Proposal for Verification of the Haldane Phase Using Trapped Ions

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A proposal to use trapped ions to implement spin-one *XXZ* antiferromagnetic chains as an experimental tool to explore the Haldane phase is presented. We explain how to reach the Haldane phase adiabatically, demonstrate the robustness of the ground states to noise in the magnetic field and Rabi frequencies, and propose a way to detect them using their characteristics: an excitation gap and exponentially decaying correlations, a nonvanishing nonlocal string order, and a double degenerate entanglement spectrum. Scaling up to higher dimensions and more frustrated lattices, we obtain richer phase diagrams, and we can reach spin liquid phase, which can be detected by its entanglement entropy which obeys the boundary law.

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The ability to verify condensed matter physics, high energy physics, and cosmological effects via controllable quantum platforms has attracted a lot of attention in recent years [1]. Many different experimental systems have been designed for this purpose, ranging from cold atomic gases in optical lattices [2], trapped ions [3], cavity QED [4], superconducting circuits [5], and linear optics [6] to spins on a diamond surface [7]. The realization of spin chains in these platforms has attracted special attention [5,8]. In particular, trapped ions have been targeted as a promising system for the implementation of spin chain Hamiltonians [9], where in many cases, experimental work [10] was triggered by theoretical proposals [11,12].

In a linear trap, the ions are captured electromagnetically, micrometers apart from one another [13,14]. A variety of cooling techniques have been proposed and used for trapped ions [15], which are utilized to reach the fidelities needed for quantum gates. Moreover, the system permits precise spin-manipulation capabilities and accurate readouts. It enables operating in decoherence-free subspaces [16] and gives rise to unprecedented fidelities [17].

In the last three decades, a great deal of theoretical effort has been invested in understanding the physics of the Haldane phase, which appears in one-dimensional systems: chains of integer spins [18–21], chains of interacting bosons [21,22], and chains of interacting fermions [23], where there exists an effective mapping between the varying systems. In this brief introduction, we will stress several important insights of this fascinating field.

According to Haldane's conjecture [18], integer-spin Heisenberg antiferromagnetic (AFM) chains have a finite energy gap between the ground and excited states and short-range correlation functions, as opposed to the half-integer spin chains which are gapless and have long-range correlations. This crucial difference between the two phenomena in one dimension is carried by the different topological θ term in the action of the spin-coherent state path integral [24,25]. It adds a contribution of $\exp(2\pi i S k)$ to the partition function, where S is the spin and k is a natural number. For integer spin, the topological θ term is not operational; thus, the theory reduces to the quantum rotor model action, where the energy gap appears. Intuitively, it can be understood at the mean-field approximation, where the partition function is governed by a mode with linear dispersion. Quantum fluctuations around this mode will additively contribute to the partition function and will alter its linear dispersion by the creation of an energy gap. On the other hand, for half-integer spin $\exp(2\pi i S k) = (-1)^k$, i.e., the topological sectors carry alternating signs, so the fluctuations' contribution is canceled, and the linear dispersion survives. Therefore, the energy gap vanishes.

Kennedy and Tasaki [19] showed that the Haldane phase in the spin-one *XXZ* AFM chain, which is described by the following Hamiltonian

$$H = \sum_i S_x^i S_x^{i+1} + S_y^i S_y^{i+1} + \lambda S_z^i S_z^{i+1} + D(S_z^i)^2, \quad (1)$$

is related to the breaking of a hidden $Z_2 \times Z_2$ symmetry that is introduced by a nonlocal unitary transformation and was followed by a nonzero, nonlocal string order parameter. Recently, Pollmann *et al.* [21,26] showed that in the most general case, the Haldane phase can be defined by a symmetry-protected double degeneracy of the entanglement spectrum. We take advantage of these qualities of the Haldane phase to form the Hamiltonian adiabatically and to measure its ground states.

In what follows, we propose a scheme to realize the Haldane phase in spin-one *XXZ* AFM chains and to verify its ground states. To the best of our knowledge, there is no experimentally convenient system for exploring the Haldane phase or higher-dimensional integer-spin Heisenberg AFM systems. Complicated experiments using neutron scattering from a spin-one AFM Heisenberg chain $CsNiCl_3$ have measured the excitation spectrum and found an energy gap [27]. However, a trapped ions platform for the realization of the Haldane phase enables much more to be said and examined in a much more experimentally feasible way.

The model.—In our model, we have N ions of mass m and charge e, in a linear trap with frequencies ω_x, ω_y, ω_z. The MHz trap frequencies are tunable so as to construct the desired geometric shape of the ion lattice. A Coulomb interaction causes the trapped ions to vibrate around fixed points $r_{i\alpha} = r^0_{i\alpha} + \Delta r_{i\alpha}$ of the ith ion in the α direction, whose geometric formation is determined by the equilibrium between the trapping forces and the Coulomb repulsion. In the harmonic approximation, the vibration Hamiltonian [28,29] is

$$H_{\rm vib} = \sum_{i,\alpha}\left(\frac{1}{2m}p^2_{i,\alpha} + \frac{1}{2}m\omega^2_\alpha\Delta r^2_{i,\alpha}\right) + \frac{e^2}{2}\sum_{i,j,\alpha,\beta} V_{ij\alpha\beta}\Delta r_{i\alpha}\Delta r_{j\beta}. \quad (2)$$

By solving the quadratic problem, we obtain the normal modes of the vibration $M^\alpha_{i,n}$ and ν^α_n, which are the eigenstates and the eigenvalues of the nth mode and the ith ion in α direction, respectively. Thus, the ion displacements are represented by these normal modes as $\Delta r_{i\alpha} = \sum_n M^\alpha_{i,n}/\sqrt{2m\nu^\alpha_n}(b^\dagger_{n\alpha} + b_{n\alpha})$, while the vibration Hamiltonian becomes

$$H_{\rm vib} = \sum_{n,\alpha}\nu^\alpha_n b^\dagger_{n\alpha} b_{n\alpha}. \quad (3)$$

The proposed scheme concentrates on the microwave quantum computing setup [11,30] where the spin degrees of freedom are modeled via the hyperfine structure (Fig. 1 left). The energy level configuration is produced by the $F = 0$ singlet state $|0\rangle$ and the triplet $F = 1$ states $|-1\rangle$, $|0'\rangle$, and $|1\rangle$ according to their spin projection on the z direction. Two resonant microwave fields with the same Rabi frequency Ω drive the transitions $|\pm 1\rangle \longleftrightarrow |0\rangle$. According to their relative phases, the dressed states, which will play the role of the modeled $S = 1$ spins, are obtained. For the vanishing initial phase difference, the dressed-state basis is the eigenvector set

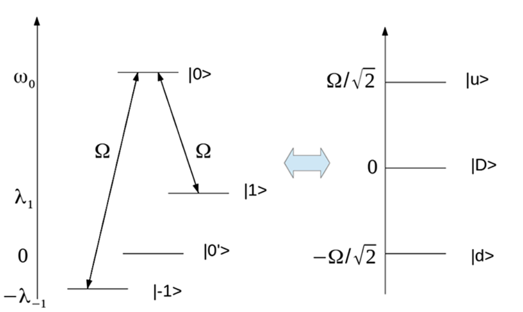

FIG. 1 (color online). Spin energy levels. Hyperfine energy levels and the driving fields (left side) that are used for preparing the effective Hamiltonian H_{XY} in the dressed-state basis (right side).

of F_x (the projection of the hyperfine spin on the x axis) and is expanded by the original basis $\{|1\rangle, |0\rangle, |-1\rangle\}$ as the following: $|u\rangle = (|1\rangle + |-1\rangle)/2 + |0\rangle/\sqrt{2}$, $|D\rangle = (-|1\rangle + |-1\rangle)/\sqrt{2}$, $|d\rangle = (|1\rangle + |-1\rangle)/2 - |0\rangle/\sqrt{2}$, with the eigenvalues $\Omega/\sqrt{2}$, 0, $-(\Omega/\sqrt{2})$, respectively (Fig. 1 right).

To induce the spin-spin interaction, which is achieved by the creation of a virtual phonon in one ion and its annihilation in another ion, there should be a term in the Hamiltonian that couples the spin and the vibration. Therefore, the very small Lamb-Dicke parameter of microwave sources requires the use of large and stable magnetic field gradients [11,31], where the field is polarized in the z direction, resulting in adding the term $g\mu_B F^i_z \partial_\alpha B_z \Delta r_{i,\alpha}$, where g is the Lande g factor and μ_B is the Bohr magneton. In the interaction picture according to the hyperfine structure, the Hamiltonian has the form of

$$H_I = \sum_{i,n,\alpha}\frac{\Omega}{\sqrt{2}}F^i_x + \nu^\alpha_n(b^\dagger_{n\alpha}b_{n\alpha} + \eta^\alpha_{in}(b^\dagger_{n\alpha} + b_{n\alpha})F^i_z), \quad (4)$$

where $\eta^\alpha_{in} = g\mu_B\partial_\alpha B_z M^\alpha_{i,n}/\sqrt{2m(\nu^\alpha_n)^3}$.

After transforming to the interaction picture according to the carrier transition $(\Omega/\sqrt{2})F_x$ and the vibration Hamiltonian [Eq. (3)], the spin-dependent force in the dressed basis is obtained:

$$H_I = \sum_{n,i,\alpha}\frac{v^\alpha_n\eta^\alpha_{in}}{2}\left(S^i_+ e^{i\frac{\Omega}{\sqrt{2}}t} + h.c\right)\left(b^\dagger_{n\alpha}e^{iv^\alpha_n} + h.c\right). \quad (5)$$

According to the rotating wave approximation (RWA), if $v^\alpha_n\eta^\alpha_{in}\sqrt{N^\alpha_n} \ll \frac{\Omega}{\sqrt{2}} \pm v^\alpha_n$, where N^α_n is the number of phonons of the nth mode in the α direction, it is sufficient to expand the Dyson series of the time propagator to the second order in H_I [32]. By doing that we obtain two terms in the effective Hamiltonian $H_{\rm eff} = H_{\rm res} + H_{XY}$:

$$H_{\rm res} = \sum_{j,n,m,\alpha,\beta} J^{\rm res}_{jnm,\alpha\beta}S^j_z\left(b^\dagger_{n\alpha}b_{m\beta} + \frac{1}{2}\delta_{\alpha,\beta}\delta_{n,m}\right)e^{-i(\nu^\alpha_n - \nu^\beta_m)t}, \quad (6)$$

$$H_{XY} = \sum_{i,j}J^{\rm eff}_{ij}\left((S^i_xS^j_x + S^i_yS^j_y)(1-\delta_{i,j}) - \frac{(S^j_z)^2}{2}\delta_{i,j}\right), \quad (7)$$

where

$$J^{res}_{jnm} = \frac{\Omega}{2\sqrt{2}}v^\alpha_n v^\beta_m\eta^\alpha_{jn}\eta^\beta_{jm}\left(\frac{1}{\left(\frac{\Omega}{\sqrt{2}}\right)^2 - \left(v^\alpha_n\right)^2} + \frac{1}{\left(\frac{\Omega}{\sqrt{2}}\right)^2 - \left(v^\beta_m\right)^2}\right), \quad (8)$$

$$J_{ij}^{eff} = \sum_{n,\alpha} \eta_{in}^{\alpha}\eta_{jn}^{\alpha} \frac{(v_n^{\alpha})^3}{\left(\frac{\Omega}{\sqrt{2}}\right)^2 - (v_n^{\alpha})^2} \propto \left|\vec{r}_i^{\,0} - \vec{r}_j^{\,0}\right|_{i\neq j}^{-\alpha}, \quad (9)$$

which includes a power-law spin-spin interactions instead of the theoretical simplification of nearest-neighbors interactions [12].

Our model can be decoupled from $H_{\mathrm{res}} \propto \sum_i S_z^i$ since its relevant ground states, as we will see later in the robustness of the ground states to noise section, has a vanishing projection over the z axis, and so we can say that they belong to the decoherence-free subspace. For the same reason, our model is decoupled from the phonons and can be stable against heating as well.

Although a constant D term like the last term in Eq. (1) was obtained, we would like to pursue a tunable D coefficient.

Generation of the anisotropy D term.—Generating a tunable D term is achieved by applying an ac Stark shift using an additional microwave field corresponding to the transition between the states $|0'\rangle \longleftrightarrow |0\rangle$ with a red detuning Δ_r (Fig. 2 left). In the dressed-state basis, $|0\rangle$ is transformed to $(1/\sqrt{2})(|u\rangle - |d\rangle)$; therefore, by moving to the interaction picture according to the carrier transition $(\Omega/\sqrt{2})S_z$ as was mentioned above, which is followed by an expansion of the Dyson series of the time propagator to the second order, all the off-diagonal terms are suppressed using RWA, if we assume that $(\Omega_r/2\sqrt{2}) \ll (\Omega/\sqrt{2}) \ll \Delta_r$. As a consequence, the anisotropy $D' = \Omega_r^2/8\Delta_r$ term is obtained, which is slightly different from D since the model is θ rotated while generating the λ term.

Generation of the Ising-like λ term.—Instead of producing the Ising-like anisotropy term $\lambda S_z^i S_z^{i+1}$ explicitly as was done with the anisotropy D term, we will use the following "trick." We will add $H_2 = \Omega' S_{z,\theta} = \Omega'(S_z \cos\theta - S_y \sin\theta)$ to the Hamiltonian, which is $\Omega' F_{x,\theta} = \Omega'(F_x \cos\theta - F_y \sin\theta)$ in the original basis. This is done by generating each term separately. The first term $\Omega' F_x \cos\theta = \Omega' S_z \cos\theta$ can be set aside from the previous microwave transitions that produce H_{XY}. The second term $\Omega' S_y \sin\theta$ (Fig. 2 right) is produced by applying two microwave fields corresponding to the transitions $|\mp 1\rangle \longleftrightarrow |0\rangle$ with $\pm((\Omega/\sqrt{2}) - \Omega' \cos\theta)$ detunings, $\pm(\pi/2)$ initial phases, and the same Rabi frequency $\sqrt{2}\Omega' \sin\theta$, respectively, after transforming to the interaction picture according to $((\Omega/\sqrt{2}) - \Omega' \cos\theta)F_x$ and neglecting the fast rotating terms. By applying a θ rotation around the x axis, the operators are transformed as follows: $S_{z,\theta} \to S_z$, $S_z \to S_z \cos\theta + S_y \sin\theta$, $S_y \to S_y \cos\theta - S_z \sin\theta$, and S_x is not changed, and the new dressed-state basis is θ rotated around the x axis as well. If we move to the interaction picture corresponding to the new term we have built and use RWA where we require $(\eta_{in}^{\alpha} v_n^{\alpha}/2) \ll \Omega'$, we will end up with the following effective Hamiltonian:

$$H_{\mathrm{eff}} = \sum_{i\neq j} J_{ij}^{\mathrm{eff}} \left\{ (S_x^i S_x^j + S_y^i S_y^j)\left(\frac{1+\cos^2\theta}{2}\right) + S_z^i S_z^j \frac{\sin^2\theta}{2} \right\} + \sum_i \left(\frac{\Omega_r^2}{8\Delta_r} - \frac{J_{ii}^{\mathrm{eff}}}{2}\right)\left(\frac{1+\cos^2\theta}{2} - \sin^2\theta\right)(S_z^i)^2, \quad (10)$$

where for simplicity we assume that $\Omega' \ll \Omega$, although it is not a real physical restriction. We have obtained a spinone XXZ AFM Hamiltonian with power-law spin-spin interactions and the following tunable parameters: the anisotropy D parameter, the Ising-like λ parameter and the power-law α exponent. The presence of the long-range interactions, which replace the nearest-neighbors interactions, gives rise to nontrivial longer-range corrections; however, it does not destroy the Haldane phase [20].

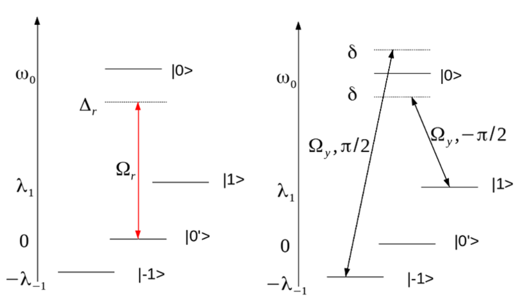

FIG. 2 (color online). Driving fields for creating the D and λ coefficients. Left side: Generating the D term in Eq. (1) is done by an ac Stark shift via the red detuned Δ_r transition. Right side: The transitions that are required for generating $\Omega' S_y \sin\theta$ and applying the "trick" where $\delta = ((\Omega/\sqrt{2}) - \Omega' \cos\theta)$ is the detuning, $\pm(\pi/2)$ are the initial phases, and $\Omega_y = \sqrt{2}\Omega' \sin\theta$ is the Rabi frequency.

Reaching the Haldane phase adiabatically.—In order to prepare the system in the Haldane phase and to reach its ground state, we first start with the large D Hamiltonian and generate its ground topologically trivial state which is a tensor product of local $|D\rangle$ states in every site [11,30]. Then we switch on the required Hamiltonian adiabatically (i.e., slower than the energy gap) by lowering the D coefficient. For an infinite chain, a problem arises when we reach a second-order phase transition in the phase diagram (Fig. 3) where the energy gap closes [24] and the adiabatic approximation cannot hold. To overcome this obstacle, we take advantage of the fact that the Haldane phase is an asymmetry-protected topological phase. Far enough from the $D \to H$ phase transition, we will adiabatically turn on a perturbation in the Hamiltonian that breaks all the symmetries in this system. Then, when we are in the region of the Haldane phase, we will adiabatically turn this perturbation off. In that way, we will not cross any second-order phase border, and, hence, we will always stay in the adiabatic approximation.

The Haldane phase is protected by the following symmetries: a bond centered spatial inversion $\vec{S}_j \to \vec{S}_{-j+1}$,

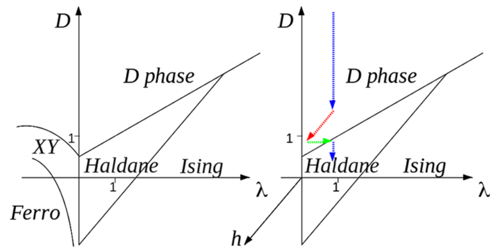

FIG. 3 (color online). Phase diagram and adiabatic path. Phase diagram of the Haldane phase, where the phases are the large D phase, the Haldane phase, the Ising-like phase, the XY phase, and the ferromagnetic phase. The colored arrows represent the adiabatic path from the large D phase to Haldane phase. The blue arrow stands for lowering the D coefficient, the red arrow stands for switching on the symmetry breaking h term while still lowering D, and the green arrow stands for switching off the h term while lowering D.

a time reversal symmetry $\vec{S}_j \to -\vec{S}_j$, or the dihedral D_2 symmetry, which is the π rotations around the x, y, and z axes. In order to break all these symmetries, we add a perturbation term $H_{\text{pert}} = -h\sum_i (-1)^i S_z^i$. This term can be produced by individual addressing [31] using a microwave frequency comb with a staggered phase. Taking advantage of the magnetic field gradient along the chain axis, each ion experiences a different Zeeman splitting, and only the right frequency from the comb can interact with it.

Before crossing the phase transition, the symmetry breaking perturbation should be turned on adiabatically, while still lowering the D coefficient in the Hamiltonian. Then, the perturbation should be turned off adiabatically, reaching the plane $h = 0$ in the Haldane phase domain. If the time duration of this procedure is shorter than the coherence time, the ground state in the Haldane phase should be achieved with high fidelity.

Detecting and measuring the ground states.—As a topological phase, the Haldane phase does not obey the Landau paradigm and cannot be characterized by a local order parameter. However, there are other properties that can characterize it: (1) an excitation gap and exponentially decaying correlations $C_{ij}^\alpha = \langle S_i^\alpha S_j^\alpha \rangle - \langle S_i^\alpha \rangle^2$, (2) a nonvanishing nonlocal string order $O_{\text{string}}^\alpha(H) = \lim_{|i-j|\to\infty} \langle -S_i^\alpha \exp [i\pi \sum_{l=i+1}^{j-1} S_l^\alpha] S_j^\alpha \rangle$, where $\langle\rangle$ denotes the expectation value in the ground state, and (3) a symmetry-protected double degenerate entanglement spectrum obtained by dividing the systems into two parts, tracing out one of them and diagonalizing the reduced density matrix [33].

Using trapped ions as a platform to verify the Haldane phase allows us to measure every spin state with high fidelity and accuracy. The tomography of an exponentially growing Hilbert space with the size of the system is time consuming. Yet, efficient, tailored reconstruction methods [34] make it possible to calculate the correlation functions and the string order. Thanks to the power-law interaction in our model, we should find a power-law tail [20] in the correlation functions in addition to the exponentially decay $C_{ij}^\alpha = Ae^{-(|i-j|/\xi)} + B|i-j|^{-a}$. By implementing a less time-consuming method, we can calculate the entanglement spectrum of the ground state and determine whether it is double degenerate, according to the Haldane phase signature.

Robustness of the ground states to noise.—Preparing the system in the Haldane phase is done adiabatically. Note that this can take more time than the time scale set by the noise sources. The main noise sources here are the fluctuating magnetic field in the z direction and the fluctuations in the Rabi frequencies of the driving fields. If during the path we represented above, the ground states are in the decoherence-free subspace, the specific subspace of Hilbert space that is invariant under these fluctuations (to the first order), we will be able to walk on that path adiabatically.

The dressed basis used for representing the spins is the eigenstates of F_x $\{|u\rangle, |D\rangle, |d\rangle\}$. These states are robust to the magnetic noise in the z direction, as $\langle s|o_{(t)}F_z|s\rangle = 0$ for $|s\rangle \in \{|u\rangle, |D\rangle, |d\rangle\}$. Moreover, the ground state of the large D phase which is the topologically trivial state of $|D\rangle$ states in every site and the ground state of the Haldane phase [19] which has the same number of sites occupied with $|u\rangle$ and $|d\rangle$ are robust to the fluctuations in the Rabi frequencies, as $o_{(t)}F_x|GS\rangle = o_{(t)}S_z|GS\rangle = 0$. The remarkable structure of the ground states explains why our model is decoupled from $H_{\text{res}} \propto S_z^i$ [Eq. (6)], and we do not have to work hard to dynamically decouple it by a π-pulse sequence [35].

Higher dimensions.—We can scale up to higher dimensions with frustrated geometries. The frustration is only increased due to the presence of the long-range interactions in the presented model. Our suggested platform gives rise to a new research of exploring the quantum spin liquid phases that exhibit long-range entanglement patterns and hidden global topological orders. Their highly nonlocal ground states would be robust to the local noise sources, giving rise to the realization of the topologically protected quantum computation.

Using the linear Paul trap, by lowering the transverse trap frequency ω_z enough, we observe a phase transition from the linear formation to a frustrated zigzag n-ladder formation [29]. As the fabrication technology progresses, arbitrary geometries could be produced for the ion lattice [36]. The ground state of the spin liquid phase is characterized (and as a result can be detected) by the entanglement entropy that obeys the boundary law $S = a\sigma_R - \gamma$, where σ_R is the $(d-1)$-dimensional volume surrounding the region, and $\gamma = \log\, d$ is a fixed value in the topologically ordered phase, which is independent of the lattice's geometry [37].

Summary.—We proposed a scheme for realization of spin-one Heisenberg AFM systems. The Hamiltonian can be generated adiabatically, starting from the large D phase until it reaches the spin liquid phase (or the Haldane phase in $d = 1$). The ground states are robust to the magnetic noise, decoupled from the fluctuations in the

Rabi frequencies, and can be detected by their characteristics as was mentioned above.

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