Noise and Disturbance in Quantum Measurements: An Information-Theoretic Approach

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We introduce information-theoretic definitions for noise and disturbance in quantum measurements and prove a state-independent noise-disturbance tradeoff relation that these quantities have to satisfy in any conceivable setup. Contrary to previous approaches, the information-theoretic quantities we define are invariant under the relabelling of outcomes and allow for the possibility of using quantum or classical operations to "correct" for the disturbance. We also show how our bound implies strong tradeoff relations for mean square deviations.

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Heisenberg's uncertainty principle (HUP) states, loosely speaking, that in quantum theory a measurement process cannot measure one observable accurately, such as the position, without causing a measurable disturbance to another incompatible observable, such as the momentum. Notwithstanding the crucial role played by Heisenberg's principle in modern science, it took a long time between its first exposition [1,2] and its rigorous formalization in terms of noise and disturbance operators [3–5]. The statistical spreads of these operators are measurable quantities, and hence tradeoff relations satisfied by these spreads yield precise mathematical translations of Heisenberg's intuition [3,6,7] that have recently been experimentally tested in a number of scenarios [8–14]. The use of noise and disturbance operators allows for a detailed, state-dependent formulation of HUP, able to capture the idea of "how accurate" a measurement is with respect to one dynamical variable and "how delicate" the same measurement is with respect to another dynamical variable.

In this Letter, we will explore a different approach to HUP, focused not on the change *per se* in a system's dynamical variables, but on the loss of correlation introduced by this change. In doing so, we will make use of ideas from information theory such as a "guessing strategy" and error correction, and our definitions will be given in terms of information-theoretic quantities like entropies and conditional entropies. While we focus on the noise-disturbance context here, our approach also yields tradeoff relations for joint measurements.

In order to understand the difference between the present approach and the previous one, let us consider, for example, the case of noise. While the noise, in its conventional form of root-mean-square deviation, is a statistical measure of the distance between a given system observable and the quantity actually measured [6,15], here we will only be interested in how well one can infer (i.e., guess) the value of a system observable from a given measurement outcome. That is to say, we will look only at the degree of correlation between the measurement and the observable, irrespective of how the corresponding outcomes and values are numerically labeled.

Analogously, when characterizing the disturbance, we will consider the measurement process as a source of noise for the system, and the degree to which such noise can be corrected (for a given observable) will give us our definition of disturbance.

Our measures of noise and disturbance will therefore quantify the unavoidable loss of correlations, i.e., the irreversible components of noise and disturbance. In particular, our definitions and results are invariant under reversible operations, such as relabeling of outcomes and unitary time evolutions. In contrast, the conventional approach using root-mean-square deviations is not invariant, as such operations can change numerical values and indeed the system observables of interest.

Information-theoretic approaches to HUP-like questions have already been proposed in a variety of forms [16–19]. However, these focus on the disturbance of the system state *per se*, with tradeoff relations which are functions only of the initial state of the system and the measuring apparatus. In contrast, we define noise and disturbance with respect to two system observables, and our tradeoff relation depends on the degree to which such observables are compatible, in the spirit of the original HUP. Moreover, our definitions are functions only of the two observables and the measuring apparatus, leading to a state-independent tradeoff relation.

We note that a state-independent noise-disturbance relation has recently been given, for the case of position and momentum observables, in the conventional context of root-mean-square noise and disturbance [20]. In contrast, our information-theoretic relation applies to arbitrary observables and leads to stronger results.

Our proposal.—For simplicity, consider two nondegenerate observables X and Z of a finite-dimensional quantum system S, with corresponding sets of eigenstates $\{|\psi^x\rangle\}$ and $\{|\varphi^z\rangle\}$, respectively. The system is subjected to a measuring apparatus $\mathcal{M}$. Our aim is to introduce an operational context for what it means for "$\mathcal{M}$ to measure X accurately," and for "$\mathcal{M}$ to disturb a subsequent measurement of Z." This will lead to sensible information-theoretic definitions of noise and disturbance, $\mathsf{N}(\mathcal{M}, X)$ and $\mathsf{D}(\mathcal{M}, Z)$, in terms of operational measurement statistics, which satisfy the following.

Theorem 1: For any measuring apparatus $\mathcal{M}$ and any nondegenerate observables X and Z, the following tradeoff between noise $\mathsf{N}(\mathcal{M}, X)$ and disturbance $\mathsf{D}(\mathcal{M}, Z)$ holds:

$$\mathsf{N}(\mathcal{M}, X) + \mathsf{D}(\mathcal{M}, Z) \geq -\log c, \tag{1}$$

where $c := \max_{x,z} |\langle\psi^x|\varphi^z\rangle|^2$ and the log is in base 2.

Theorem 1 clearly expresses the idea that, whenever observables X and Z are not compatible (i.e., $c < 1$), it is impossible to accurately measure one of them without at the same time disturbing the other. Significant generalizations and applications of this result will be given further below.

In order to proceed, we imagine two corresponding correlation experiments that can be performed with $\mathcal{M}$. The first experiment consists of a source producing eigenstates $|\psi^x\rangle$ of X at random, feeding these states into the apparatus $\mathcal{M}$, and determining how correlated the observed outcomes m are with the eigenvalues ξ_x of X (Fig. 1). If it is possible, from m, to guess ξ_x perfectly, then there is perfect correlation, and we say that $\mathcal{M}$ can measure X accurately, by making the corresponding optimal guess [21]. Hence, the noise $\mathsf{N}(\mathcal{M}, X)$ will be zero. In general, the noise will increase as the probability of correctly guessing ξ_x decreases. The experiment thus assesses the average performance of the apparatus in discriminating between different values of X.

In the second experiment, we imagine the source instead producing eigenstates $|\varphi^z\rangle$ of Z at random, and feeding these states through the apparatus $\mathcal{M}$ (Fig. 2). The task is then to guess, as accurately as possible, the eigenvalue ζ_z of the input state $|\varphi^z\rangle$. We first permit an arbitrary operation $\mathcal{E}$, on both the classical outcome m and the "disturbed" output quantum system S', to allow for the possibility of "correcting" any reversible disturbance by $\mathcal{M}$, before making a guess $\hat{z}$ corresponding to the outcome of a measurement of Z. If it is possible to guess perfectly, then there is perfect correlation between $\hat{z}$ and the input eigenvalue ζ_z, and we say that $\mathcal{M}$ does not disturb Z. Hence, the disturbance $\mathsf{D}(\mathcal{M}, Z)$ will be zero. In general the disturbance will increase as the correlation decreases.

The main difference between the first and the second correlation experiments is that, in the second one, we are allowed to use both the classical outcome observed and the output quantum system S'. This is because "disturbance" can only be meaningfully defined with respect to a measurement of Z that happens after the measurement process described by $\mathcal{M}$ has occurred. Thus, one is allowed to base a guess on all of the data that emerges from the measuring apparatus. Alternatively, the second experiment can be understood in terms of "error correction": before guessing z, one tries to "undo," as accurately as possible, the action of the apparatus, seen as a noisy channel with both quantum and classical outputs.

The notion of disturbance we consider is, therefore, related to the "irreversible" character of a quantum measurement: any reversible dynamical evolution is automatically corrected during the correction stage. It therefore captures the idea of "unavoidable" disturbance, in strong contrast to the conventional formulation in terms of root-mean-square deviations, where any change in the value of a system's dynamical variables is considered as a nontrivial disturbance.

Quantifying noise.—As discussed above, we require the information-theoretic noise $\mathsf{N}(\mathcal{M}, X)$ to represent the quality of the correlation between which eigenstate of X was input and the measurement outcome m. For a given input $|\psi^x\rangle$, this correlation is determined by the conditional probability distribution $p(m|\psi^x)$, which can be measured via the experimental setup in Fig. 1. Since we are interested

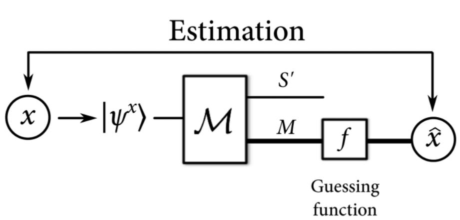

FIG. 1. Noise with respect to X is measured by the ability to guess correctly, from the measurement outcome $M = m$, in which eigenstate $|\psi^x\rangle$ of X the system was initially prepared. The guessed value $\hat{x}$ will, in general, be some function $f(m)$ of the measurement outcome. Optimization over f is allowed.

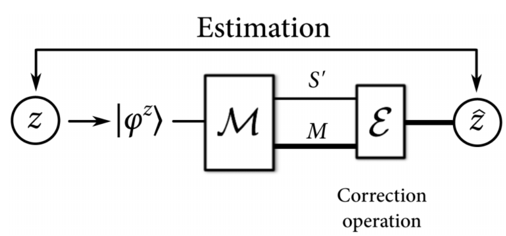

FIG. 2. Disturbance with respect to Z is measured by the ability to guess correctly, from the outcome variable M and the "disturbed" output quantum system S', in which eigenstate $|\varphi^z\rangle$ of Z the system was initially prepared. It is permitted to apply a quantum operation $\mathcal{E}$ to attempt to correct or minimize any disturbance, prior to a measurement of Z (with outcome $\hat{z}$). Optimization over $\mathcal{E}$ is allowed.

in the average noise performance, we have to introduce an *a priori* distribution on the eigenstates $|\psi^x\rangle$. By fixing the prior to be the uniform one, i.e., $p(x) = 1/d$ where d denotes the dimension of the Hilbert space of the system, we obtain the joint input-output probability distribution

$$p(m,x) = p(x)p(m|\psi^x) = \frac{1}{d}p(m|\psi^x). \tag{2}$$

This characterizes the correlation between X and M, and leads to the following.

Definition 1.—The information-theoretic noise of the instrument $\mathcal{M}$ as a measurement of X is defined as $\mathsf{N}(\mathcal{M},X) \coloneqq H(X|M)$, where $H(X|M)$ denotes the conditional entropy computed from the joint probability distribution $p(m,x)$ in Eq. (2).

Note that $\mathsf{N}(\mathcal{M},X)$ can be interpreted as the average uncertainty as to which eigenvalue of X was input, given the outcome of the measurement. Our definition can be further justified, in the precise sense that the noise $\mathsf{N}(\mathcal{M},X)$ is small if and only if the outcome m identifies the eigenvalues of X accurately. In fact, as we show in the Supplemental Material [22], standard arguments in information theory [23–26] imply that, given a guessing function $\hat{x} = f(m)$ with its total error probability $p_e \coloneqq \Pr\{\hat{X} \neq X\} = \sum_x \sum_{\hat{x}\neq x} p(\hat{x},x)$, it holds that

$$\mathsf{N}(\mathcal{M},X) \to 0 \text{ iff } \min_f p_e \to 0. \tag{3}$$

Quantifying disturbance.—Let us now consider a second nondegenerate observable $Z = \sum_z \zeta_z |\varphi^z\rangle\langle\varphi^z|$ of system S. While the noise depends only on the measurement outcome, the disturbance can depend, in principle, on both the classical outcome (M) and the quantum output system (S') of $\mathcal{M}$. However, the conceptual framework is analogous to that for noise: we imagine that eigenstates of Z are acted upon by the measurement process $\mathcal{M}$. We then require the information-theoretic disturbance to quantify the extent to which the action of $\mathcal{M}$ reduces the information about which eigenstate $|\varphi^z\rangle$ was initially selected.

However, if we want to quantify truly unavoidable disturbance, we have to allow any possible action aimed at recovering this information, after the measurement process $\mathcal{M}$ has taken place. We therefore allow an optimization over all possible correction procedures, before any attempt to estimate z is conducted. A general correction procedure is modeled by a completely positive trace-preserving map $\mathcal{E}$, reconstructing the initial system S from the output system S' and the measurement record M (Fig. 2). The final estimation of z can then be performed via a standard (i.e., von Neumann) measurement of Z, since any additional optimization can be incorporated into the correction channel $\mathcal{E}$, and no more than d outcomes are needed to discriminate between the input eigenstates. The information-theoretic disturbance will therefore depend on the joint probability distribution given by

$$p(\hat{z},z) \coloneqq p(z)p(\hat{z}|\varphi^z) = \frac{1}{d}p(\hat{z}|\varphi^z), \tag{4}$$

which characterizes the correlation between z and $\hat{z}$. In the above equation, as we did before for the case of noise, we are selecting the eigenstates of Z uniformly at random.

We can now formalize the above discussion as follows.

Definition 2.—The information-theoretic disturbance that the apparatus $\mathcal{M}$ introduces on any subsequent attempt to measure the observable Z, is defined as $\mathsf{D}(\mathcal{M},Z) \coloneqq \min_{\mathcal{E}} H(Z|\hat{Z})$, where the conditional entropy $H(Z|\hat{Z})$ is computed from the joint probability distribution $p(\hat{z},z)$ in Eq. (4), and the minimum is taken over all possible completely positive trace-preserving maps $\mathcal{E}$.

As for the information-theoretic noise above, this measure quantifies the average uncertainty of Z, given the outcome of the estimate. Eq. (3) can similarly be applied to justify our definition. In fact, besides Eq. (3), the notion of disturbance we have introduced can be given an alternative interpretation, directly related to the idea that the measurement process irreversibly disturbs the measured system. Defining the probability of error as $p_e = \sum_z \sum_{\hat{z}\neq z} p(\hat{z},z)$, the probability of guessing correctly, $1 - p_e$, is nothing but the average fidelity of correction, i.e., $1 - p_e = d^{-1}\sum_z F\{(\mathcal{E}\circ\mathcal{M})(|\varphi^z\rangle\langle\varphi^z|), |\varphi^z\rangle\langle\varphi^z|\}$, where $F\{\rho,\sigma\} \coloneqq \mathrm{Tr}\,[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]^2$ is the fidelity between states ρ and σ [27,28].

Information-theoretic noise-disturbance relation.—Definitions 1 and 2 lead to the noise-disturbance relation, Eq. (1), as shown in the Supplemental Material [22]. The proof is based on a mapping of the statistics of the two estimation procedures in Figs. 1 and 2 (which require separate inputs of the eigenstates of X and Z), to the measurement statistics of a single maximally entangled state, and applying the Maassen-Uffink entropic uncertainty relation [29] to this state.

Useful quantum lower bound on disturbance.—Given an apparatus $\mathcal{M}$ and two observables X and Z, both the noise $\mathsf{N}(\mathcal{M},X)$ and disturbance $\mathsf{D}(\mathcal{M},Z)$ can in principle be computed. However, while the noise can be computed directly from the data, the definition of disturbance involves an optimization over all possible correction procedures. Such an optimization can, in general, be very hard to perform. Such a problem can, however, be encompassed simply by noticing that any correction followed by an estimation of Z is nothing but a postprocessing of systems S' and M into the estimated variable $\hat{Z}$. We can therefore apply the quantum data-processing inequality [30–32] to arrive at the lower bound $\mathsf{D}(\mathcal{M},Z) \geq H(Z|S'M)$ for the disturbance, where the conditional quantum entropy $H(A|B)$ is defined as the difference of the von Neumann entropies corresponding to the combined system AB and system B. As we prove in the Supplemental Material [22], it is also possible to refine the bound of Theorem 1 to

$$\mathsf{N}(\mathcal{M}, X) + H(Z|S'M) \geq -\log\, c. \tag{5}$$

The idea is that one can use the Stinespring representation theorem [33] to purify the action of the measurement channel $\mathcal{M}$ to an isometry $V: \mathcal{H}_S \to \mathcal{H}_{S'} \otimes \mathcal{H}_M \otimes \mathcal{H}_E \otimes \mathcal{H}_{\bar{M}}$, where the additional systems E and $\bar{M}$ represent the environment and the environment's redundant copy of M, respectively, and then apply the recently discovered complementarity relations in the presence of quantum memory [34,35].

Generalizations.—Since the proof of the tradeoff relation, Eq. (1), only requires that $\hat{X}$ and $\hat{Z}$ are joint estimates of X and Z [22], it follows as an immediate corollary that

$$\mathsf{N}(\mathcal{M}, X) + \mathsf{N}(\mathcal{M}, Z) \geq -\log\, c \tag{6}$$

for any joint estimate of X and Z via measurement apparatus $\mathcal{M}$. This information-theoretic joint-measurement tradeoff relation may be contrasted to the joint-measurement information exclusion relation of Hall [36]. In particular, contrarily to the latter, Eq. (6) is state independent; i.e., it constrains the inherent degree to which the measurement apparatus can simultaneously perform as an X-measuring device and a Z-measuring device.

Moreover, we can generalize the tradeoff relation in the theorem to degenerate observables X and Z. One can use essentially the same arguments as in the nondegenerate case, with a suitable entropic uncertainty relation [37], to replace the constant c in Theorem 1 with $c' = \max_{x,z}||X_x Z_z||^2{}_\infty$, where X_x and Z_z are the spectral projectors corresponding to distinct eigenvalues of X and Z, respectively (see Supplemental Material [22]).

Our information-theoretic approach also yields tradeoff relations for root-mean-square deviations. In particular, for the joint probability distribution $p(m,x)$ in Eq. (2), consider the alternative measure of noise defined by $V^X_{\mathsf{N}} \coloneqq \sum_{m,x} p(m,x)[\hat{x} - \xi_x]^2$, where $\hat{x} = f(m)$ is the estimate of x from measurement outcome m. Thus, this measure is just the mean square deviation (MSD) of the estimate of the input eigenvalue from its true value. One may similarly define a measure of disturbance by the MSD $V^Z_{\mathsf{D}} \coloneqq \sum_{\hat{z},z} p(\hat{z},z)[\hat{z} - \zeta_z]^2$. As shown in the Supplemental Material [22], these quantities are equal to squares of the noise and disturbance measures defined by Ozawa [3] for the particular case of a maximally mixed system state $\rho_S = d^{-1}\mathbb{1}_S$. If the spacing between eigenvalues of X (Z) is a multiple of some value s_X (s_Z), then the tradeoff relation,

$$\left[V^X_{\mathsf{N}} + \frac{(s_X)^2}{12}\right]\left[V^Z_{\mathsf{D}} + \frac{(s_Z)^2}{12}\right] \geq \left(\frac{s_X s_Z}{2\pi e c}\right)^2, \tag{7}$$

follows as a corollary to Theorem 1 (see Supplemental Material [22]). It follows, for example, that if X and Z are the Pauli spin operators for a qubit system, then V^X_{N} and V^Z_{D} cannot both vanish. This cannot, in contrast, be concluded from known noise-disturbance tradeoff relations for the maximally mixed state [3,7].

A generalization to continuous observables is not straightforward operationally, as the corresponding eigenkets are not physical states. However, as described in the Supplemental Material [22], it is possible to formally take limits to obtain the tradeoff relation,

$$\mathsf{N}(\mathcal{M}, Q) + \mathsf{D}(\mathcal{M}, P) \geq \log\, \pi e \hbar, \tag{8}$$

for position Q and momentum P. Moreover, defining MSDs V^Q_{N} and V^P_{D} as above, this bound further implies the Heisenberg-type noise-disturbance relation $V^Q_{\mathsf{N}} V^P_{\mathsf{D}} \geq \hbar^2/4$ in the same way that the usual Heisenberg uncertainty relation follows from the entropic uncertainty relation for Q and P [38]. Note this is similar in form to, but stronger than, the relation recently obtained by Busch *et al.* [20], as the latter is for the product of the maximum possible deviations, rather than for the product of the mean deviations. In both cases, however, the measures of noise and disturbance for position and momentum are purely formal, with no operational counterparts. Hence it appears that state-dependent noise-disturbance and joint-measurement relations [3,6,7,10,15,36,39] may be preferable for continuous observables.

Finally, while Theorem 1 relies on the entropic uncertainty relation due to Maassen and Uffink, any such relation, such as those recently obtained by Puchala *et al.* [40] and Coles *et al.* [41], will similarly lead to a corresponding tradeoff relation for the information-theoretic noise and disturbance.

Conclusion.—We have obtained an information-theoretic characterization of Heisenberg's uncertainty principle, which for the first time characterizes the inherent degree to which a given measurement apparatus must disturb one observable to gain information about another observable, independently of the state of the system undergoing measurement. Our proposed measures of noise and disturbance quantify the irreversible loss of correlations introduced by the measurement apparatus, and are invariant under operations such as the relabeling of outcomes and invertible evolutions. Further, in the case of discrete observables, they can be operationally determined, as per Figs. 1 and 2. Our main theorem has a number of generalizations, including extensions to tradeoff relations for joint measurements, degenerate observables, root-mean-square deviations, and continuous observables, and yields stronger constraints than previous results in the literature.

We believe the above fundamental results will have diverse applications in quantum information theory and quantum metrology, for example, and will also motivate experimental confirmation of the strong information-theoretic form of the Heisenberg uncertainty principle presented here.

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