2014)

In many wave systems, propagation of steadily traveling solitons or kinks is prohibited because of resonances with linear excitations. We show that wave systems with resonances may admit an infinite number of traveling solitons or kinks if the closest to the real axis singularities of a limiting asymptotic solution in the complex upper half plane are of the form $z_\pm = \pm\alpha + i\beta$, $\alpha \neq 0$. This quite general statement is illustrated by examples of the fifth-order Korteweg-de Vries equation, the discrete cubic-quintic Klein-Gordon equation, and the nonlocal double sine-Gordon equations.

DOI: 10.1103/PhysRevLett.112.054103 PACS numbers: 05.45.Yv

Introduction.—Nonlinear localized traveling waves such as bright or dark solitons are key concepts for many branches of modern physics, including nonlinear optics, theory of magnets, theory of Josephson junctions, etc. It is known that in many dispersive systems the presence of these nonlinear entities is strongly restricted due to resonances with linear excitations. These resonances take place in wave systems of various origin, such as the fifth-order Korteweg-de Vries equation [1], nonlinear lattices [2–4], and models with complex dispersion and nonlocal interactions [5,6]. As a result, it is quite typical that in such wave systems the localized excitations either do not exist at all or they only exist for specific values of some external parameters. In the last case, the nonlinear excitations are called embedded solitons (i.e., solitons "embedded" into the spectrum of linear waves). These embedded solitons have been discovered in hydrodynamics, nonlinear optics, and other fields of physics [7].

To give an example, consider an operator equation

$$L_\varepsilon u = F(u), \tag{1}$$

for a function $u(\xi)$ where L_ε is a Fourier multiplier operator in ξ space with even symbol $\hat{L}(k)$ in k space, $F(u)$ is a nonlinear function and ε is a parameter. The prototypical examples of problems leading to Eq. (1) are the generalized Korteweg-de Vries equation,

$$u_t + [F(u)]_x + M_\varepsilon u_x = 0, \tag{2}$$

or discrete or nonlocal Klein-Gordon equations,

$$u_{tt} - M_\varepsilon u + F(u) = 0, \tag{3}$$

for $u(x,t)$, where M_ε is a Fourier multiplier operator in x space. Above, $\xi = x - vt$ is the traveling wave coordinate and the operator L_ε in Eq. (1) includes both M_ε and v. We assume that in both cases $\varepsilon = 0$ implies a degeneration of the problem with $L_0 = \partial_\xi^2$.

Consider a solitary wave $u(\xi)$, which is asymptotic to the equilibrium state $u \equiv 0$ as $\xi \to \pm\infty$ (the case of a kink wave which is asymptotic to a pair of equilibrium states $u \equiv u_\pm$ as $\xi \to \pm\infty$ can be analyzed in a similar way). Then, the resonances correspond to the real roots of the dispersion equation near $u \equiv 0$

$$\hat{L}_\varepsilon(k) = F'(0). \tag{4}$$

If, for some value of ε, there exist a single pair of real roots $k = \pm k_0$ in Eq. (4), we are in a situation where the resonance prohibits propagation of regular solitons in Eq. (2) and Eq. (3) and the embedded solitons may appear. In this case, the velocity v of the soliton, typically, is not arbitrary but should be "adjusted" to avoid "gluing" with linear modes. In general, v belongs to some discrete set. This set may be empty (i.e., no localized waves propagate), or include a finite or infinite number of values. The case where Eq. (4) has more then one pair of real roots is more complex and the presence of localized excitations in this case is highly doubtful.

In this Letter, we address the following question: Are there some conditions which would indicate the existence of infinitely many embedded solitons described by Eq. (1)? If this is possible, can we describe this infinite set asymptotically? We answer this question positively and present sufficient conditions for the existence of a countable infinite sequence of embedded solitons. These embedded solitons are all single-humped or single-kinked.

The main assumption for our construction of embedded solitons is that the limiting solution of Eq. (1) as $\varepsilon \to 0$, being extended in a complex plane, should have a pair of symmetric singularities in the upper half plane. We give an asymptotic formula for values $\{\varepsilon_n\}$ as $n \to \infty$, for which embedded solitons exist. In terms of Eq. (2) and Eq. (3), this means a presence of an infinite number of velocities v for localized excitations. Surprisingly, it has been observed that the asymptotic formula predicts parameters of the lowest embedded solitons from this sequence with reasonable accuracy.

We note that the idea that two symmetric singularities in the upper half plane can be related to the countable infinite sequence of tangential intersections of stable and unstable manifolds can be found in [8] for the primary intersection point of the two-dimensional symplectic maps. In this Letter, we generalize this principle to a more general class of physically relevant systems.

Main result.—Consider Eq. (1), where $u(\xi)$ is a real-valued function defined on $\mathbb{R}$. Assume that L_ε is a real operator which depends continuously on the real parameter ε and satisfies $L_0 = \partial_\xi^2$. The Fourier symbol $\hat{L}_\varepsilon(k)$ is supposed to be an even function of k.

Now, assume that (a) the equation $F(u) = 0$ has zero solution $u = 0$ with $F'(0) > 0$, (b) the dispersion equation (4) has only one pair of real roots $k = \pm k(\varepsilon)$ such that $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and (c) the equation

$$u'' = F(u) \tag{5}$$

has an even localized solution $\tilde{u}(\xi)$ such that $\tilde{u}(\xi) \to 0$ as $\xi \to \pm\infty$. In addition, the key assumption of our asymptotic theory is that the solution $\tilde{u}(\xi)$ can be continued into the complex plane and the closest singularities to the real axis of $\tilde{u}(\xi)$ in the upper half plane are given by the pair $z_\pm = \pm\alpha + i\beta$, with α, $\beta > 0$, which is symmetric with respect to the imaginary axis.

Then, we expect the existence of an infinite sequence of values $\{\varepsilon_n\}$ such that for each $\varepsilon = \varepsilon_n$, Eq. (1) has a soliton solution $u(\xi)$ with $u(\xi) \to 0$ as $\xi \to \pm\infty$, and this sequence obeys the following asymptotic law:

$$k(\varepsilon_n) \sim (n\pi + \varphi_0)/\alpha, \tag{6}$$

where φ_0 is a phase constant that depends on L_ε and $\tilde{u}$.

This result can be extended naturally to the case of the kink solutions of Eq. (1) connecting a pair of equilibrium states $u = u_\pm$, such that $F(u_\pm) = 0$ and $F'(u_-) = F'(u_+) > 0$. In this case, $F'(u_\pm)$ appears instead of $F'(0)$ in the dispersion relation (4), whereas the differential equation (5) is assumed to have a kink solution $\tilde{u}(\xi)$ such that $\tilde{u}(\xi) \to u_\pm$ as $\xi \to \pm\infty$.

Note that the conditions above are sufficient, but not necessary. Another way for an infinite number of embedded solitons to exist is associated with a sequence of "bound states" of single solitons or kinks separated by a number of linear oscillations "trapped" between them [2,5]. In this case, our analysis is inapplicable.

Justification.—Let us give some heuristic arguments for justification of the main result. Introduce $v(\xi) = u(\xi) - \tilde{u}(\xi)$, $N(v) = F(\tilde{u} + v) - F(\tilde{u}) - F'(\tilde{u})v$, and $H_\varepsilon = L_0 - L_\varepsilon$. Then, we have

$$[L_\varepsilon - F'(\tilde{u})]v = H_\varepsilon \tilde{u} + N(v). \tag{7}$$

Based on our assumptions, we take for granted that (a) the function $H_\varepsilon \tilde{u}(\xi)$ can be continued into the upper complex half plane and its closest to the real axis singularities are $z_\pm = \pm\alpha + i\beta$, whereas (b) the homogeneous linearized equation

$$[L_\varepsilon - F'(\tilde{u})]v = 0 \tag{8}$$

has a pair of solutions $\varphi_\varepsilon^\pm(\xi) = e^{\pm ik(\varepsilon)\xi}\psi_\varepsilon^\pm(\xi)$, where $k(\varepsilon)$ is the only positive root of Eq. (4) and $\psi_\varepsilon^\pm(\xi) \to 1$ as $\varepsilon \to 0$. The latter hypothesis is quite natural, since $v(\xi) = e^{\pm ik(\varepsilon)\xi}$ are solutions of $[L_\varepsilon - F'(0)]v = 0$.

Then for small ε, the term $H_\varepsilon \tilde{u}$ in the right-hand side of the inhomogeneous equation (7) dominates and the solvability condition for this inhomogeneous equation [9] can be written approximately as the orthogonality condition

$$0 = J_\pm(\varepsilon) \approx \int_{-\infty}^{\infty} e^{\pm ik(\varepsilon)\xi} H_\varepsilon \tilde{u}(\xi) d\xi. \tag{9}$$

The asymptotic value of the integral in (9) as $\varepsilon \to 0$ is determined by the closest to the real axis singularities of the integrand in the complex plane (the Darboux principle, see [10]). Since $H_\varepsilon \tilde{u}(\xi)$ is even, bounded, and real valued for real ξ, the main contribution comes from $z_\pm = \pm\alpha + i\beta$. Also we have $J_+ = J_- \equiv J$.

In the simplest case, when the integrand has poles of order n in the points $z_\pm$, the result is simply a sum of the residues in these poles multiplied by $2\pi i$. In a more complicated case, the singularities $z_\pm$ of $H_\varepsilon \tilde{u}$ can be rational or transcendental branch points. For both cases, since $H_\varepsilon \tilde{u}(x)$ is even and real for real x, we can write

$$H_\varepsilon \tilde{u}(\xi) \sim C_\pm(\varepsilon) e^{i\pi\kappa/2} (\xi - z_\pm)^\kappa, \qquad \xi \to z_\pm, \tag{10}$$

where $C_-(\varepsilon) = \overline{C_+(\varepsilon)}$, κ is a real number, $\kappa \neq 0, 1, 2, \ldots$. It is natural to assume that $C_+(\varepsilon) \sim C_0 \varepsilon^q$ as $\varepsilon \to 0$ for some values of C_0 and q. Then, applying standard formulas [11] in the asymptotic limit $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, we conclude that

$$J(\varepsilon) \sim \frac{4\pi\varepsilon^q |C_0| e^{-\beta k(\varepsilon)}}{[k(\varepsilon)]^{\kappa+1}\Gamma(-\kappa)} \cos\,[\alpha k(\varepsilon) + \phi_0], \tag{11}$$

where $\varphi_0 = \arg(C_0)$. Consequently, zeros of $J(\varepsilon)$ obey the asymptotic formula (6) with $\varphi_0 = \pi/2 - \phi_0$.

If $\tilde{u}(\xi)$ is a symmetric kink solution of Eq. (5), the reasoning remains the same up to the point that $H_\varepsilon \tilde{u}(\xi)$ is now odd in ξ. We note that the integral condition similar to (9) appeared before in the context of existence of embedded solitons [12].

Examples.—The validity of the main result has been confirmed by many numerical studies. Below we give three illustrative examples that concern problems of different physical origins.

Example 1: Consider the equation

$$\varepsilon^2 u'''' + u'' - u + ru^2 - u^3 = 0, \qquad (12)$$

where ε is a parameter. Equation (12) arises in hydrodynamics where it describes traveling waves for the fifth-order KdV equation [1]. If $\varepsilon = 0$ and $r > 3/\sqrt{2}$, Eq. (12) has an exact soliton solution

$$\tilde{u}(\xi) = \frac{3}{\sqrt{r^2 - \frac{9}{2}} \cosh \xi + r}. \qquad (13)$$

Closest singularities in the upper complex half-plane to the real axis are the simple poles

$$z_{\pm}(r) = \pm \mathrm{arctanh} \frac{3}{\sqrt{2} r} + i\pi. \qquad (14)$$

Since $H_\varepsilon = -\varepsilon^2 \partial_\xi^4$, we note that the singularities of $H_\varepsilon \tilde{u}$ are situated in (14) and are poles of order $n = 5 = -\kappa$. The expansion (10) holds with $C(\varepsilon) = \varepsilon^2 C_0$, where C_0 is purely imaginary. Therefore, $\phi_0 = \pi/2$ in Eq. (11) and $\varphi_0 = 0$ in Eq. (6).

The dispersion relation (4) reads as $\varepsilon^2 k^4 - k^2 - 1 = 0$ and it has one positive root $k_0(\varepsilon)$ such that $k_0(\varepsilon) \sim 1/\varepsilon$ as $\varepsilon \to 0$. According to the main result, we expect that there exists an infinite sequence of values $\{\varepsilon_n\}$ such that Eq. (12) has soliton solutions for $\varepsilon = \varepsilon_n$ with the asymptotic formula

$$\pi n \varepsilon_n \sim \alpha = \mathrm{arctanh} \frac{3}{\sqrt{2} r} \quad \text{as} \quad n \to \infty. \qquad (15)$$

Numerical computations strongly support this prediction. Figure 1 shows the values $\alpha/(\pi\varepsilon_n)$ which approach integers for larger values of n. The profiles of the three lowest solitons corresponding to points A, B, and C are shown on the inserts by solid lines, together with the limiting soliton (13) by dotted lines. The discrepancy reduces quickly for larger values of n.

Example 2: Consider the nonlocal double sine-Gordon equation

$$u_{tt} = \int_{\mathbb{R}} K_\varepsilon(|x-y|) u_{yy} dy + \sin u + 2a \sin 2u, \qquad (16)$$

where $a > 0$ is a parameter. In particular, this equation arises in nonlocal Josephson electrodynamics where it

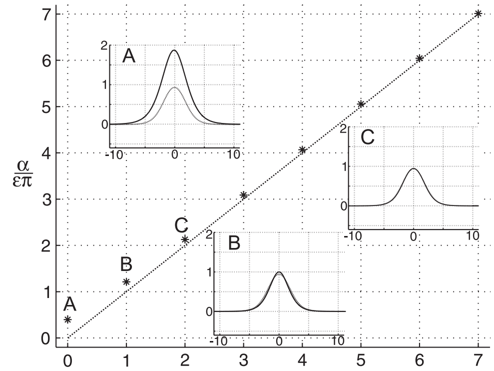

FIG. 1. Soliton solutions of Eq. (12) with $r = 2.3$. Values of $\alpha/(\pi\varepsilon_n)$ are shown by asterisks. Profiles of the first three solitons are shown on the inserts by solid lines, the dotted line shows the limiting soliton (13).

describes layered structures [6] (the second sine harmonic is important if they include, for instance, ferromagnetic layers, [13]). A list of possible kernels K_ε which arise in Josephson models can be found in [14]. We assume that K_0 is the Dirac distribution such that Eq. (16) with $\varepsilon = 0$ reduces to the classical double sine-Gordon equation. If we denote the Fourier transform of K_ε by $\hat{K}_\varepsilon(k)$, then $\hat{K}_\varepsilon(k) \to \hat{K}_0 = 1$ as $\varepsilon \to 0$.

Traveling wave solutions $u(\xi) = u(x - vt)$ of Eq. (16) satisfy the equation

$$v^2 u_{\xi\xi} = \int_{\mathbb{R}} K_\varepsilon(|\xi - \xi'|) u_{\xi'\xi'} d\xi' + \sin u + 2a \sin 2u. \qquad (17)$$

If $\varepsilon = 0$, Eq. (17) reads

$$(1 - v^2) u'' = \sin u + 2a \sin 2u, \qquad (18)$$

where $v^2 < 1$ is assumed. We consider 2π-kink solutions with boundary conditions at infinity: $\lim_{\xi \to -\infty} u(\xi) = 0$, $\lim_{\xi \to +\infty} u(\xi) = 2\pi$. For $a > 0$, Eq. (18) has an exact 2π-kink solution

$$\tilde{u}(\xi) = \pi + 2 \arctan \left[\frac{1}{\sqrt{1+4a}} \sinh \left(\frac{\sqrt{1+4a}}{\sqrt{1-v^2}} \xi \right) \right], \qquad (19)$$

and the closest singularities to the real axis are the two logarithmic branching points $z_\pm = \pm\alpha + i\beta$, where

$$\alpha = \frac{\sqrt{1-v^2}}{2\sqrt{1+4a}} \mathrm{arccosh}(1 + 8a), \qquad \beta = \frac{\pi\sqrt{1-v^2}}{2\sqrt{1+4a}}.$$

The dispersion relation

$$-v^2 k^2 + k^2 \hat{K}_\varepsilon(k) = 1 + 4a$$

is assumed to have a single pair of real roots $k = \pm k(\varepsilon)$ for all $v^2 < 1$. In particular, if K_ε is the Kac-Baker kernel

$$K_\varepsilon(|\zeta|) = \frac{1}{2\varepsilon}\exp\left(-\frac{|\zeta|}{\varepsilon}\right), \qquad \hat{K}_\varepsilon(k) = \frac{1}{1+\varepsilon^2 k^2}, \quad (20)$$

then this assumption is satisfied and

$$k_0(\varepsilon) \sim \frac{\sqrt{1-v^2}}{\varepsilon v}, \quad \text{as} \quad \varepsilon \to 0.$$

Let us present arguments that Eq. (17) with the kernel (20) admits an infinite sequence of the 2π-kink solutions. We note that this equation can be reduced to the system of differential equations

$$v^2 u_{\xi\xi} = q + \sin u + 2a \sin 2u,$$
$$-\varepsilon^2 q_{\xi\xi} + q = u_{\xi\xi}, \quad (21)$$

where an additional variable q is introduced. For the limiting kink $\tilde{u}$, we denote a solution of the second equation of the system (21) by $\tilde{q}$. Now since $H_\varepsilon \tilde{u} = \tilde{u}'' - \tilde{q}$, we understand that $H_\varepsilon \tilde{u}$ has a double pole in $\tilde{u}''$ at $z_\pm = \pm\alpha + i\beta$ in addition to the logarithmic branching points in $\tilde{q}$. Hence, $n = 2 = -\kappa$ and the expansion (10) hold with $C(\varepsilon) \to C_0$ as $\varepsilon \to 0$, where C_0 is real. Therefore, $\phi_0 = 0$ in Eq. (11) and $\varphi_0 = \pi/2$ in Eq. (6).

According to the main result, we expect that there exists an infinite sequence of values $\{\varepsilon_n\}$ such that Eq. (17) with the kernel (20) has a 2π-kink solution for $\varepsilon = \varepsilon_n$ with the asymptotic formula as $n \to \infty$,

$$\pi(1+2n)\varepsilon_n \sim \delta = \frac{(1-v^2)\operatorname{arccosh}(1+8a)}{v\sqrt{1+4a}}. \quad (22)$$

Using an appropriate shooting method [15], we compute numerically the values of ε, for which there exist 2π-kink solutions of Eq. (17) with the kernel (20). Numerical calculations strongly confirm the existence of the sequence $\{\varepsilon_n\}$ as well as its asymptotic properties (22). The values of $\delta/(\pi\varepsilon_n)$ for $a = 1/8$ and $v = 0.1$ are given in Table I and approach closer to odd integers for larger values of n.

Evidently, each value ε_n depends on the parameter v; however, from the physical viewpoint, the inverse functions $v_n(\varepsilon)$ are more important. Figure 2 represents the dependence of the velocities v_n versus ε for the first three 2π-kink solutions. The corresponding profiles of the 2π-kinks (solid lines) at the points A, B, and C are shown in the inserts together with the limiting kink (19) (dotted line). The difference between the actual kink and the limiting kink (19) is not visible already for kinks at points B and C.

TABLE I. The values of $\delta/(\pi\varepsilon_n)$ for which Eq. (17) with the kernel (20) admits the 2π-kink solution for $a = 1/8$ and $v = 0.1$.

$1+2n$	1	3	5	7	9	11
$\delta/(\pi\varepsilon_n)$	3.7168	4.9763	6.3699	7.8595	9.4541	11.1396

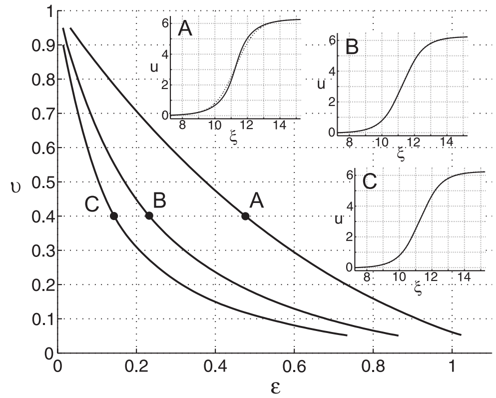

FIG. 2. Kink solutions of Eq. (17) with the kernel (20). Values of v_n versus ε are shown for the first three solutions. Profiles of the first three kinks are shown in the inserts by solid lines, the dotted line shows the limiting kink (19).

Example 3: The discrete Klein-Gordon equation is one of the basic equations describing lattice dynamics in various contexts, from solid state physics to biophysics [2]. Traveling waves of the discrete Klein-Gordon equation satisfy the equation

$$v^2 u_{\xi\xi} = \varepsilon^{-2}[u(\xi+\varepsilon) - 2u(\xi) + u(\xi-\varepsilon)] + F(u), \quad (23)$$

where ε is the spacing between lattice sites and F is a nonlinear function. Bistable nonlinearity

$$F(u) = u(1-u^2)(1+\gamma u^2), \qquad \gamma > 0, \quad (24)$$

may support kinks which satisfy the boundary conditions $\lim_{\xi\to\pm\infty} u(\xi) = \pm 1$. If $\gamma = 0$, Eq. (23) corresponds to the classical ϕ^4 model, where no traveling kinks were previously found [4]. We anticipate that for $\gamma > 0$ and v fixed there exists an infinite sequence of traveling kinks for discrete values of parameter ε.

If $\varepsilon = 0$, Eq. (23) reads

$$(1-v^2)u'' + u(1-u^2)(1+\gamma u^2) = 0, \quad (25)$$

where $v^2 < 1$ is assumed. Equation (25) has the exact kink solution

$$\tilde{u}(\xi) = \frac{\sqrt{3+\gamma}\tanh(\eta\xi)}{\sqrt{3(1+\gamma) - 2\gamma\tanh^2(\eta\xi)}}, \qquad \eta = \frac{\sqrt{1+\gamma}}{\sqrt{2(1-v^2)}}, \quad (26)$$

and for $\gamma > 0$, the closest singularities to the real axis are the two square root branching points $z_\pm = \pm\alpha + i\beta$, where

$$\alpha = \frac{\sqrt{1-v^2}}{\sqrt{2(1+\gamma)}}\operatorname{arccosh}\left(\frac{3+5\gamma}{3+\gamma}\right), \qquad \beta = \frac{\pi\sqrt{1-v^2}}{\sqrt{2(1+\gamma)}}.$$

TABLE II. The values of ε for which Eq. (23) admits kink solutions for $\gamma = 5$ and $v = 0.6$.

$3+4n$	3	7	11	15
$\chi/(\pi\varepsilon_n)$	3.5303	7.3547	11.1520	15.0329

Note that if $\gamma = 0$, then the only singularity of the exact solution (26) occurs at the imaginary axis; hence, no traveling kinks exist for $\gamma = 0$ [4].

If $\gamma > 0$ the dispersion relation

$$-v^2k^2 + 4\varepsilon^{-2}\sin^2(k\varepsilon/2) + 2(1+\gamma) = 0$$

has a single pair of real roots $k = \pm k(\varepsilon)$ for $v \in (v_0, 1)$ for some $v_0 \equiv v_0(\gamma) > 0$. More than one pair of roots exist (always, an odd number) for $v \in (0, v_0)$. We note that $k(\varepsilon) \sim p_0/\varepsilon$ as $\varepsilon \to 0$, where p_0 is a positive root of the transcendental equation $-v^2p_0^2 + 4\sin^2(p_0/2) = 0$. This equation has a single pair of real roots for $v \in (\tilde{v}_0, 1)$ where $\tilde{v}_0 \approx 0.22$ [4].

For the limiting kink $\tilde{u}$, we have

$$H_\varepsilon\tilde{u} = \tilde{u}'' - \varepsilon^{-2}[\tilde{u}(\xi+\varepsilon) - 2\tilde{u}(\xi) + \tilde{u}(\xi-\varepsilon)].$$

The first term yields the expansion (10) with $\kappa = -5/2$ and $C(\varepsilon) \sim C_0e^{5i\pi/4}$ as $\varepsilon \to 0$ near the singular points $z_\pm = \pm\alpha + i\beta$, where C_0 is purely imaginary. The second term yields the expansion (10) with $\kappa = -1/2$ and $C(\varepsilon) \sim \varepsilon^{-2}C_0e^{i\pi/4}$ as $\varepsilon \to 0$, where C_0 is purely imaginary. Both terms yield equal contribution in ε that depends on v. Nevertheless, for both terms, $\phi_0 = -\pi/4$ in Eq. (11) and $\varphi_0 = 3\pi/4$ in Eq. (6).

According to the main result, we expect that there exists an infinite sequence of values $\{\varepsilon_n\}$ such that Eq. (23) has a kink solution for ε_n with the asymptotic formula

$$\pi(4n+3)\varepsilon_n \sim \chi = 4p_0\alpha \quad \text{as} \quad n \to \infty. \qquad (27)$$

Using Newton's method with the fourth-order finite-difference approximation of the second derivative, we compute numerically the values of ε, for which kink solutions of Eq. (23) exist. Numerical calculations strongly confirm the existence of the sequence $\{\varepsilon_n\}$ as well as its asymptotic properties (27). The values of $\chi/(\pi\varepsilon_n)$ for $\gamma = 5$ and $v = 0.6$ are given in Table II with satisfactory agreement.

Conclusion.—We have shown on three prototypical examples that an infinite sequence of traveling solitons or kinks in wave systems with resonances is related to singularities in the complex plane of the leading-order asymptotic solution. This simple but universal observation reveals the reason why traveling solitons or kinks have increased mobility in some nonlinear systems with resonances but not in others.

The authors are grateful to A. S. Malishevskii for useful discussions. The work of G. L. A. and E. V. M. was supported by the Russian federal program "Scientific and educational personnel of the innovative Russia", Grant No. 14.B37.21.1273. The work of D. E. P. is supported by the ministry of education and science of Russian Federation (Project No. 14.B37.21.0868).

[1] T. Kawahara, J. Phys. Soc. Jpn. **33**, 260 (1972); J. K. Hunter and J. Scheule, Physica (Amsterdam) **32D**, 253 (1988); A. R. Champneys and M. D. Groves, J. Fluid Mech. **342**, 199 (1997); A. R. Champneys, Nonlinearity **14**, 87 (2001); A. Tovbis and D. Pelinovsky, Nonlinearity **19**, 2277 (2006).

[2] Y. Zoloratyuk, J. C. Eilbeck, and A. V. Savin, Physica (Amsterdam) **108D**, 81 (1997); P. G. Kevrekidis, Physica (Amsterdam) **183D**, 68 (2003); A. A. Aigner, A. R. Champneys, and V. M. Rothos, Physica (Amsterdam) **186D**, 148 (2003).

[3] A. V. Savin, Y. Zolotaryuk, and J. C. Eilbeck, Physica (Amsterdam) **138D**, 267 (2000).

[4] G. Iooss and D. E. Pelinovsky, Physica (Amsterdam) **216D**, 327 (2006); O. F. Oxtoby, D. E. Pelinovsky, and I. V. Barashenkov, Nonlinearity **19**, 217 (2006).

[5] G. L. Alfimov, V. M. Eleonsky, and N. E. Kulagin, Chaos **2**, 565 (1992); G. L. Alfimov, V. M. Eleonsky, N. E. Kulagin, and N. V. Mitzkevich, Chaos **3**, 405 (1993); G. L. Alfimov, V. M. Eleonsky, and L. M. Lerman, Chaos **8**, 257 (1998).

[6] Yu. M. Aliev, K. N. Ovchinnikov, V. P. Silin, and S. A. Uryupin, Zh. Eksp. Teor. Fiz. **107**, 972 (1995) S. Savel'ev, V. Yampol'skii, A. Rakhmanov, and F. Nori, Rep. Prog. Phys. **73**, 026501 (2010).

[7] A. R. Champneys, B. A. Malomed, and M. J. Friedman, Phys. Rev. Lett. **80**, 4169 (1998); A. R. Champneys, B. A. Malomed, J. Yang, and D. J. Kaup, Physica (Amsterdam) **152D–153D**, 340 (2001); T. Wagenknecht and A. R. Champneys, Physica (Amsterdam) **177D**, 50 (2003).

[8] V. G. Gelfreich, V. F. Lazutkin, and M. B. Tabanov, Chaos **1**, 137 (1991); V. Gelfreich and C. Simo, Discrete Contin. Dyn. Syst. B **10**, 511 (2008).

[9] V. Vougalter and V. Volpert, Proc. Edinb. Math. Soc. **54**, 249 (2011); Anal. Math. Phys. **2**, 473 (2012).

[10] J. Boyd, *Chebyshev and Fourier Spectral Methods* (Dover Publishers, New York, 2001).

[11] J. D. Murray, *Asymptotic Analysis* (Springer, New York, 1984).

[12] D. Kaup and B. Malomed, Physica (Amsterdam) **184D**, 153 (2003); M. Syafwan, H. Susanto, S. M. Cox, and B. A. Malomed, J. Phys. A **45**, 075207 (2012).

[13] A. A. Golubov, M. Yu. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).

[14] A. A. Abdumalikov, G. L. Alfimov, and A. S. Malishevskii, Supercond. Sci. Technol. **22**, 023001 (2009).

[15] G. L. Alfimov and E. V. Medvedeva, Phys. Rev. E **84**, 056606 (2011).