Resurgence in Quantum Field Theory: Nonperturbative Effects in the Principal Chiral Model

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We explain the physical role of nonperturbative saddle points of path integrals in theories without instantons, using the example of the asymptotically free two-dimensional principal chiral model (PCM). Standard topological arguments based on homotopy considerations suggest no role for nonperturbative saddles in such theories. However, the resurgence theory, which unifies perturbative and nonperturbative physics, predicts the existence of several types of nonperturbative saddles associated with features of the large-order structure of the perturbation theory. These points are illustrated in the PCM, where we find new nonperturbative "fracton" saddle point field configurations, and suggest a quantum interpretation of previously discovered "uniton" unstable classical solutions. The fractons lead to a semiclassical realization of IR renormalons in the circle-compactified theory and yield the microscopic mechanism of the mass gap of the PCM.

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In general, observables in quantum field theories (QFTs) receive perturbative and nonperturbative contributions. The perturbative contributions summarize information about quantum fluctuations around the trivial perturbative saddle point (vacuum) of the path integral, while the nonperturbative contributions come from quantum fluctuations around the nontrivial nonperturbative (NP) saddle points. In this Letter, we develop a deeper understanding of what the structure of perturbation theory implies for the nature and existence of NP saddle points of path integrals.

We illustrate these ideas with the two-dimensional (2D) $SU(N)$ principal chiral model (PCM). The PCM is an asymptotically free matrix field theory and is believed to have a dynamically generated mass gap determined by the strong scale $\Lambda = \mu e^{-[4\pi/g^2(\mu)N]}$, with μ the renormalization scale; see, e.g., [1–3]. The PCM models many features of 4D Yang-Mills (YM) theory, but historically it has received less attention [4] than its vector-model cousin, the $\mathbb{CP}^{N-1}$ model, because (i) since $\pi_2[SU(N)] = 0$, there are no topologically stable instanton configurations which may lead to NP factors such as e^{-t/g^2}, and (ii) its large-N limit is not analytically tractable [5]. The $\mathbb{CP}^{N-1}$ model has neither of these issues, but (i) is also shared with many other 2D QFTs, such as $O(N > 3)$ and $Sp(N)$ models, which are relevant to condensed matter physics [6].

However, the divergent structure of the perturbative series in the PCM is very similar to YM or $\mathbb{CP}^{N-1}$. After regularization and renormalization, the perturbative series has at least two types of factorial divergences. One is due to the combinatorics of the Feynman diagrams, while the other is known as the IR and UV renormalon divergence [2,7] and comes from the low and high momenta in phase space integrals. Resummation of the perturbative series using the standard technique of Borel summation leads to singularities in the Borel plane. IR renormalons render the Borel sum ambiguous and ill defined, because there is a subset of factorially divergent terms that do not alternate in sign. These problems are ubiquitous in asymptotically free QFTs, including YM and QCD (see, e.g., [8]) as well as in string theory and matrix models [9].

It is generally believed that in quantum mechanics (QM) and QFT, the ambiguities in the summation of perturbative series due to the growth in the number of Feynman diagrams cancel against ambiguities in the contributions from NP instanton–anti-instanton saddle points, $[\mathcal{I}\bar{\mathcal{I}}]$, $a_n \sim n!/(S_{[\mathcal{I}\bar{\mathcal{I}}]})^n$ [7]. On the other hand, the semiclassical meaning of IR renormalons has been unclear until recently, when it was shown that renormalons may also be continuously connected to new semiclassical NP saddle points [10,11]. In the weak coupling regime of circle-compactified deformed YM and QCD(adj) in 4D, and the $\mathbb{CP}^{N-1}$ model in 2D, it was shown that Belavin-Polyakov-Schwarz-Tyupkin instantons fractionalize into N monopole instantons $\mathcal{M}_i$ [12,13] and N kink instantons $\mathcal{K}_i$ [14,15], respectively. Correlated $[\mathcal{K}_i\bar{\mathcal{K}}_i]$ and $[\mathcal{M}_i\bar{\mathcal{M}}_i]$ events control the IR renormalon singularities in these theories and render physical observables unambiguous through the mechanism of resurgence [10,11,16].

However, the PCM has neither instantons nor fractional instantons [17]. In fact, the PCM has no known stable NP saddles which could lead to NP factors such as e^{-t/g^2}. This produces a deep puzzle. Since the perturbative series is divergent and non-Borel summable, an attempt to do Borel resummation results in ambiguities of the form $\pm i e^{-t_i/g^2}$. If

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the theory is to be semiclassically meaningful and well defined according to the criterion of Refs. [10,11], such NP ambiguities *must* cancel; i.e., there must exist NP saddles whose amplitude is proportional to $\pm ie^{-t_i/g^2}$. This is a highly nontrivial prediction of the resurgence theory applied to QFT. But, since there are no instantons, what are these NP saddles?

Thus the perturbative similarity (in particular, the non-Borel summability due to IR renormalons) between the PCM and other asymptotically free theories such as YM and $\mathbb{CP}^{N-1}$ appears to be in conflict with their NP difference: YM and $\mathbb{CP}^{N-1}$ have nontrivial homotopy groups and hence have instantons, while the PCM has trivial homotopy, $\pi_2[SU(N)] = 0$, and no stable instantons. This suggests that naive topological considerations based solely on homotopy are insufficient to fully characterize NP saddles and miss a large class of important NP saddle points. Some early hints that nontopological field configurations may make important contributions in 2D field theories appeared in, e.g., Refs. [18,19], but an explicit identification of such configurations was not achieved at that time. In this work, we combine the resurgence theory with a physical principle of continuity and show the existence of new NP saddles in the path integral of the PCM, which we refer to as "fractons" following the important early work in Ref. [20]. Our analysis easily generalizes to other theories, such as $O(N > 3)$ or $Sp(N)$ models, which also have no instantons.

Unitons.—The PCM has classical action

$$S_b = \frac{N}{2\lambda}\int_M d^2x\, \mathrm{tr}\partial_\mu U \partial^\mu U^\dagger, \qquad U \in SU(N), \quad (1)$$

where $\lambda = g^2 N$ is a dimensionless coupling constant and we work in Euclidean space with $M = \mathbb{R}^2$ and $\mathbb{R}\times S^1$. The PCM has the symmetry $SU(N)_L \times SU(N)_R$ acting as $U \to g_L U g_R^\dagger$. There are no instantons, but there exist "unitons," *finite-action* solutions to the *second-order* Euclidean equations of motion, discovered in the seminal work of Uhlenbeck [21]. Further properties are discussed in Ref. [22]. These uniton solutions, which are harmonic maps from S^2 into $SU(N)$, did not receive much attention in the QFT literature, mainly because they are unstable; small fluctuations lead to a decrease in the action [23]. Topologically, this instability is expected, since instantons would be smooth maps from compactified base space $M = \mathbb{R}^2 \cup \infty = S^2$ to the target space $\mathcal{T} = SU(N)$ and classified by the second homotopy group $\pi_2[SU(N)]$. But $\pi_2[SU(N)] = 0$, so there are no topologically stable instantons.

However, the notion that all finite action saddles contributing to path integrals of, e.g., 2D sigma models are classified by $\pi_2[\mathcal{T}]$ is incorrect, even if $\pi_2[\mathcal{T}]$ is nontrivial, as pointed out in Ref. [15] for the $\mathbb{CP}^{N-1}$ model, which has exact finite action non-Bogomolny-Prasad-Sommerfield (BPS) solutions, and where the connection to resurgence was also emphasized. One way to think about such solutions is to observe that in the path integral the relevant quantity is the topology of the infinite-dimensional space of field configurations $\pi_n[\mathrm{Maps}\colon S^2 \to \mathcal{T}] = \pi_{n+2}[\mathcal{T}]$, and not just the number of disconnected components of $\mathrm{Maps}\colon S^2 \to \mathcal{T}$ counted by $\pi_d(\mathcal{T})$ [24]. From this perspective, unitons are associated to $\pi_3[SU(N)]$. In fact, the issue with focusing only on $\pi_2[\mathcal{T}]$ can already be seen in QM. Consider an instanton in QM with a periodic potential and coupling g, where instantons are classified by their winding number $W \in \pi_1(S^1) = \mathbb{Z}$, and the basic instanton solution has $W = 1$. This is a solution to the first-order BPS equation and possesses an exact zero mode, its position. The amplitude for this event is $\mathcal{I} \sim e^{-S_\mathcal{I} + i\Theta}$, where Θ is the topological Θ angle [6]. Now, consider a correlated instanton–anti-instanton event $[\mathcal{I}\bar{\mathcal{I}}]$. This is *homotopically indistinguishable* from the perturbative vacuum, since $W = 1 + (-1) = 0$, but its action is $S = 1 + 1 = 2$, in units of the instanton action. Yet the separation between the two instantons is a negative quasizero mode, and the action of this saddle decreases with decreasing separation. To write the two-event amplitude, one must integrate over the quasizero mode. Naively, when $\arg(g^2) = 0$ this integration is dominated by short distances and is ill defined. However, doing the quasizero mode integration at $\arg(g^2) = 0^\pm$, we find $[\mathcal{I}\bar{\mathcal{I}}]_\pm \sim (\log(1/g^2) - \gamma \pm i\pi)e^{-2S_\mathcal{I}}$, a twofold ambiguous result. This is a manifestation of the fact that $\theta = \arg(g^2) = 0$ is a Stokes line. The resurgence theory explains that the purely imaginary ambiguous part of the nonperturbative amplitude cures the ambiguity associated with the non-Borel summability of the perturbative series, i.e., $\mathrm{Im}(\mathbb{B}_{0,\theta=0^\pm} + [\mathcal{I}\bar{\mathcal{I}}]_\pm \mathbb{B}_{2,\theta=0^\pm}) = 0$, where $\mathbb{B}_{0,\theta=0^\pm}$ and $\mathbb{B}_{2,\theta=0^\pm}$ are left or right Borel sums of the formal perturbative series describing quantum fluctuations around the perturbative and nonperturbative $[\mathcal{I}\bar{\mathcal{I}}]_\pm$ saddle points of the path integral, respectively [25]. For a fuller discussion of this cancellation mechanism, see [11]. Thus, the "instability" of the $[\mathcal{I}\bar{\mathcal{I}}]$ saddle point, i.e., a negative mode in the fluctuation operator, is in fact a positive feature, not a deficiency. Without it, the theory would be ill defined.

Unitons, as finite action non-BPS field configurations (just like $[\mathcal{I}\bar{\mathcal{I}}]$ events), must be summed over in a semiclassical analysis of the path integral. The uniton action is quantized in units of $S_\mathcal{U} \equiv 8\pi/g^2$ [22]. The minimal uniton solution is easy to obtain. Let $v(z) \in \mathbb{C}^N$, with $z = x_1 + ix_2$, $x_\mu \in M$, be a single instanton solution in $\mathbb{CP}^{N-1}$ [26]. Then the minimal uniton in the $SU(N)$ PCM is given by $U(z, \bar{z}) = e^{i\pi/N}(1 - 2\mathbb{P})$, where $\mathbb{P}$ is the projector: $\mathbb{P}_{ij} = v_i v_j^\dagger / v^\dagger \cdot v$. Figure 1 (left) depicts a small uniton in $SU(2)$.

Our results here suggest that the uniton amplitude provides a substitute for instantons in theories with a trivial homotopy group. We write the amplitude associated with the basic uniton event and observe its relation to the strong scale

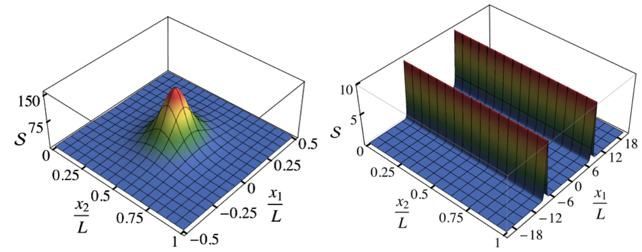

FIG. 1 (color online). Action densities $\mathcal{S}$ for small (left) and large (right) $SU(2)$ unitons in the setting described in the text. The large uniton splits into two fractons.

$$\mathcal{U}(\mu) \sim \det[O_{\mathcal{U}}(\mu)]e^{-[8\pi/g^2(\mu)]}, \qquad \Lambda^{2\beta_0} = \mu^{2\beta_0}\mathcal{U}, \quad (2)$$

where det $O_{\mathcal{U}}$ encodes the Gaussian fluctuations around the uniton saddle point. In contrast, for theories with instantons, we would have $\Lambda^{\beta_0} = \mu^{\beta_0}\mathcal{I}$, and $\beta_0 = N$ is the one-loop β function of the theory.

Fractons.—We cannot directly study the dynamics of the theory on $\mathbb{R}^2$, because the PCM becomes strongly coupled at large distances, just like YM and $\mathbb{CP}^{N-1}$. However, there exists a way to continuously connect the strongly coupled PCM on $\mathbb{R}^2$ to a weakly coupled calculable regime, analogous to the double-trace deformation of the YM theory [27]. Then, we can address many NP questions in weak coupling and, by adiabatic continuity, extract universal results valid even at strong coupling. This setup also permits us to see the fractionalizaton of unitons into their constituents, the *fractons*. The construction involves introducing the twisted boundary conditions $U(x_1, x_2 + L) = e^{iH}U(x_1, x_2)e^{-iH}$, $e^{iH} = \mathrm{Diag}[e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_N}]$, or equivalently turning on an x_2 component of a background gauge field for the $SU(N)_V$ symmetry so that $\partial_\mu U \to D_\mu U = \partial_\mu U + i\delta_{\mu 2}L^{-1}[H, U]$. Then we study the dynamics of the action $S_b = (N/2\lambda)\int_M d^2x\,\mathrm{tr}|D_\mu U|^2$. It turns out that there is a unique choice for the μ_i, determined by the condition of unbroken $\mathbb{Z}_N$ center symmetry, such that the small-L theory is continuously connected to its large-L limit, without phase transitions or rapid crossovers as a function of L. For more details, see [28].

Working with the special small-L theory defined above, the easiest way to show the splitting of a uniton into fractons is the following. Let $v_{\mathrm{tw}}(z)$ be a single instanton solution in $\mathbb{CP}^{N-1}$ on $\mathbb{R}\times S^1$ in the presence of twisted boundary conditions, which exhibits fractionalization of an instanton into kink instantons as the size moduli of the instanton is changed from small to large [11,14]. Then, the uniton in the $SU(N)$ PCM is given by $U_{\mathrm{tw}}(z,\bar{z}) = e^{i\pi/N}(1 - 2\mathbb{P}_{\mathrm{tw}})$, where $\mathbb{P}_{\mathrm{tw}} = v_{\mathrm{tw}}(v_{\mathrm{tw}})^\dagger/(v_{\mathrm{tw}})^\dagger v_{\mathrm{tw}}$. Figures 1 (right) and 2 depict the fracton constituents of a uniton for $N = 2$, 3, 4.

It is straightforward to construct explicit solutions corresponding to isolated fractons in the $SU(N)$ PCM. For instance, for $SU(2)$, by using Hopf coordinates θ, ϕ_1, ϕ_2, the action is

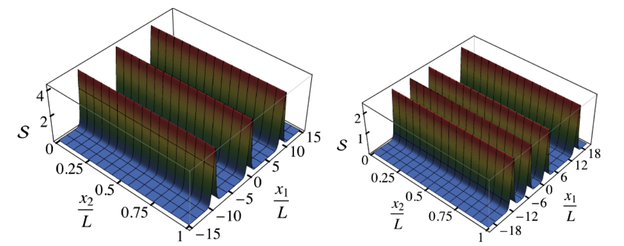

FIG. 2 (color online). Action densities $\mathcal{S}$ for large $SU(3)$ and $SU(4)$ unitons, which split into three and four fractons, respectively.

$$S = \frac{1}{g^2}\int_M [(\partial_\mu\theta)^2 + \cos^2\theta(\partial_\mu\phi_1)^2 + \sin^2\theta(\partial_\mu\phi_2 + \xi\delta_{\mu 2})^2],$$

where $\xi = 2\pi(\mu_2 - \mu_1)L^{-1}$. In the small-$L$ regime, forgetting about the high Kaluza-Klein modes, we land on QM with a nontrivial potential on the $SU(2)$ manifold: $S = (L/g^2)\int_{\mathbb{R}}[\dot{\theta}^2 + \cos^2\theta\dot{\phi}_1^2 + \sin^2\theta\dot{\phi}_2^2 + \xi^2\sin^2\theta]$, where the crucial existence of the potential term is due to the nontrivial background holonomy. The equations of motion associated with this action admit the solution $\phi_{1,2} = \phi_{1,2}^0$ and $\theta(x_1; x_1^{(0)}) = 2\mathrm{arccot}[e^{-\xi(x_1 - x_1^{(0)})}]$. The constants of integrations $\{\phi_1^0, \phi_2^0, x_1^{(0)}\}$ are the three zero modes associated with a fracton. These configurations, which emerge for $\xi \neq 0$, are stable, since they are just instantons from the point of view of the small-L effective field theory. The small-L effective potential on the $SU(2)$ manifold yields an effective "topology" stabilizing the fractons in this regime. Hence, in the small-L limit in which we work, the individual fracton contributions to the path integral are free of ambiguities.

As in the gauge theory on $\mathbb{R}^3 \times S^1$ and the $\mathbb{CP}^{N-1}$ model on $\mathbb{R}\times S^1$, where there exist Kaluza-Klein (KK) monopole instantons [12,13] and KK kink instantons [14], respectively, which are associated with the affine root of the $SU(N)$ algebra, there is also a KK fracton in the PCM. By taking this into account, there are N basic types of fractons in the $SU(N)$ PCM in a $\mathbb{Z}_N$ symmetric background, each of which carries $1/N$ of the action of a uniton. Namely,

$$\mathcal{F}_i \sim e^{-[8\pi(\mu_{i+1}-\mu_i)/g^2]} \sim e^{-(8\pi/g^2N)}, \qquad \mathcal{U} = \prod_{i=1}^{N}\mathcal{F}_i. \quad (3)$$

The surprise here with respect to earlier work [12–15,29] is that we are now considering a theory which does not have instantons. Since each fracton carries three zero modes, and each uniton is composed from N fractons, the number of the combined zero and quasizero modes of a uniton must be $3N$. This is analogous to what happens in the $\mathbb{CP}^{N-1}$ model, where each instanton has $2N$ exact zero modes, and each kink instanton has two zero modes.

Renormalon and uniton ambiguities on $\mathbb{R}^2$.—The IR renormalon divergence and ambiguities on $\mathbb{R}^2$ can be determined in two different ways. One is by studying a subclass of planar Feynman diagrams. The number of planar diagrams grows only exponentially [30,31], but a subset of such diagrams contribute factorially due to momentum integration at large orders; hence, the effect is present at large N as well [8]. Another way is using the S matrix and Bethe-ansatz equations (starting with the standard assumptions thereof, such as that the theory is gapped). The approaches must give the same answer, but for the PCM the second approach has been the main one pursued, with the result that the nonalternating late terms in the perturbative series diverge as $n!(\lambda/8\pi)^n$, meaning that the perturbative series is non-Borel resummable [2]. This produces an ambiguity of the form $\pm i e^{-(8\pi k/g^2 N)}$. The IR renormalon singularities found in Ref. [2] lie on the positive real Borel axis at

$$\mathbb{R}^2: \; t_k^+ = 8\pi k/N = k[g^2 S_{\mathcal{U}}]/\beta_0, \qquad k \in \mathbb{Z}^+. \tag{4}$$

The appearance of the 't Hooft coupling in the IR renormalon ambiguity means that, unlike instanton–anti-instanton and uniton ambiguities, it does not go away in the large-N limit. Also note that the leading IR renormalon singularity is proportional to the square of the strong scale, $e^{-[8\pi/g^2(Q)N]} \sim (\Lambda/Q)^2$, where Q is the Euclidean momentum.

In theories with instantons and a nontrivial homotopy group π_d, the leading IR renormalon ambiguity is approximately $e^{-S_{\mathcal{I}\bar{\mathcal{I}}}/\beta_0} = e^{-2S_{\mathcal{I}}/\beta_0}$ and is exponentially larger than the $[\mathcal{I}\bar{\mathcal{I}}]_\pm$ ambiguity, as emphasized by 't Hooft [7]. In the PCM, the relation between the leading renormalon and uniton ambiguity is $e^{-(8\pi/g^2 N)} \sim e^{-S_{\mathcal{U}}/\beta_0}$. Since $\pi_2[\mathcal{T}]$ is trivial for the PCM, there is nothing preventing the uniton amplitude from canceling the ambiguity in the Borel resummation due to a singularity in the Borel plane associated with the perturbative sector. This is to be contrasted with instantons, which carry a topological charge, and cannot cure perturbative series ambiguities due to singularities in the Borel plane. Indeed, on $\mathbb{R}^2$, we expect a Borel plane singularity related to the combinatorics of Feynman diagrams, which we conjecture is connected to the uniton amplitude. The cancellation mechanism for the IR renormalon ambiguities on $\mathbb{R}^2$ is unknown, but after spatial compactification to small $\mathbb{R} \times S^1$ the theory is under control. Below, we provide a microscopic mechanism of cancellation on $\mathbb{R} \times S^1$ in the regime of the theory continuously connected to $\mathbb{R}^2$.

Continuity and cancellation of semiclassical renormalon ambiguities on $\mathbb{R} \times S^1$.—At small L, the theory reduces to QM, which is continuously connected to the 2D QFT. Consider the ground state energy $\mathcal{E}$. The late terms of the perturbative series for $\mathcal{E}$ involve a nonalternating divergent subseries. Upon left or right Borel resummation $\mathcal{S}_{0^\pm}\mathcal{E}$ at $g^2 + i0^\pm$, we find a twofold ambiguous result [28]:

$$\mathcal{S}_{0^\pm}\mathcal{E}(g^2) = \mathrm{Re}\,\mathbb{B}_0 \mp i\frac{32\pi}{g^2 N} e^{-(16\pi/g^2 N)} \tag{5}$$

reflecting the non-Borel summability of the theory on the $\arg(g^2) = 0$ Stokes line. This is the semiclassical realization of the renormalon ambiguity. The associated semiclassical singularities in the Borel plane are located at

$$\mathbb{R} \times S^1: \; t_k^{+,\mathrm{s.c.}} = \frac{16\pi k}{N} = 2 \times k \times \frac{g^2 S_{\mathcal{U}}}{\beta_0}, \qquad k \in \mathbb{Z}^+ \tag{6}$$

diluted by a factor of 2 with respect to $\mathbb{R}^2$ but parametrically in the same neighborhood as the IR renormalon singularities of 't Hooft seen in (4).

Remarkably, as predicted by the resurgence theory of Ecalle [16], this ambiguity cancels against the fracton-antifracton correlated events, for which the leading amplitude at $g^2 + i0^\pm$ are given by [28]

$$\begin{aligned}[\mathcal{F}_i\bar{\mathcal{F}}_i]_{\theta=0^\pm} &= \mathrm{Re}[\mathcal{F}_i\bar{\mathcal{F}}_i] + i\,\mathrm{Im}[\mathcal{F}_i\bar{\mathcal{F}}_i]_{\theta=0^\pm} \\ &= \left[\log\left(\frac{\lambda}{16\pi}\right) - \gamma\right]\frac{16}{\lambda}e^{-(16\pi/\lambda)} \\ &\quad \pm i\frac{32\pi}{\lambda}e^{-(16\pi/\lambda)}. \end{aligned} \tag{7}$$

This leads to the cancellation of the nonperturbative ambiguities coming from the perturbative series against the ambiguity that arises from the NP saddle. That is,

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}[\mathcal{F}_i\bar{\mathcal{F}}_i]_{\theta=0^\pm} = 0. \tag{8}$$

This is a QFT example of Borel-Écalle resummation, a generalization of Borel resummation to account for the Stokes phenomenon [10,11,32].

Mass gap on $\mathbb{R} \times S^1$ *and* $\mathbb{R}^2$.—A speculation by 't Hooft that IR renormalons may be related to the mass gap and confinement in QCD [7] finds a concrete realization in our approach. The leading ambiguity on $\mathbb{R}^2$ is proportional to $e^{-[8\pi/g^2(Q)N]} \sim \Lambda^2/Q^2$, and recent works [10,11] have shown that, in the semiclassical domain, it is always "half" of the renormalon which leads to a mass gap. If we assume that this semiclassical fact extrapolates to the strongly coupled domain, we observe that indeed $e^{-[4\pi/g^2(Q)N]} \sim \Lambda/Q$, proportional to the first power of the strong scale. In our current example, in the semiclassical domain, the mass gap is a one-fracton (half-renormalon) effect and is given by $m_g \sim (1/LN)e^{-(8\pi/\lambda)} \sim \Lambda(\Lambda LN)$ for $LN\Lambda \lesssim 2\pi$. In future work, it would be important to understand fully the origin of the dilution factor highlighted in Eq. (5) as the theory moves continuously from the semiclassical domain to the strongly coupled domain.

Conclusions.—The resurgence theory shows that in the principal chiral model (an asymptotically free theory without stable instantons) naive homotopy considerations are insufficient to classify saddle points in the path integral.

Requiring the model to be well defined in the sense of Borel-Écalle summability [10,11], together with the physical principle of continuity and spatial compactification [27], leads to the existence of new non-BPS fracton solutions giving a semiclassical realization of IR renormalons and also provides a quantum interpretation to the classical uniton solutions. The fracton contributions to the path integral of the PCM give the microscopic origin of the mass gap of the theory.

These results illustrate the general lesson that all nonperturbative saddle points, not only self-dual and stable ones, can make vital contributions to path integrals in theories with or without instantons. While our discussion focused here on the PCM, similar ideas apply to $O(N)$ and $Sp(2N)$ sigma models [33] and also to pure Yang-Mills theory, where the resurgent approach provides a more refined classification of NP saddles relative to the naive homotopy classification.

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